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STIEFEL–WHITNEY CLASSES OF MANIFOLDS

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1. Introduction. The purpose of this paper is to axiomatize the Stiefel–Whitney classes of closed manifolds and using the axioms to give a proof of Wu's theorem ([3]):

THEOREM. *If M^n is a closed n manifold and $v = 1 + v_1 + \cdots + v_n$ with $v_i \in H^i(M; Z_2)$ defined by*

$$\langle v_i \cup x, [M]\rangle = \langle Sq^i x, [M]\rangle$$

for all $x \in H^{n-i}(M; Z_2)$, then $w = Sqv$ is the Stiefel–Whitney class of M.

The axioms for the Stiefel–Whitney classes are as follows.

For each closed manifold M^n there is a class $\alpha(M) = \alpha_0(M) + \cdots + \alpha_n(M)$ where $\alpha_i(M) \in H^i(M; Z_2)$ satisfying:

(1) If $i: M \to N$ is an imbedding with trivial normal bundle, then $\alpha(M) = i^*\alpha(N)$,

(2) $\alpha(M \times N) = \pi_M^*(\alpha(M)) \cup \pi_N^*(\alpha(N))$, and

(3) $\alpha(RP(n)) = (1 + a)^{n+1}$, where $a \in H^1(RP(n); Z_2)$ is the non-zero class.

It is well-known (See [2]) that the Stiefel–Whitney classes satisfy these properties, and it will be shown that if $\alpha(M)$ satisfies these properties, then $\alpha(M) = w(M)$.

A different axiomatization has been given by J. D. Blanton and P. A. Schweitzer, "Axioms for characteristic classes of manifolds", Proc. of Symp. in Pure Math., Amer. Math. Soc. 27 (1975), volume I, 349–356. Their axioms use all manifolds, so cannot be used to prove Wu's theorem which is only meaningful for closed manifolds.

2. Axiomatics. Suppose one is given classes $\alpha(M) \in H^*(M; Z_2)$ for each closed manifold M^n which satisfy properties 1, 2, and 3.

LEMMA 1. $\alpha_0(M) = 1$, *i.e., is the unit class.*

Proof. If P is a point and $i: P \to RP(n)$ then i is an imbedding with trivial normal bundle so $\alpha_0(P) = 1$. Then for any $f: P \to M$, f is an imbedding with trivial normal bundle, so $f^*\alpha_0(M) = 1$. Thus $\alpha_0(M) = 1$.*

LEMMA 2. $\alpha(S^n)=1$.

Proof. For $n=0$ this is $\alpha_0(S^0)=1$. If $n>0$, S^n imbeds in S^{n+1} with trivial normal bundle, so $\alpha(S^n)=i^*(1+\alpha_{n+1}(S^{n+1}))=1$.*

Now consider a smooth vector bundle η over a closed manifold M and let $S=S(\eta\oplus 1)$ be the unit sphere bundle of the Whitney sum of η and a trivial line bundle. Considering $S\subset E(\eta)\times R$ as the pairs (x,t) with $\|x\|^2+t^2=1$, let $\sigma: M\to S$ by $\sigma(y)=(0y,1)$ where $0y$ is the zero vector over y. Then $\sigma: M\to S$ is an imbedding with normal bundle η.

Define $\alpha(\eta)=\sigma^*(\alpha(S))/\alpha(M)$.

Note. $\alpha(M)$ is invertible since $\alpha_0(M)=1$, and $\alpha_0(\eta)=1$.

