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ORLICZ SPACE CONVERGENCE OF MARTINGALES OF RADON-NIKODYM DERIVATIVES GIVEN A σ-LATTICE

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Let $\{M_k\}$ be an increasing sequence of sub σ-lattices of a σ-algebra $\mathscr{A}$, and let M be the σ-lattice generated by $\bigcup_k M_k$. Let $L\Phi$ be an associated Orlicz space of $\mathscr{A}$-measurable functions, where Φ does not necessarily satisfy the Δ_2-condition. Given $h \in L\Phi$, let f_k be the Radon-Nikodym derivative of h given M_k. Necessary and sufficient conditions are given on h to insure that $\{f_k\}$ converges in $L\Phi$ to f, where f is the Radon-Nikodym derivative of h given M. The situation where f is valued in a Banach space with basis is also examined.

1. Introduction. If λ and μ are countably additive set functions defined on a σ-lattice of sets, then the Radon-Nikodym derivative of λ with respect to μ has been defined by Johansen [4]. We may consider this derivative as a conditional expectation of a function with respect to the σ-lattice in the case where λ is absolutely continuous with respect to μ. Hence we may define martingales in this setting. The relation between martingales and Orlicz spaces has been studied by Darst and DeBoth [3] in the case where the Orlicz function Φ satisfied the Δ_2-condition. In this paper we drop the Δ_2-condition and give necessary and sufficient conditions for all martingales to converge to the appropriate function. We also consider the extension of this theory to Banach space valued set functions.

2. Notation. Let M be a sub σ-lattice of a σ-algebra $\mathscr{A}$ of subsets of a nonempty set Ω, and let λ and μ be countably additive, real valued set functions defined on $\mathscr{A}$. Then f is a derivative of λ with respect to μ on M if f is an extended real-valued function defined on Ω such that

(1) f is M-measurable ($[f > a]$ belongs to M for every real a)

(2) $\lambda(A \cap [f < b]) \leqq b\mu(A \cap [f < b])$ for all $A \in M$, $b \in R$.

(3) $\lambda(B^c \cap [f > a]) \geqq a\mu(B^c \cap [f > a])$ for all $B \in M$, $a \in R$.

Now suppose μ is a finite, nonnegative measure on $\mathscr{A}$, and $h \in L^1(\Omega, \mathscr{A}, \mu)$. Let $\lambda(E) = \int_E h d\mu$ for $E \in \mathscr{A}$. Then λ is a bounded signed measure on $\mathscr{A}$. If f is the Radon-Nikodym derivative of λ with respect to μ on M, then we use the notation $f = E(h, M)$. This notation is used since f is the conditional expectation of h given M

in the case $h \in L^2(\Omega, \mathscr{A}, \mu)$. (See [1].)

The theory of Orlicz spaces may be found in detail in [5]. We will describe here only the facts we need.

Let $\Phi(x)$ be an even, real-valued function defined on R such that $\Phi(0) = 0$. Recall Φ satisfies the Δ_2-condition in case there is a constant $K > 0$ such that $\Phi(2x) \leqq K\Phi(x)$ for all $x \in R$. If

$$\psi(y) = \max_{x \geqq 0} [x|y| - \Phi(x)] ,$$

then ψ is called the complementary function to Φ.

Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space. We denote by $L^\Phi = L^\Phi(\Omega, \mathscr{A}, \mu)$ the space of (equivalence classes of) $\mathscr{A}$-measurable, real-valued functions f on Ω such that $\int_\Omega \Phi(f/N)d\mu < \infty$ for some $N > 0$. L^Φ is a Banach space under either of the following equivalent norms:

$$||f|| = \inf \left\{N: \int_\Omega \Phi\left(\frac{f}{N}\right)d\mu \leqq 1\right\}$$

$$|||f||| = \sup \left\{\left|\int_\Omega fg d\mu\right|: \int_\Omega \psi(g)d\mu \leqq 1\right\} .$$

Using Jensen's inequality, it is easy to see that $L^\Phi \subset L^1$. Hence if $h \in L^\Phi$, then $f = E(h, M)$ is defined.

