PACIFIC JOURNAL OF MATHEMATICS
Vol. 68, No 1, 1977

CHARACTERIZATIONS OF CERTAIN MAPS OF CONTRACTIVE TYPE

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The following result is obtained. Let f be a self map on a nonempty complete metric space (X, d). Then the following conditions are equivalent: (i) For any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $d(f(x), f(y)) < \epsilon$ whenever $\epsilon \leqq d(x, y) < \epsilon + \delta(\epsilon)$. (ii) There exists a function w of $[0, \infty)$ into $[0, \infty)$ such that $w(s) > s$ for all $s > 0$, w is lower semicontinuous from the right on $(0, \infty)$ and $w(d(f(x), f(y))) \leqq d(x, y)$, $x, y \in X$.

1. Introduction. In 1969, E. Keeler and A. Meir [3] obtained the following result.

THEOREM A. (Keeler and Meir). *Let f be a self map on a nonempty complete metric space (X, d). Suppose that for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $d(f(x), f(y)) < \epsilon$ whenever $\epsilon \leqq d(x, y) < \epsilon + \delta(\epsilon)$. Then f has a unique fixed point x_0 and $\{f^n(x)\}$ converges to x_0 for all x in X.*

Theorem A generalized the following result of D. W. Boyd and J. S. W. Wong [1] (and therefore, an earlier result of E. Rakotch [4]).

THEOREM B. (Boyd and Wong). *Let f be a self map on a nonempty complete metric space (X, d). Suppose that there exists a self map Φ on $[0, \infty)$ such that Φ is upper semicontinuous from the right, $\Phi(t) < t$ for $t > 0$ and f is Φ-contractive*:

$$d(f(x), f(y)) \leqq \Phi(d(x, y)), \qquad x, y \in X.$$

Then f has a unique fixed point x_0 and $\{f^n(x)\}$ converges to x_0 for all x in X.

In this paper, equivalent conditions in terms of monotone transformations are obtained. These will show that the essential difference between Theorems A and B is a matter of imposing monotone transformations on the left side or right side of certain inequalities.

2. Main results.

THEOREM 1. *Let f be a self map on a nonempty complete metric space (X, d). Then the following conditions are equivalent*:

(i) *For any* $\epsilon > 0$, *there exists* $\delta(\epsilon) > 0$ *such that* $d(f(x), f(y)) < \epsilon$ *whenever* $\epsilon \leqq d(x, y) < \epsilon + \delta(\epsilon)$.

(ii) *There exists a self map* w *of* $[0, \infty)$ *into* $[0, \infty]$ *such that* $w(s) > s$ *for all* $s > 0$, *w is lower semicontinuous from the right on* $(0, \infty)$ *and*

$$w(d(f(x), f(y))) \leqq d(x, y), \quad x, y \in X.$$

Proof. (i) $\Rightarrow$ (ii). Let $\epsilon > 0$. (i) implies that f is contractive: $d(f(x), f(y)) < d(x, y)$ for distinct x, y in X. So

$$(*) \qquad d(f(x), f(y)) < \epsilon \text{ whenever } d(x, y) < \delta(\epsilon).$$

Define $w(0) = 0$ and

$$w(\epsilon) = \sup\{\delta(\epsilon) > 0 \colon \delta(\epsilon) \text{ satisfies } (*)\}.$$

Then w is an increasing function of $[0, \infty)$ into $[0, \infty]$ such that $w(s) > s$ for all $s > 0$. Also w is semicontinuous from the right. We need only prove that

$$w(d(f(x), f(y))) \leqq d(x, y), \qquad x, y \in X.$$

Suppose not. Then

$$w(d(f(x), f(y))) > d(x, y)$$

for some x, y in X. Thus

$$\epsilon \equiv d(f(x), f(y)) > 0 \text{ and } d(x, y) < w(\epsilon).$$

By the choice of w, $d(f(x), f(y)) < \epsilon$, a contradiction.

(ii) $\Rightarrow$ (i). Let $\epsilon > 0$. Since w is lower semicontinuous from the right at ϵ, there exists $\delta_1(\epsilon) > 0$ such that

$$\frac{\epsilon + w(\epsilon)}{2} < w(s) \text{ whenever } \epsilon \leqq s < \epsilon + \delta_1(\epsilon).$$

Let $\delta(\epsilon) = \min\{\delta_1(\epsilon), (w(\epsilon) - \epsilon)/2\}$. Suppose that $\epsilon \leqq d(x, y) < \epsilon + \delta(\epsilon)$. We need only to prove that $d(f(x), f(y)) < \epsilon$. Suppose not. Then by the contractivity of f,

$$\epsilon \leqq d(f(x), f(y)) < \epsilon + \delta(\epsilon) \leqq \epsilon + \delta_1(\epsilon).$$

So

$$\begin{aligned}\frac{\epsilon + w(\epsilon)}{2} &< w(d(f(x), f(y))) \\ &\leqq d(x, y) \\ &\leqq \epsilon + \delta(\epsilon) \\ &\leqq \epsilon + \frac{w(\epsilon) - \epsilon}{2} \\ &= \frac{\epsilon + w(\epsilon)}{2},\end{aligned}$$

a contradiction.

As shown above, Theorem 1 gives Theorem A. Intuitively, one would think that the conditions on f in Theorem B and (ii) of Theorem 1 should be equivalent. However, Theorem B is a special case of, and is not equivalent to Theorem A **[3]**. In other words, there is no symmetry in "right and left" in the sense that the fixed point theorems obtained depend on the sides—left or right—on which we impose monotone transformations. However, the following shows that such symmetry does exist if we restrict ourselves to the case where w in (ii) of Theorem 1 is lower semicontinuous (or Φ in Theorem B is upper semicontinuous).

THEOREM 2. *Let f be a self map on a nonempty complete metric space (X, d). Then the following conditions are equivalent:*

(i) *There exists a self map Φ on $[0, \infty)$ such that $\Phi(t) < t$ for $t > 0$, Φ is increasing, continuous and f is Φ-contractive.*

(ii) *There exists a self map w on $[0, \infty)$ such that $w(s) > s$ for $s > 0$, w is lower semicontinuous and*

$$w(d(f(x), f(y))) \leqq d(x, y), \qquad x, y \in X.$$

For related fixed point theorems for function f satisfying conditions in Theorem 2, we refer the reader to **[2]** and **[5]**.

Added in proof: Indication of a proof for Theorem 2 is given in **[6]**: Chi Song Wong, Maps of Contractive Type, Proceedings of the Seminar on fixed point theory and its applications, Academic Press (1976), 197–207.

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Received October 21, 1975 and in revised form July 16, 1976.

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