RINGS WHOSE PROPER CYCLIC MODULES ARE QUASI-INJECTIVE

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A ring $R$ with identity is a right PCQI-ring (PCI-ring) if every cyclic right $R$-module $C \neq R$ is quasi-injective (injective). Left PCQI-rings (PCI-rings) are similarly defined. Among others the following results are proved: (1) A right PCQI-ring is either prime or semi-perfect. (2) A nonprime nonlocal ring is a right PCQI-ring iff every cyclic right $R$-module is quasi-injective or $R \cong \left( \begin{array}{cc} D & D \\ 0 & D \end{array} \right)$, where $D$ is a division ring. In particular, a nonprime nonlocal right PCQI-ring is also a left PCQI-ring. (3) A local right PCQI-ring with maximal ideal $M$ is a right valuation ring or $M^2 = (0)$. (4) A prime local right PCQI-ring is a right valuation domain. (5) A right PCQI-domain is a right Ore-domain. Faith proved (5) for right PCI-domains. If $R$ is commutative then some of the main results of Klatt and Levy on pre-self-injective rings follow as a special case of these results.

Since, in a commutative Dedekind domain $D$, for each nonzero ideal $A$, $D/A$ is a self-injective ring, or equivalently $D/A$ is a quasi-injective $D$-module, every commutative Dedekind domain is a PCQI-ring. An example of a PCQI-ring which is not a Dedekind domain is given in Levy [14]. Commutative PCQI-rings are precisely the pre-self-injective rings characterized by Klatt and Levy [11]. PCI-rings have recently been investigated by Faith [4]. Right self-injective right PCQI-rings are $qc$-rings which have been studied by Ahsan [1] and Koehler [13].

1. Definitions and preliminaries. Throughout all modules are unitary and right unless specified. An $R$-module $X$ is called injective relative to an $R$-module $M$ if for each short exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ the sequence $0 \rightarrow \text{Hom}_R(M/N, X) \rightarrow \text{Hom}_R(M, X) \rightarrow \text{Hom}_R(N, X) \rightarrow 0$ is exact. $X$ is called quasi-injective if $X$ is injective relative to itself. Any $R$-module injective relative to all $R$-modules is called injective. Relative projectivity is defined dually.

A ring $R$ is called a right $q$-ring if each of its right ideals is quasi-injective (see Jain, Mohamed, and Singh [9]). For more results, see [7], [8], [13], [15]. Dually, a ring $R$ is called a right $q^*$-ring if each cyclic right $R$-module is quasi-projective (see Koehler [12]).

A ring $R$ is right $qc$-ring if each cyclic right $R$-module is quasi-injective (see Ahsan [1]). A well-known result of Osofsky [16] states
that $R$ is semisimple artinian iff each cyclic $R$-module is injective. Koehler [13] showed that $R$ is a right $qc$-ring iff $R$ is a finite direct sum of rings each of which is semisimple artinian or a rank 0 duo maximal valuation ring. As a consequence, every $qc$-ring is both a $q$-ring and $q^*$-ring.

In this paper the classes of rings initially called $q$-rings, $q^*$-rings, and $qc$-rings have been called $Q$-rings, $Q^*$-rings, and $QC$-rings respectively.

Let $J(R)$ denote the radical of a ring $R$. $R$ is called semiperfect if $R/J(R)$ is semisimple artinian and idempotents modulo $J(R)$ can be lifted to $R$. If $R$ is semiperfect, then there exists a finite maximal family of primitive orthogonal idempotents $\{e_i\}_{1 \leq i \leq n}$ such that $R = \bigoplus_{i=1}^{n} e_i R$.

$R$ is called a local ring if it has a unique maximal right ideal which must be the radical $J(R)$.

$R$ is a right valuation ring if the set of all right ideals is linearly ordered. $R$ is a maximal valuation ring if every family of pairwise solvable congruences of the form $x \equiv x_a \pmod{A_a}$ has a simultaneous solution where $x_a \in R$ and each $A_a$ is an ideal in $R$. $R$ is called an almost maximal valuation ring if each of its proper homomorphic images is a maximal valuation ring.

A ring is right duo if every right ideal is two-sided. A ring $R$ has rank 0 if every prime ideal is a maximal ideal. By duo rings or valuation rings, we shall mean both right and left.

3. General results.

**Sublemma 1.** Let $I$ be a right ideal in a ring $R$ such that $R/I \cong R$. Then $R = I \oplus J$, where $J$ is a right ideal, and thus $I = eR$, $e = e^2 \in R$.

**Proof.** $R/I \cong R$ implies $R/I$ is projective, and hence $I$ is a direct summand of $R$.

**Proposition 2.** Let $R$ be a right $PCQI$-ring. If $I$ is a right ideal of $R$ such that $R/I \cong R$, then $I$ is contained in every nonzero two-sided ideal of $R$.