3. Martingale convergence theorems.

PROPOSITION 1. *If $h \in L^\Phi$, and $f = E(h, M)$, then $f \in L^\Phi$; in fact, $||f|| \leqq ||h||$.*

Proof. The argument used in [3, Thm. 1] can be trivially extended to show that $\int_\Omega \Phi(f/N)d\mu \leqq \int_\Omega \Phi(h/N)d\mu$. Hence if $N = ||h||$, we have $\int_\Omega \Phi(f/N)d\mu \leqq 1$, implying $||f|| \leqq N = ||h||$.

Suppose that $\{M_k\}_{k=1}^\infty$ is an increasing sequence of σ-lattices of subsets of Ω, and M is the σ-lattice generated by $\bigcup_{k=1}^\infty M_k$. Denote by $\mathscr{A}_k$ the σ-algebra generated by M_k. Let $h \in L^\Phi$ and h_k be the $\mathscr{A}_k$-measurable function such that $\int_E h d\mu = \int_E h_k d\mu$ for all $E \in \mathscr{A}_k$. Let $f_k = E(h_k, M_k)$. We call $\{f_k, M_k\}_{k=1}^\infty$ a martingale.

It was shown in [3, Thm. 2] that if Φ satisfied the Δ_2-condition, then $\{f_k\}$ converges to $f = E(h, M)$ in the space L^Φ. We now drop the Δ_2-condition.

LEMMA 2. *If E_Φ denotes the norm closure of the bounded functions in L^Φ, then $g \in E_\Phi$ if and only if $\int_\Omega \Phi(g/N)d\mu < \infty$ for all $N > 0$. $E_\Phi = L^\Phi$ if and only if Φ satisfies the Δ_2-condition.*

Proof. See [5].

THEOREM 3. *Let $h \in L^{\Phi}(\Omega, \mathscr{A}, \mu)$. Then the following statements are equivalent*:

(a) $h \in E_{\Phi}$

(b) *Every martingale* $\{f_k, M_k\}$ *converges to* $f = E(h, M)$ *in* L^{Φ} *norm.*

(c) *Every martingale* $\{f_k, M_k\}$ *converges to* $f = E(h, M)$ *weakly in* L^{Φ}.

Proof. I. (a) implies (b). If g is any function and M is a positive integer, let

$$g^M(x) = \begin{cases} g(x) & \text{if } |g(x)| \leqq M \\ 0 & \text{if } |g(x)| > M . \end{cases}$$

Darst and DeBoth, in [3, Thm. 2], established

(4) $\int_{[|f_k|>a]} \Phi(f_k/n)d\mu \leqq \int_{[|f_k|>a]} \Phi(h/n)d\mu$ for any $a \geqq 0$ and for all $k \geqq 1$.

Hence

(5) $\|f_k\chi_{[|f_k|>a]}\| \leqq \|h\chi_{[|f_k|>a]}\|$. But since each f_k is a Radon-Nikodym derivative of h, we have

(6) $\mu([|f_k| > a]) \leqq |\lambda|([|f_k| > a]) \leqq |\lambda|(\Omega)$. Hence $\mu([|f_k| > a]) \to 0$ as $a \to \infty$ uniformly in k. Since $h \in E_{\Phi}$, h has an absolutely continuous norm, hence

(7) $\|h\chi_{[|f_k|>a]}\| \to 0$ as $a \to \infty$ uniformly in k. Referring back to (5), we conclude

(8) $\|f_k^M - f_k\| = \|f_k\chi_{[|f_k|>M]}\| \to 0$ as $M \to \infty$ uniformly in k. Now let $M > 0$ be temporarily fiexd. Let $\varepsilon > 0$, $\delta > 0$, and consider

$$\int_{\Omega} \Phi\Big(\frac{f^M - f_k^M}{\varepsilon}\Big)d\mu \leqq \Phi\Big(\frac{2M}{\varepsilon}\Big)\mu([|f^M - f_k^M| > \delta]) + \Phi(\delta)\mu(\Omega) .$$