LEMMA 3. *If $f: M\to N$ is an imbedding with normal bundle ν, then $\alpha(\nu)=f^*(\alpha(N))/\alpha(M)$.*

Proof. Let $g: M\to N\times S^1$ by $g(m)=(f(m),1)$. Then g is an imbedding with normal bundle $\nu\oplus 1$ and extends to an imbedding

$$h: D(\nu\oplus 1)\to N\times S^1$$

of the disc bundle. Clearly g is homotopic to $h\circ\sigma$, $\sigma: M\to S(\nu\oplus 1)$ and $h: S(\nu\oplus 1)\to N\times S^1$ is an imbedding with trivial normal bundle. Now

$$\alpha(\nu)=\frac{\sigma^*\alpha(S)}{\alpha(M)}=\frac{\sigma^*h^*\alpha(N\times S^1)}{\alpha(M)}=\frac{g^*\alpha(N\times S^1)}{\alpha(M)}$$
$$=\frac{g^*(\pi_N^*(\alpha(N))\cup\pi_S^*1(\alpha(S^1)))}{\alpha(M)}=\frac{g^*\pi_N^*(\alpha(N))}{\alpha(M)}=\frac{f^*(\alpha(N))}{\alpha(M)}.^*$$

COROLLARY 1. *If $f: M\to N$ is a smooth map $f^*(\alpha(N))=\alpha(f^*\tau_N)$, where τ_N is the tangent bundle of N.*

Proof. $g: M\to M\times N$ given by $g(m)=(m,f(n))$ is an imbedding with normal bundle $f^*\tau_N$. Thus

$$\alpha(f^*\tau_N)=g^*(\alpha(M\times N))/\alpha(M)=g^*(\pi_M^*(\alpha(M))\cup\pi_N^*(\alpha(N)))/\alpha(M)$$
$$=\alpha(M)\cup f^*(\alpha(N))/\alpha(M)=f^*(\alpha(N)).^*$$

COROLLARY 2. $\alpha(M)=\alpha(\tau_M)$.

Proof. Apply Corollary 1 to the identity map.*

COROLLARY 3. *If ρ is a bundle over M, $\alpha(\rho \oplus \tau_M) = \alpha(\rho) \cup \alpha(\tau_M)$.*

Proof. $\sigma\colon M \to S = S(\rho \oplus 1)$ is an imbedding with normal bundle ρ so $\alpha(\rho) = \sigma^*\alpha(S)/\alpha(M)$ or $\alpha(\rho)\cdot\alpha(\tau_M) = \sigma^*(\alpha(S)) = \alpha(\sigma^*\tau_S) = \alpha(\rho \oplus \tau_M)$.*

LEMMA 4. *If ξ and η are smooth vector bundles over M, $\alpha(\xi \oplus \eta) = \alpha(\xi) \cup \alpha(\eta)$.*

Proof. Consider the composite

$$f\colon M \xrightarrow{\Delta} M \times M \xrightarrow{\sigma_1 \times \sigma_2} S(\xi \oplus 1) \times S(\eta \oplus 1) = X.$$

This is an imbedding with normal bundle $\xi \oplus \eta \oplus \tau_M$ so $f^*(\alpha(X)) = \alpha(\xi \oplus \eta \oplus \tau_M) \cup \alpha(M) = \alpha(\xi \oplus \eta) \cup \alpha(M) \cup \alpha(M)$. On the other hand,

$$\begin{aligned} f^*(\alpha(X)) &= f^*(\pi_1^*(\alpha(S(\xi \oplus 1))) \cup \pi_2^*(\alpha(S(\eta \oplus 1)))) \\ &= f^*\pi_1^*(\alpha(S(\xi \oplus 1))) \cup f^*\pi_2^*(\alpha(S(\eta \oplus 1))) \\ &= \sigma_1^*(\alpha(S(\xi \oplus 1))) \cup \sigma_2^*(\alpha(S(\eta \oplus 1))) \\ &= \alpha(\xi) \cup \alpha(M) \cup \alpha(\eta) \cup \alpha(M).* \end{aligned}$$

LEMMA 5. *If ρ is a smooth vector bundle over N and $f\colon M \to N$ is a smooth map, then $f^*(\alpha(\rho)) = \alpha(f^*\rho)$.*