**Proof.** Let $S$ be a nonzero two-sided ideal of $R$. Then $R/S$ is a $qc$-ring, hence is semiperfect. Let $f: R/I \to R$ be an isomorphism. Since $1 + I$ generates $R/I$, $\bar{R} = xR$, where $x = f(1 + I)$. Then $I = \text{ann } x = \{r \in R | xr = 0\}$. So there exists $y \in R$ such that $xy = 1$. Since $R/S$ is semiperfect, $(x + S)(y + S) = 1 + S = (y + S)(x + S)$. Then $1 - yx \in S$. Let $a \in I$, i.e., $xa = 0$. Then $(1 - yx)a = a - yxa = a$, hence $a \in S$. So $I \subseteq S$. 
PROPOSITION 3. Let $R$ be a right PCQI-ring. Then either $R$ is a prime ring or $R$ is semiperfect with nil radical.

Proof. Suppose $R$ is not prime, and $P \neq 0$ is a prime ideal. Then $R/P$ is a qc-ring, and hence a q-ring. So $R/P$ is simple artinian [9]. Thus $P$ is maximal, hence primitive. So the Jacobson radical is nil.

Since $R$ is not prime, there exist nonzero ideals $A, B$ such that $AB = 0$. Since $R$ is a right PCQI-ring, $R/A$ and $R/B$ are semiperfect, hence each of them has finitely many prime ideals. Since every prime ideal of $R$ contains $A$ or $B$, it follows that $R$ has finitely many prime ideals as well. Thus $R/J(R)$ is semisimple artinian, and since $J(R)$ is nil, $R$ is semiperfect.

4. Nonlocal semiperfect PCQI-rings. By Proposition 3, all nonprime right PCQI-rings are semiperfect, so the results of this section hold for the class of nonprime nonlocal right PCQI-rings. The case of local right PCQI-rings is discussed in the next section.

LEMMA 4. Let $R$ be a semiperfect ring. Then $R/A$ is a proper cyclic right $R$-module, for all nonzero right ideals $A$.

Proof. There exists a positive integer $n$ such that $R$ is a direct sum of $n$ indecomposable right $R$-modules, and $R$ cannot be expressed as a direct sum of more than $n$ right $R$-modules. Now, if $R/A \cong R$, then, by Lemma 1, $R = A \oplus B$ and $B \cong R$. So $A = (0)$, proving the lemma.

Let $R$ be a nonlocal semiperfect ring, and let $\{e_i\}_{1 \leq i \leq n}$ be a maximal set of primitive orthogonal idempotents in $R$. Then $R = \bigoplus_{1 \leq i \leq n} e_i R$ and $n \geq 2$. Throughout this section, $e_i$'s will denote primitive idempotents. We shall often use a well-known fact that if $A \oplus B$ is a quasi-injective module then any monomorphism $A \rightarrow B$ splits.

LEMMA 5. Let $R$ be a semiperfect nonlocal right PCQI-ring. If $\sigma \in \text{Hom}_R(e_i R, e_j R)$ such that $\sigma \neq 0$, where $i \neq j$, then $\ker \sigma = (0)$.

Proof. Suppose $\ker \sigma \neq (0)$, where $0 \neq \sigma \in \text{Hom}_R(e_i R, e_j R)$, $i \neq j$. Then $R/\ker \sigma \cong \bigoplus_{1 \leq i \leq n} e_i R \times \text{Im} \sigma$, and $R/\ker \sigma$ is quasi-injective. Since $\text{Im} \sigma \subseteq e_j R$, the inclusion map $i: \text{Im} \sigma \rightarrow \bigoplus_{1 \leq i \leq n} e_i R$ is a monomorphism. Since $R/\ker \sigma$ is quasi-injective, the inclusion map splits. So $\text{Im} \sigma$ is a direct summand of $e_j R$, hence $\text{Im} \sigma = e_j R$. Since $e_j R$ is projective, $\sigma: e_i R \rightarrow e_j R$ splits. Thus $\ker \sigma = (0)$. 
Lemma 6. Let \( R \) be a semiperfect nonlocal right PCQI-ring with decomposition \( \bigoplus_{i=1}^{n} e_i R \), where \( n > 2 \). Then \( \text{Hom}_R(e_i R, e_j R) \neq 0 \) iff \( e_i R \cong e_j R \), i.e., \( e_i Re_i \neq 0 \) iff \( e_j R \cong e_i R \).