Brunk and Johansen, [2, Thm. 2.8], have established that $f_k \to f$ a.e. Hence we may choose δ so small and then k_0 so large that

$$\int_{\Omega} \Phi\Big(\frac{f^M - f_k^M}{\varepsilon}\Big)d\mu \leqq 1 \quad \text{for} \quad k \geqq k_0 .$$

This implies $\|f^M - f_k^M\| \leqq \varepsilon$ for $k \geqq k_0$, so

(9) $\|f^M - f_k^M\| \to 0$ as $k \to \infty$. Finally, since $\int_{\Omega} \Phi(f/N)d\mu \leqq \int_{\Omega} \Phi(h/N)d\mu$ for all $N > 0$, Lemma 2 guarantees that $f \in E_{\Phi}$ whenver $h \in E_{\Phi}$. Hence by [5, Lemma 10.1],

(10) $\|f - f^M\| \to 0$ as $M \to \infty$. Consequently, given $\varepsilon > 0$, we use (10) and (8) to choose M large enough so that $\|f - f^M\| < \varepsilon/3$

and $||f_k^M - f_k|| < \varepsilon/3$ for all k. Then using (9), we let k_0 be so large that $||f^M - f_k^M|| < \varepsilon/3$ for $k \geqq k_0$. Then $||f - f_k|| \leqq ||f - f^M|| + ||f^M - f_k^M|| + ||f_k^M - f_k|| < \varepsilon$ for $k \geqq k_0$, which establishes I.

II. (b) implies (c) trivially.

III. (c) implies (a). We will show that if $h \notin E_\Phi$, then there is a martingale $\{f_k, M_k\}$ such that $\{f_k\}$ does not converge weakly to $f = E(h, M)$.
Let $E_k = [|h| \leqq k]$, and let $M_k = \{B: B = A \cap E_k, A \in \mathscr{A}\} \cup \{E_k^c\}$. Then M_k is a σ-lattice, and $M = \bigcup_{k=1} M_k = \mathscr{A}$. It is clear that $f_k = E(h_k, M_k) = h^k$. Hence $f_k \in E_\Phi$ for all k. Now since $M = \mathscr{A}$, it follows that $f = h$, which is not in E_Φ. By the Hahn-Banach theorem there is a continuous linear functional L on L^Φ such that $L(f) = 1$ but $L(g) = 0$ for all $g \in E_\Phi$. Hence the sequence $\{f_k\}$ does not converge weakly to f. Theorem 3 is established.

There is a type of convergence under which the the martingale $\{f_k, M_k\}$ will always converge to f. We say that $\{u_n\} \subset L_\Phi$ converges E_Ψ-weakly to u if $\int_\Omega u_n v d\mu \to \int_\Omega u v d\mu$ for every $v \in E_\Psi$, where Ψ is the complimentary function to Φ. The following result may be found in [5, Thm. 14.6]:

THEOREM 4. *Suppose the sequence $\{u_n\} \subset L^\Phi$ converges in measure to u, and there is a constant $M > 0$ such that $||u_n|| \leqq M$ for all n. Then $u \in L^\Phi$ and $\{u_n\}$ converges E_Ψ-weakly to u.*

COROLLARY 5. *If $h \in L^\Phi$, $f = E(h, M)$, and $\{f_k, M_k\}$ is a martingale, then the sequence $\{f_k\}$ converges E_Ψ-weakly to f.*

Proof. We have already seen that $||f_k|| \leqq ||h||$ for all k. Also, $f_k \to f$ a.e., hence also in measure. The result follows from Theorem 4.

4. A martingale convergence theorem for vector valued measures. In this section we define the Radon-Nikodym derivative of a bounded countably additive set function valued in a Banach space $\underline{X}$ with a Schauder basis with respect to a nonnegative measure given a σ-lattice. Then we prove a martingale convergence theorem.