Proof. Let $g\colon M \to S = S(\rho \oplus 1)$ be the composite of $f\colon M \to N$ and $\sigma\colon N \to S$. Then

$$\begin{aligned} g^*(\alpha(S)) &= f^*\sigma^*(\alpha(S)) = f^*(\alpha(N) \cup \alpha(\rho)), \\ &= f^*(\alpha(N)) \cup f^*(\alpha(\rho)) \end{aligned}$$

and

$$\begin{aligned} g^*(\alpha(S)) &= g^*(\alpha(\tau_S)) = \alpha(g^*\tau_S) = \alpha(f^*\sigma^*\tau_S) \\ &= \alpha(f^*(\tau_N \oplus \rho)) = \alpha(f^*\tau_N \oplus f^*\rho) \\ &= \alpha(f^*\tau_N) \cup \alpha(f^*\rho). \end{aligned}$$

Since $f^*(\alpha(N)) = \alpha(f^*\tau_N)$, the result follows.*

LEMMA 6. *If ξ is the nontrivial line bundle over $RP(n)$, then $\alpha(\xi) = 1 + a$.*

Proof. If $i: RP(n) \to RP(n+1)$ is the standard inclusion, then i is an imbedding with normal bundle ξ, so $\alpha(\xi) = i^*(1+a)^{n+2}/(1+a)^{n+1} = 1 + a.$*

PROPOSITION 1. *$\alpha(M)$ is the Stiefel–Whitney class of the manifold M.*

Proof. From Lemmas 4, 5, and 6, $\alpha(\eta)$ is the Stiefel–Whitney class of the bundle η, for these axioms characterize the Stiefel–Whitney classes of smooth bundles. From Corollary 2, one has $\alpha(M) = \alpha(\tau_M) = w(\tau_M).$*

3. Wu's theorem. For a closed n-manifold M^n, let $v(M) = v_0(M) + \cdots + v_n(M) \in H^*(M; Z_2)$ be the classes defined by

$$\langle v_i(M) \cup x, [M] \rangle = \langle Sq^i x, [M] \rangle$$

for all $x \in H^{n-i}(M; Z_2)$. Let $\alpha(M) = Sq(v(M))$.

LEMMA 7. *If $i: M \to N$ is an imbedding with trivial normal bundle, then $i^*(v(N)) = v(M)$, and $i^*(\alpha(N)) = \alpha(M)$.*

Proof. If $m = \dim M$, $n = \dim N$, let

$$i_*: H^j(M; Z_2) \to H^{j+n-m}(N; Z_2)$$

be the Gysin homomorphism defined by

$$\langle i_*(x) \cup y, [N] \rangle = \langle x \cup i^*(y), [M] \rangle$$

for all $x \in H^j(M; Z_2)$ and $y \in H^{m-j}(N; Z_2)$. Let i be extended to an imbedding

$$j: M \times D^{n-m} \to N$$

and let

$$c: N \to M \times D^{n-m}/M \times S^{n-m-1}$$

be the collapse map sending all points not in the interior of the image of j to the base point. Identify $\tilde{H}^*(M \times D^{n-m}/M \times S^{n-m-1}; Z_2)$ with $H^*(M; Z_2) \otimes \tilde{H}^*(D^{n-m}, S^{n-m-1}; Z_2)$ and then

$$i_*(x) = c^*(x \otimes \sigma)$$

where $\sigma \in H^{n-m}(D^{n-m}, S^{n-m-1}; Z_2)$ is the nonzero class.