Proof. Let \( \sigma \in \text{Hom}_R(e_i R, e_j R) \) such that \( \sigma \neq 0 \). By Lemma 5, \( \ker \sigma = 0 \). Since \( n > 2 \), \( e_i R \bigoplus e_j R \cong R / \bigoplus_{k \neq i, j} e_k R \) is quasi-injective. Then \( \sigma \) splits, and \( 0 \neq \text{Im} \sigma \) is a direct summand of \( e_j R \). So \( \text{Im} \sigma = e_j R \), and \( \sigma \) is an isomorphism. The converse is trivial.

Proposition 7. Let \( R \) be a semiperfect nonlocal right PCQI-ring with decomposition \( R = \bigoplus_{i=1}^{n} e_i R \), where \( n > 2 \). Then \( R \) is a qc-ring.

Proof. For each \( i \), \( e_i R \cong R / \bigoplus_{k \neq i} e_k R \). So \( e_i R \) is quasi-injective, for each \( i \). Let \( A_i \) be the sum of all those \( e_i R \) which are isomorphic to each other. Then \( R = \bigoplus_{i} A_i \). We claim that \( A_i \) is a two-sided ideal of \( R \), for each \( i \). Clearly \( A_i \) is a right ideal. Consider \( e_j R \) such that \( e_j R \not\subseteq A_i \). Define \( f: e_i R \to e_j R \), where \( e_i R \subseteq A_i \), by \( f(e_ir) = e_j xe_i r \), for \( x \in R \). Then \( f \in \text{Hom}_R(e_i R, e_j R) \). Since \( e_i R \) and \( e_j R \) are not isomorphic, \( f = 0 \) by Lemma 6. So, for \( e_i R \subseteq A_i \), \( e_j RA_i = 0 \). So \( RA_i \subseteq A_i \). Since \( A_i \) is a finite direct sum of isomorphic quasi-injective right ideals, \( A_i \) is quasi-injective, hence a qc-ring. Thus, by Koehler [13], \( R \) is a qc-ring.

Proposition 8. Let \( R \) be a semiperfect right PCQI-ring such that \( R = e_1 R \bigoplus e_2 R \). If \( e_i R \cong e_2 R \), then \( R \) is a qc-ring.

Proof. Now \( e_i R \cong e_2 R \) and \( R / e_i R \cong R / e_2 R \), hence \( e_i R \) and \( e_2 R \) are quasi-injective. Since \( e_i R \cong e_2 R \), \( R = e_1 R \bigoplus e_2 R \) is quasi-injective, hence right self-injective. So \( R \) is a qc-ring.

Proposition 9. Let \( R \) be a semiperfect right PCQI-ring such that \( R = e_1 R \bigoplus e_2 R \). If \( e_1 Re_2 = 0 \) and \( e_2 Re_1 = 0 \), then \( R \) is a qc-ring.

Proof. If \( e_1 Re_2 = 0 \) and \( e_2 Re_1 = 0 \), then \( e_1 R \) and \( e_2 R \) are two-sided ideals of \( R \). Thus \( e_1 R \cong R / e_2 R \) and \( e_2 R \cong R / e_1 R \) are qc-rings. Then \( R = e_1 R \bigoplus e_2 R \) is a qc-ring.

Proposition 10. Let \( R \) be a semiperfect right PCQI-ring such that \( R = e_1 R \bigoplus e_2 R \). If \( e_1 Re_2 \neq 0 \) and \( e_2 Re_1 \neq 0 \), then \( R \) is a qc-ring.

Proof. \( e_1 Re_2 \neq 0 \) and \( e_2 Re_1 \neq 0 \) imply that there exist nonzero homomorphisms, hence monomorphisms by Lemma 5, from \( e_i R \) to \( e_j R \) and from \( e_j R \) to \( e_i R \). Thus, by Bumby [2], \( e_i R \cong e_j R \), and Proposition 8 yields the result.
PROPOSITION 11. Let $R = e_1R \oplus e_2R$ be a semiperfect right PCQI-ring where $e_1R \neq e_2R$ and exactly one of $e_1Re_2$ or $e_2Re_1$ is zero. Then $R$ is nonprime with nil radical.

Proof. It follows from the fact that if $e_1Re_2 \neq 0$, then $e_1Re_2$ is a nilpotent ideal.

THEOREM 12. Let $R$ be a nonlocal right PCQI-ring. Then $R$ is semiperfect iff $R$ is nonprime or simple artinian.


THEOREM 13. Let $R$ be a semiperfect nonlocal ring. Then $R$ is a right PCQI-ring iff either (i) $R = \bigoplus_{i=1}^r R_i$, where $R_i$ is semisimple artinian or a rank 0 duo maximal valuation ring or (ii) $R = \left( \begin{array}{cc} D & D \\ 0 & D \end{array} \right)$, where $D$ is a division ring.