Let $\underline{X}$ be a Banach space with a Schauder basis $\{e_i\}_{i=1}^\infty$ of unit vectors. Recall that there exists a constant $K > 0$ such that

(11) $||\sum_{i=1}^n c_i e_i||_{\underline{X}} \leqq K||\sum_{i=1}^\infty c_i e_i||_{\underline{X}}$ for all n, and all $\sum_{i=1}^\infty c_i e_i \in \underline{X}$.

Suppose $(\Omega, \mathscr{A}, \mu)$ is a finite measure space and M is a sub σ-lattice of $\mathscr{A}$. If $\lambda: M \to \underline{X}$ is countably additive, we may write $\lambda = \sum_{i=1}^\infty \lambda_i e_i$, where each $\lambda_i: M \to R$ is countably additive.

DEFINITION 6. Let $f_i(x)$ be the Radon-Nikodym derivative of λ_i with respect to μ on M. Then we call $f(x) = \sum_{i=1}^{\infty} f_i(x)e_i$ the Radon-Nikodym derivative of $\lambda = \sum_{i=1}^{\infty} \lambda_i e_i$ with respect to μ on M.

Suppose $h: \Omega \rightarrow \underline{\bar{X}}$ is given by $h(x) = \sum_{i=1}^{\infty} h_i(x)e_i$, where each $h_i: \Omega \rightarrow R$, and suppose further that $\int_{\Omega} ||h(x)||_{\underline{\bar{X}}} d\mu < \infty$. Then $\lambda(E) = \int_E h(x)d\mu$ defines an $\underline{\bar{X}}$-valued set function on $\mathscr{A}$. Hence λ may also be written $\lambda(E) = \sum_{i=1}^{\infty} \lambda_i(E)e_i$. It is routine to verify that $\lambda_i(E) = \int_E h_i(x)d\mu$ for each i. In view of this, we make the following

DEFINITION 7. If $h(x) = \sum_{i=1}^{\infty} h_i(x)e_i$ is integrable, and $f(x) = \sum_{i=1}^{\infty} E(h_i, M)e_i$, then we call $f(x)$ the Radon-Nikodym derivative of h on M. We denote $f = E(h, M)$.

Denote by $L^{\Phi}(\Omega, \underline{\bar{X}})$ the space of functions f defined on Ω such that $||f(x)||_{\underline{\bar{X}}}$ is in $L^{\Phi}(\Omega, \mathscr{A}, \mu)$, and $E_{\Phi}(\Omega, \underline{\bar{X}})$ the space of functions f such that $||f(x)||_{\underline{\bar{X}}}$ is in $E_{\Phi}(\Omega, \mathscr{A}, \mu)$. Then a sequence $\{f_n\}$ converges to f in $L'(\Omega, \underline{\bar{X}})$ if $||f_n - f||_{\underline{\bar{X}}}$ converges to 0 in $L^{\Phi}(\Omega, \mathscr{A}, \mu)$.

THEOREM 8. *If* $h(x) = \sum_{i=1}^{\infty} h_i(x)e_i$ *is in* $L^{\Phi}(\Omega, \underline{\bar{X}})$, *and* $\sum_{i=1}^{\infty} ||h_i|| < \infty$, *then* $f = E(h, M)$ *is in* $L^{\Phi}(\Omega, \underline{\bar{X}})$.

Proof. Recall that $||E(g, M)|| \leqq ||g||$ for any $g \in L^{\Phi}(\Omega, \mathscr{A}, \mu)$. Let ψ be the complimentary function to Φ, and let g be a nonnegative $\mathscr{A}$-measurable function on Ω such that $\int_{\Omega} \psi(g)d\mu \leqq 1$. Let $C = \sum_{i=1}^{\infty} ||h_i||$. Then $\int_{\Omega} ||f(x)||_{\underline{\bar{X}}} g(x)d\mu = \int_{\Omega} ||\sum_{i=1}^{\infty} f_i(x)e_i||_{\underline{\bar{X}}} g(x)d\mu \leqq \int_{\Omega} (\sum_{i=1}^{\infty} |f_1(x)|)g(x)d\mu = \sum_{i=1}^{\infty} \int_{\Omega} |f_i(x)| g(x)d\mu \leqq \sum_{i=1}^{\infty} |||f_i||| \leqq 2\sum_{i=1}^{\infty} ||f_i|| \leqq 2\sum_{i=1}^{\infty} ||h_i|| = 2C$. Hence $||f(x)||_{\underline{\bar{X}}} \in L^{\Phi}(\Omega, \mathscr{A}, \mu)$, so $f \in L^{\Phi}(\Omega, \underline{\bar{X}})$.