Note. To see this c_* sends the fundamental class of N to $[M] \otimes$ (dual σ) for it is degree one on top dimensional simplices. Letting $V = N$-interior $(j(M \times D^{n-m}))$, and $W = M \times D^{n-m}$, $(N, V) \cong (M \times D^{n-m}, M \times S^{n-m-1})$ by excision, and the module structure of $H^*(N, V; Z_2)$ as $H^*(N; Z_2)$ module factors through $H^*(jW; Z_2)$, i.e.

$$\begin{array}{ccc} H^*(N; Z_2) \otimes H^*(N, V; Z_2) & \xrightarrow{\cup} & H^*(N, V; Z_2) \\ \downarrow & & \downarrow \cong \\ H^*(jW; Z_2) \otimes H^*(jW, jW \cap V; Z_2) & \xrightarrow{\cup} & H^*(jW, jW \cap V; Z_2) \end{array}$$

commutes. Thus $y \cup c^*(x \otimes \sigma) = c^*(i^*(y) \cup x \otimes \sigma)$. (See [1]).

Then

$$\begin{aligned} \langle i^*(v_j) \cup x, [M] \rangle &= \langle v_j \cup i_*(x), [N] \rangle \\ &= \langle Sq^j i_*(x), [N] \rangle \\ &= \langle Sq^j c^*(x \otimes \sigma), [N] \rangle \\ &= \langle c^*(Sq^j (x \otimes \sigma)), [N] \rangle \\ &= \langle c^*((Sq^j x) \otimes \sigma), [N] \rangle \\ &= \langle i_*(Sq^j x) \cup 1, [N] \rangle \\ &= \langle Sq^j x, [M] \rangle \\ &= \langle v_j \cup x, [M] \rangle \end{aligned}$$

for all x, so $i^*(v_j(N)) = v_j(M)$. Squaring gives $i^*\alpha(N) = i^*Sqv(N) = Sqi^*v(N) = Sqv(M) = \alpha(M)$.*

LEMMA 8. *$v(M \times N) = \pi_M^* v(M) \cup \pi_N^* v(N)$ and $\alpha(M \times N) = \pi_M^* \alpha(M) \cup \pi_N^* \alpha(N)$.*

Proof. Identify $H^*(M \times N; Z_2)$ with $H^*(M; Z_2) \otimes H^*(N; Z_2)$ so that

$$\langle \pi_M^*(a) \cdot \pi_N^*(b), [M \times N] \rangle = \langle a, [M] \rangle \cdot \langle b, [N] \rangle,$$

and the result follows easily from the axioms for Sq^i.*

LEMMA 9. $v(RP(n)) = \Sigma_{i=0}^{n} \binom{n-i}{i} a^i$ and $\alpha(RP(n)) = (1+a)^{n+1}$.

Proof. Since $Sq^i a^{n-i} = \binom{n-i}{i} a^n$, $v_i = \binom{n-i}{i} a^i$. In $H^*(RP(\infty); Z_2)$ define classes

$$v(n) = \sum_{i=0}^{n} \binom{n-i}{i} a^i.$$

Then

$$\begin{aligned} v(n+1) &= \sum_{i=0}^{n+1} \binom{n+1-i}{i} a^i \\ &= \sum_{i=0}^{n+1} \left\{ \binom{n-i}{i} + \binom{n-i}{i-1} \right\} a^i \\ &= \sum_{i=0}^{n+1} \left\{ \binom{n-i}{i} + \binom{n-1-(i-1)}{i-1} \right\} a^i \\ &= v(n) + av(n-1), \end{aligned}$$

with $v(0) = v(1) = 1$. Letting $\alpha(n) = Sqv(n)$,

$$\begin{aligned} \alpha(n+1) &= Sq(v(n) + av(n-1)) \\ &= \alpha(n) + (a + a^2)\alpha(n-1) \end{aligned}$$

with $\alpha(0) = \alpha(1) = 1$.

Then it is easy to see $\alpha(n) = (1+a)^{n+1} - a^{n+1}$ by induction. Since $\alpha(RP(n))$ is the pull back of $\alpha(n)$ and $a^{n+1} = 0$, the result follows.*

From Proposition 1, one then has $\alpha(M) = w(M)$ for all M, which is Wu's theorem.

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