Proof. Let $R$ be a right PCQI-ring. By Propositions 7-10, $R$ is a qc-ring unless $R = e_1R \oplus e_2R$, where $e_1R$ and $e_2R$ are not isomorphic and exactly one of $e_1Re_2$ or $e_2Re_1$ is zero, say $e_1Re_2 \neq 0$ and $e_2Re_1 = 0$. If $R$ is a QC-ring, we get (i) by Koehler [13]. Otherwise, we have $R \cong \left( \begin{array}{cc} e_1Re_2 & e_2Re_2 \\ 0 & e_1Re_2 \end{array} \right)$. We claim that $e_1Re_1$ and $e_2Re_2$ are isomorphic division rings and $M = e_1Re_2$ is a $(D, D)$-bimodule such that $\dim_M M = 1 = \dim_M D$, where $D \cong e_1Re_1 \cong e_2Re_2$. Clearly $e_1Re_2$ is nilpotent ideal and since it is nonzero, $R$ is not prime. So, by Proposition 3, the radical $N$ of $R$ is a nil ideal. Thus $e_2Ne_2$ is nil. We claim that $e_2Ne_2 = 0$. Let $e_2xe_2 \in e_2Ne_2$. Define $\sigma: e_2R \rightarrow e_2R$ by $\sigma(e_1y) = e_1xe_2y$. Then $\sigma \in \text{Hom}_R(e_1R, e_2R)$, and since $e_2xe_2$ is nilpotent, $\sigma$ is not a monomorphism. So ker $\sigma \neq (0)$. Since $\text{Hom}_R(e_1R, e_2R) \neq 0$, there exists an embedding $\gamma: e_2R \rightarrow e_1R$. Now $\gamma \sigma: e_2R \rightarrow e_1R$, and since ker $\sigma \neq (0)$, ker $\gamma \sigma \neq (0)$. By Lemma 5, $\gamma \sigma = 0$. Since $\gamma$ is a monomorphism, we have $\sigma = 0$. Thus $e_2xe_2 = 0$, and $e_2Ne_2 = 0$. So $e_2Re_2$ is a division ring. Further $e_2Re_2 = e_1R$ since $e_2Re_1 = (0)$. Thus $e_2N = 0$, and $e_2R$ is a minimal right ideal. Now $e_1R$ is uniform because it is quasi-injective and indecomposable. Since $0 \neq e_1Re_2R$ is the sum of the images of all $R$-homomorphisms of $e_1R$ into $e_1R$, the fact that $e_1R$ is minimal and $e_2R$ is uniform yields that $e_1Re_2R$ itself is the unique minimal right subideal of $e_1R$, is isomorphic to $e_2R$, and is contained in every nonzero right subideal of $e_1R$. We claim that $e_1Ne_1 = 0$. Let $0 \neq e_1xe_1 \in e_1Ne_1$. Since $N$ is nil, $e_1xe_1$ is nilpotent. Then $\sigma: e_1R \rightarrow e_1R$ defined by $\sigma(e_1r) = e_1xe_1r$ is an endo-
morphism of \(e_iR\) with \(\ker \sigma \neq (0)\). Let \(A = \ker \sigma\). Then \(e_iRe_iR \subseteq A\), and we have \(e_ixe_iRe_i = (0)\). On the other hand, \(e_iRe_iR \subseteq e_ixe_iR\) yields that \(e_ixe_iRe_i \neq (0)\). This is a contradiction. Hence \(e_iNe_i = (0)\), and \(e_iRe_i\) is a division ring. Now using the fact that \(\text{Hom}_R(e_iR, e_iR)\) is a division ring and that \(e_iR\) is quasi-injective, it follows that every member of \(\text{Hom}(e_iRe_iR, e_iRe_iR)\) admits a unique extension to an endomorphism of \(e_iR\). Further, every endomorphism of \(e_iR\) maps \(e_iRe_iR\) into itself since \(e_iRe_iR\) is the unique minimal subideal of \(e_iR\). Thus \(\text{Hom}(e_iRe_iR, e_iRe_iR) \cong \text{Hom}(e_iR, e_iR)\). Since \(e_iRe_iR \cong e_iR\), we obtain \(e_iRe_i \cong e_iRe_i\).

Now \(e_iN = e_iNe_i\) because \(e_iNe_i = (0)\). Since \(e_iRe_iR \subseteq e_iN\), we get \(e_iN = e_iRe_i = e_iRe_iR\). Thus \(M = e_iRe_i\) is a one-dimensional right vector space over \(D = e_iRe_i\). We show that \(M\) is also a one-dimensional left \(e_iRe_i\)-space. Let \(X = \left(\begin{array}{c}e_iR \\ M \\ 0 \\ 0 \end{array}\right) \cong R/A\), where \(A = \left(\begin{array}{c}0 \\ 0 \\ 0 \\ D \end{array}\right)\).