Let $\{M_k\}_{k=1}^{\infty}$ be an increasing sequence of sub σ-lattices of $\mathscr{A}$, and let M be the σ-lattice generated by $\bigcup_{k=1}^{\infty} M_k$. If $f^k = E(h, M_k)$, then $\{f^k, M_k\}$ is called a martingale.

THEOREM 9. *Suppose* $h \in E_{\Phi}(\Omega, \underline{\bar{X}})$ *and* $\sum_{i=1}^{\infty} ||h_i|| < \infty$. *If* $\{f^k, M_k\}$ *is a martingale, and* $f = E(h, M) = \sum_{i=1}^{\infty} f_i e_i$, *then* $f^k \rightarrow f$ *as* $k \rightarrow \infty$ *in* $L'(\Omega, \underline{\bar{X}})$.

Proof. Since, by (11), $|h_i(x)| \leqq 2K||h(x)||_{\underline{\bar{X}}}$ for each i, we have

$$\int_{\Omega} \Phi\left(\frac{|h_i(x)|}{N}\right)d\mu \leqq \int_{\Omega} \Phi\left(\frac{||h(x)||_{\underline{\bar{X}}}}{N(2K)^{-1}}\right)d\mu$$

for all $N > 0$. Referring to Lemma 2, this implies $h_i \in E_{\Phi}$ for each i. Hence also $f_i \in E_{\Phi}$ for each i.

Let $\varepsilon > 0$. Since, by hypothesis, $\sum_{i=1}^{\infty} ||h_i|| < \infty$, we have

$\sum_{i=1}^{\infty}||f_i|| < \infty$ also. Let p be a positive integer such that $\sum_{i=p+1}^{\infty}||f_i|| < \varepsilon/8$. Since $f_i^k \to f_i$ in $L^{\psi}(\Omega, \mathscr{A}, \mu)$ for each i, (Thm. 3), we can find a positive integer Q such that for $q \geqq Q$, $|||f_i^q - f_i||| < \varepsilon/2p$, $i = 1, \cdots, p$.

Let g be a nonnegative, $\mathscr{A}$-measurable function such that $\int_\Omega \psi(g)d\mu \leqq 1$. Then for $q \geqq Q$,

$$\begin{aligned}
&\left|\int_\Omega ||f^q(x) - f(x)||_{\underline{X}}\, g(x)d\mu\right| \\
&= \int_\Omega ||\sum_{i=1}^{\infty}(f_i^q(x) - f_i(x))e_i||_{\underline{X}}\, g(x)d\mu \\
&\leqq \int_\Omega (\sum_{i=1}^{\infty}|f_i^q(x) - f_i(x)|)g(x)d\mu \\
&= \sum_{i=1}^{p}\int_\Omega |f_i^q(x) - f_i(x)|\, g(x)d\mu \\
&\quad + \sum_{i=p+1}^{\infty}\int_\Omega f_i^q(x) - f_i(x)|\, g(x)d\mu \\
&\leqq \sum_{i=1}^{p}|||f_i^q - f_i||| + \sum_{i=p+1}^{\infty}(|||f_i^q||| + |||f_i|||) \\
&\leqq \frac{\varepsilon}{2} + 2\sum_{i=p+1}^{\infty}(||f_i^q|| + ||f_i||) \\
&\leqq \frac{\varepsilon}{2} + 4\sum_{i=p+1}^{\infty}||f_i|| \leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.
\end{aligned}$$

Hence $|||\,||f^q - f||_{\underline{X}}\,||| < \varepsilon$ for $q \geqq Q$, and the proof is complete.

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