Then \(X\) is quasi-injective. Let \(0 \neq x \in M\), and let \(y \in M\). Consider \(\sigma: \left(\begin{array}{c}0 \\ M \\ 0 \\ 0 \end{array}\right) \to \left(\begin{array}{c}0 \\ M \\ 0 \\ 0 \end{array}\right)\) defined by \(\sigma(0 x^c) = (0 y^c)\), for \(c \in D\). Then \(\sigma\) is an \(R\)-endomorphism, so it can be extended to an endomorphism \(\eta\) of \(X\). Let \(\eta(\left(\begin{array}{c}0 \\ 0 \\ 0 \\ 0 \end{array}\right)) = (a b)\). Then we have \(\left(\begin{array}{c}0 \\ y \\ 0 \\ 0 \end{array}\right) = \sigma(\left(\begin{array}{c}0 \\ x \\ 0 \\ 0 \end{array}\right)) = \eta(\left(\begin{array}{c}0 \\ x \\ 0 \\ 0 \end{array}\right)) = (0 a x)\). Thus \(y = ax\), so \(M = e_iRe_ix\). So \(M\) is a one-dimensional left vector space over \(e_iRe_i\). Thus, for each \(d \in e_iRe_i\), there exists a unique \(d' \in e_iRe_i\) such that \(dx = xd'\). Define \(\theta: e_iRe_i \to e_iRe_i\) by \(\theta(d) = d'\). Then \(\theta\) is an isomorphism, and we may identify \(d\) and \(d'\). Then \(\eta: \left(\begin{array}{c}D \\ D \\ 0 \\ 0 \end{array}\right) \to \left(\begin{array}{c}D \\ M \\ 0 \\ D \end{array}\right)\) defined by \(\eta(\left(\begin{array}{c}a \\ b \\ 0 \\ c \end{array}\right)) = (a b x)\) is an isomorphism.

Conversely, if \(R\) satisfies (i), then, by Koehler [13], \(R\) is a QC-ring, hence a PCQI-ring. If \(R\) satisfies (ii), then straightforward computation shows that \(R\) is a right PCQI-ring.

Since every right QC-ring is a left QC-ring and \(\left(\begin{array}{c}D \\ D \\ 0 \\ D \end{array}\right)\) is also a left PCQI-ring, we get the following corollary.

**COROLLARY.** A nonlocal semiperfect right PCQI-ring is also a left PCQI-ring.

5. **Local PCQI-rings.** Theorem 13 and Theorems 14, 15, and 16 which follow generalize Klatt and Levy's [11] theorems for commutative pre-self-injective rings which are not domains. Throughout this section \(M\) will denote the unique maximal right ideal of a local ring \(R\). \(M\) is then the Jacobson radical of \(R\), and \(R/M\) is a division ring.

**Theorem 14.** Let \(R\) be a local right PCQI-ring with maximal ideal \(M\). Then either \(R\) is a right valuation ring or \(M^2 = (0)\) and \(M_R\) has composition length 2.
Proof. First note that for all nonzero right ideals $A$, $R/A$ is indecomposable quasi-injective and hence uniform. Now we show that all nonzero right ideals are either minimal or essential. Let $A$, $B$ be nonzero right ideals such that $A \cap B = (0)$. We claim that $A$ is minimal. Let $C$ be a nonzero right ideal properly contained in $A$. Then $R/C$ is quasi-injective and not uniform since $A/C \cap (B + C)/C = 0$. This is a contradiction, so $A$ is minimal. Similarly, $B$ is minimal. In particular, it follows that any maximal independent family of minimal right ideals can contain at most two members.

If $\text{Soc } R = (0)$, then all nonzero right ideals are essential. Let $A$, $B$ be two nonzero right ideals. If neither $A \subseteq B$ nor $B \subseteq A$, then $R/A \cap B$ is quasi-injective but not uniform since $A/(A \cap B) \cap B/(A \cap B) = (0)$. As before, this is a contradiction. So either $A \subseteq B$ or $B \subseteq A$.

If $\text{Soc } R$ consists of a unique minimal right ideal then it is clear that $R$ is a right valuation ring.

Finally, suppose $\text{Soc } R = A \oplus B$, where $A$, $B$ are minimal right ideals. Then $R$ cannot be prime. Let $x \in M$, and consider $xR$. If $xR$ is not minimal, then $xR$ is quasi-injective and decomposable. Then $xR = A \oplus B$. In any case, for all $x \in M$, $x \in \text{Soc } R$. This implies that $M^2 = (0)$, and the composition length of $M$ is 2, completing the proof.

The next two theorems give the structure of non-prime local right PCQI-rings. Prime local PCQI-rings are discussed in the next section.

**Theorem 15.** For a nonprime right valuation ring $R$, the following are equivalent:

(i) $R$ is a right PCQI-ring.

(ii) $R$ is a right duo almost maximal valuation ring of rank 0 such that any left ideal containing a nonzero right ideal is two-sided.

Proof. (i) ⇒ (ii). Since $R$ is not prime, $M$ is nil by Proposition 3. So, if $xR$ is a nontrivial principal right ideal of $R$, $xR$ is quasi-injective. Since $xR$ is essential in $R$, the injective hull of $xR$ is the same as that of $R$. Hence, by Johnson and Wong [10], $RxR \subseteq xR$. So $xR$ is a two-sided ideal of $R$. Thus $R$ is a right duo ring. Since each proper homomorphic image of a PCQI-ring is a QC-ring, the proof of (i) ⇒ (ii) as well as that of (ii) ⇒ (i) is completed by a theorem of Koehler [13].

**Theorem 16.** For a local ring $R$ with $M^2 = (0)$ and the composition length of $M_R$ equal to 2, the following are equivalent:
(i) \( R \) is a right PCQI-ring.

(ii) For each nonzero right ideal \( A \) in \( R \) and for each \( m_1, m_2 \in A \), the congruence \( xm_1 \equiv m_2 \pmod{A} \) has a solution, \( x = \alpha \), such that \( \alpha A \subseteq A \).

**Proof.** Under the hypothesis the only nonzero right ideals \( A \) of \( R \) different from \( M \) and \( R \) are minimal right ideals, and \( M/A \) is a simple right \( R \)-module.

(i) \implies (2) Let \( A \) be a nontrivial right ideal in \( R \), and let \( m_1, m_2 \in R \) such that \( m_1, m_2 \in A \). Then \( \bar{m}_1 R = M/A = \bar{m}_2 R \), and the mapping \( \sigma : M/A \to M/A \) which sends \( \bar{m}_1 r \) to \( \bar{m}_2 r \) is a well-defined \( R \)-homomorphism. Since \( R/A \) is quasi-injective, \( \sigma \) can be lifted to \( \sigma^* \in \text{Hom}_R(R/A, R/A) \). Let \( \sigma^*(\bar{1}) = \bar{a} \). Then \( \bar{a}m_1 = \bar{m}_2 \). Hence \( xm_1 \equiv m_2 \pmod{A} \) has a solution \( x = \alpha \). Clearly \( \alpha A \subseteq A \).

(ii) \implies (i) We only need to prove that if \( A \) is a nontrivial right ideal of \( R \) and \( \sigma : M/A \to R/A \), is a nonzero \( R \)-homomorphism, then \( \sigma \) can be extended to an \( R \)-homomorphism \( \sigma^* : R/A \to R/A \). Let \( m \in M \), where \( m \in A \). Then \( M/A = \bar{m} R \). Also, \( \sigma(M/A) = M/A \). Let \( \sigma(\bar{m}) = \bar{m} r \). Since \( M^* = (0), r \in M \). So \( r \) is invertible, and \( mr \in A \). Let \( \alpha \in R \) be chosen such that \( \alpha m = mr \pmod{A} \), and \( \alpha A \subseteq A \). Then \( \sigma^*(\bar{r}) = \bar{\alpha} R \) is well-defined, and it extends \( \sigma \), completing the proof.

The example which follows shows that a local right PCQI-ring is not necessarily a left PCQI-ring.

**Example.** Let \( F \) be a field which has a monomorphism \( \rho : F \to F \) such that \( [F : \rho(F)] > 2 \). Take \( x \) to be an indeterminate over \( F \).

Make \( V = xF \) into a right vector space over \( F \) in a natural way. Let \( R = \{(a, x\beta) \mid a, \beta \in F \} \). Define

\[
(a_1, x\beta_1) + (a_2, x\beta_2) = (a_1 + a_2, x\beta_1 + x\beta_2)
\]

and

\[
(a_1, x\beta_1)(a_2, x\beta_2) = (a_1a_2, x(\rho(a_1)\beta_2 + \beta_1\beta_2)).
\]

Then \( R \) is a local ring with identity with the maximal ideal

\[
M = \{(0, xa) \mid a \in F \}.
\]

In fact, \( M \) is also a minimal right ideal and \( M^2 = (0) \). Thus \( R \) is a right PCQI-ring. Further, if \( \{\alpha_i\}_{i \in I} \) is a basis of \( F \) as a vector space over \( \rho(F) \) then straightforward computations yield that \( M = \bigoplus \sum R(0, xa_i) \) as a direct sum of irreducible left \( R \)-modules \( R(0, xa_i) \). Since card \( I > 2 \), it follows by Theorem 14 that \( R \) is not a left PCQI-ring.
6. Prime local $PCQI$-rings.

**Theorem 17.** Let $R$ be a prime local right $PCQI$-ring. Then $R$ is a right valuation domain, hence right semihereditary.

**Proof.** By Theorem 14, $R$ is a right valuation ring. Let $A$ denote the intersection of all nonzero two-sided ideals of $R$. The proof that $R$ is a domain falls into three cases.

(i) $A = (0)$.

Let $x, y \in R$ such that $xy = 0$. Suppose $y \neq 0$. Then $yR$ is a nonzero right ideal of $R$. Since $R$ is right valuation and $A = (0)$, $yR$ must contain a nonzero two-sided ideal of $R$. Further, each proper homomorphic image of $R$ is a local QC-ring, hence a duo ring [13]. This implies that $yR$ is two-sided. Hence $x = 0$, and $R$ is an integral domain.

(ii) $A \neq (0)$ and $A \neq M$.

Under these hypotheses, $A$ cannot be a prime ideal. So there exist $x, y \in R$ such that $xRy \subseteq A$, $x \notin A$ and $y \notin A$. Since $R$ is right valuation, $A \subseteq xR$ and $A \subseteq yR$. So both $xR$ and $yR$ are two-sided ideals. For definiteness, let $xR \subseteq yR$. Then $(xR)^2 \subseteq (xR)(yR) \subseteq AR = A$ gives that $(xR)^2 = A$ by the minimality of $A$. Also $A = A^2$, hence $(xR)^2 = (xR)^1$. It follows that $x^2R = x'R$. Then $x^2 = x'r$, for some $r \in R$, and $x^2(1 - x^2r) = 0$. So $x^2 = 0$. Thus $A = (0)$, and this case cannot occur.

(iii) $A = M$.

Let $S \subseteq R$, and let $r(S)$ denote the right annihilator of $S$ in $R$. Let $Z(R) = \{x \in R | r(x) \text{ is an essential right ideal}\}$. Then $Z(R)$ is an ideal in $R$ called the right singular ideal.

Since $R$ is a right valuation ring, $R$ is immediately a domain if $Z(R) = (0)$.

So assume that $Z(R) \neq (0)$. Then $Z(R) = M$, and each element in $M$ is a right zero divisor. So $x \in M$ implies that $xR$ is proper cyclic, hence quasi-injective. Also $xR$ is an essential right ideal in $R$. By Johnson and Wong [10], $RxR \subseteq xR$. Hence $xR$ is two-sided. So $R$ is a prime right duo ring, and it follows that $R$ is a domain.

7. $PCQI$-domains. In this section we discuss right $PCQI$-rings which are integral domains and prove that these are right Ore-domains. This generalizes the result of Faith [4]. Our proof, in this case, though it runs on the same lines as that of Faith, does not use Faith's result.

**Proposition 18.** Let $R$ be a right $PCQI$-domain, and let $I$ be a nonessential right ideal of $R$. Then $R/I$ is an injective right $R$-
module containing a copy of $R$.

**Proof.** Since $I$ is nonessential, there exists a nonzero right ideal $J$ in $R$ such that $I \cap J = 0$. Let $a \in J$ such that $a \neq 0$. Then $aR \cap I \subseteq J \cap I = 0$. Consider $r(a + I) = \{x \in R | ax \in I\}$. Clearly $r(a + I) = 0$. So $R/I$ contains a copy of $R$. Since $R/I$ is also quasi-injective, this implies that $R/I$ is injective by [17].

For a right $R$-module $A$, let $\hat{A}$ denote the injective hull of $A$.

**Proposition 19.** Let $R$ be a right PCQI-domain which is not a right Ore-domain. Then $R$ is finitely presented.

**Proof.** Let $a \in R$ such that $a \neq 0$ and $aR$ is not essential. Then $R/aR$ is injective. Since $R/aR$ contains a copy of $R$ and is injective, $R/aR$ contains a copy of $\hat{R}$. Then $R/aR = Y/aR \oplus X/aR$, where $X/aR \cong \hat{R}$. Now $Y/aR$ is cyclic. So $Y = aR + bR$, for some $b \in R$, and the short exact sequence $0 \to Y \to R \to Y/aR \cong X/aR \cong \hat{R} \to 0$ shows that $\hat{R}$ is finitely presented.

**Theorem 20.** A right PCQI-domain $R$ is a right Ore-domain.

**Proof.** Let $R$ be a right PCQI-domain. Suppose $R$ is not a right Ore-domain. Then, as in Proposition 19, there exists $a \in R$ such that $R/aR = Y/aR \oplus X/aR$, where $X/aR \cong \hat{R} \cong R/Y$ and $Y = aR + bR$. We also get that $R = X + Y$, where $X \cap Y = aR$. This yields an exact sequence $0 \to aR \to X \times Y \to R \to 0$ which splits. So $X \times Y \cong aR \times R \cong R \times R$. This implies that $Y = aR + bR$ is a finitely generated projective right ideal. Since $\hat{R} \cong R/Y$, $0 \to Y \to R \to \hat{R} \to 0$ is exact. Then $Y \otimes R \hat{R} \to R \otimes R \hat{R} \to \hat{R} \otimes R \hat{R} \to 0$ is exact. Also, a finitely generated projective $R$-module is essentially finitely related. So, by Cateforis ([3], Proposition 1.7), $(aR + bR) \otimes R \hat{R}$ is projective as an $\hat{R}$-module. Then $Y \otimes R \hat{R}$ is a direct summand of a free $\hat{R}$-module. Now $Z(\hat{R}_R) = 0$, hence $Z(Y \otimes R \hat{R}) = 0$ because $Y \otimes R \hat{R}$ is a direct summand of a free $\hat{R}$-module. Now consider $Y \otimes R \hat{R} \hat{R} \to R \otimes R \hat{R} \to R \otimes R \hat{R} \to 0$. Again, by Cateforis ([3], Lemma 1.8), $\ker i = Z(Y \otimes R \hat{R}) = 0$. So $0 \to Y \otimes R \hat{R} \hat{R} \to R \otimes R \hat{R} \to R \otimes R \hat{R} \to 0$ is exact. Since $R \otimes R \hat{R} \cong \hat{R}$, let $f: R \otimes R \hat{R} \to \hat{R}$ be the canonical isomorphism. Then $Y \otimes R \hat{R} \to \hat{R}$ is a monomorphism, and $Y \otimes R \hat{R} \cong Y \hat{R}$. Since $Y$ is finitely generated, $Y \hat{R}$ is a finitely generated right ideal of $\hat{R}$. So $Y \hat{R} = e \hat{R}$, where $e^2 = e$. Thus we have the following exact sequence: $0 \to e \hat{R} \to \hat{R} \otimes R \hat{R} \to 0$, and $\hat{R} \otimes R \hat{R} \cong \hat{R}/e \hat{R} = (1 - e) \hat{R}$. Hence $\hat{R} \otimes R \hat{R}$ is isomorphic to a direct summand of $\hat{R}$. Since $Z(\hat{R}_R) = 0$, $Z(\hat{R} \otimes R \hat{R}) = 0$. Since $\hat{R} = xR$, for some $x \in \hat{R}$, the
kernel of the canonical map \( f: \hat{R} \otimes_R \hat{R} \to \hat{R} \) defined by \( f(a \otimes b) = ab \) is contained in \( Z(\hat{R} \otimes_R \hat{R}) \) and hence must be zero. Since \( f \) is surjective, \( f \) is an isomorphism. By Silver ([18], Proposition 1.1), there exists an epimorphism in the category of rings from \( R \) to \( \hat{R} \).

Let \( M \) be a right \( \hat{R} \)-module which is quasi-injective as a right \( R \)-module. We claim that \( M \) is quasi-injective as a right \( \hat{R} \)-module. Let \( 0 \to A_\hat{R} \to M_\hat{R} \to B_\hat{R} \to 0 \) be exact. Consider \( 0 \to \text{Hom}_R(B_\hat{R}, M_\hat{R}) \to \text{Hom}_R(M_\hat{R}, M_\hat{R}) \to \text{Hom}_R(A_\hat{R}, M_\hat{R}) \). By Silver ([18], Corollary 1.3), \( \text{Hom}_R(N, N') \cong \text{Hom}_R(N, N') \), where \( N, N' \) are right \( \hat{R} \)-modules. Also \( 0 \to \text{Hom}_R(B, M) \to \text{Hom}_R(M, M) \to \text{Hom}_R(A, M) \to 0 \) is exact since \( M_\hat{R} \) is quasi-injective. Thus \( 0 \to \text{Hom}_R(B, M) \to \text{Hom}_R(M, M) \to \text{Hom}_R(A, M) \to 0 \) is exact. So \( M_\hat{R} \) is quasi-injective. Let \( K \) be a cyclic right \( R \)-module. Then \( K \) is a cyclic right \( R \)-module. Since \( R \) is a right \( PCQI \)-domain, \( K_\hat{R} \) is quasi-injective. Thus \( K_\hat{R} \) is quasi-injective. Since \( \hat{R} \) is right self-injective, \( \hat{R} \) is a \( QC \)-ring. So \( \hat{R} \) is semiperfect and simple, hence simple artinian. Thus \( \hat{R} \) is a division ring. This proves that \( R \) is a right \( \tilde{O} \)-domain.

We conclude by a remark that we have not studied arbitrary prime right \( PCQI \)-rings. This case remains open. Indeed, a characterization of right \( PCQI \)-domains has not yet been obtained.

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