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SUPERALGEBRAS OF WEAK-*DIRICHLET ALGEBRAS

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Let A be a weak-*Dirichlet algebra of $L^\infty(m)$ and let $H^\infty(m)$ denote the weak-*closure of A in $L^\infty(m)$. Muhly showed that if $H^\infty(m)$ is an integral domain, then $H^\infty(m)$ is a maximal weak-*closed subalgebra of $L^\infty(m)$. We show in this paper that if $H^\infty(m)$ is not maximal as a weak-*closed subalgebra of $L^\infty(m)$, there is no algebra which contains $H^\infty(m)$ and is maximal among the proper weak-*closed subalgebras of $L^\infty(m)$. Moreover, we investigate the weak-*closed superalgebras of A and we try to classify them. We show that there are two canonical weak-*closed superalgebras of A which play an important role in the problem of describing all the weak-*closed superalgebras of A.

1. Preliminaries. Recall that by definition **[7]**, a weak-*Dirichlet algebra is an algebra A of essentially bounded measurable functions on a probability measure space $(X, \mathscr{A}, m)$ such that (i) the constant functions lie in A; (ii) $A + \bar{A}$ is weak-*dense in $L^\infty(m)$ (the bar denotes conjugation, here and always); (iii) for all f and g in A, $\int_X fgdm = \int_X fdm \int_X gdm$. The abstract Hardy space $H^p(m)$, $1 \leqq p \leqq \infty$, associated with A are defined as follows. For $1 \leqq p \leqq \infty$, $H^p(m)$ is the $L^p(m)$-closure of A, while $H^\infty(m)$ is defined to be the weak-*closure of A in $L^\infty(m)$. For $1 \leqq p \leqq \infty$, let $H_0^p = \left\{ f \in H^p(m); \int_X fdm = 0 \right\}$.

A (weak-*closed) subalgebra B^∞ of $L^\infty(m)$, containing A, is called a superalgebra of A. Let $B_0^\infty = \left\{ f \in B^\infty; \int_X fdm = 0 \right\}$ and let I_B^∞ be the largest weak-*closed ideal of B^∞ which is contained in B_0^∞. (The existence of I_B^∞ is shown in Lemma 2 of **[6]**). If $B^\infty = H^\infty(m)$ (resp. $L^\infty(m)$), it is clear that $B_0^\infty = I_B^\infty = H_0^\infty$ (resp. $I_B^\infty = \{0\}$). In general, $I_B^\infty \subseteq H_0^\infty$ by [**6**, Lemma 2]. Let $\mathscr{L}_B^\infty$ be a self-adjoint part of B^∞, i.e. the set of all functions in B^∞ whose complex conjugates are also in B^∞.

For any subset $M \subseteq L^\infty(m)$ and $1 \leqq p < \infty$, denote by $[M]_p$ the norm closed linear span of M in $L^p(m)$ and by $[M]_*$ the weak-*closed linear span of M. For a weak-*closed superalgebra B^∞, let $B^p = [B^\infty]_p$ and let $I_B^p = [I_B^\infty]_p$ for $1 \leqq p < \infty$. For any measurable subset E of X, the function χ_E is the characteristic function of E. If $f \in L^p(m)$, denote by E_f the support set of f and by χ_f the characteristic function of E_f.

LEMMA 1. *If B^∞ is a weak-*closed superalgebra of A, then B^2 and $\bar{I}^2_B$ are orthogonal in $L^2(m)$ and $B^2 \oplus \bar{I}^2_B = L^2(m)$.*

The proof is in [**6**, Lemma 2].

LEMMA 2. (Hoffman) *Let E be a measurable subset of X such that $0 < m(E) < 1$. Then there exists k in $H^\infty(m)$ such that k is real on E while k is not constant on E.*

The proof for logmodular algebra [**1**, p. 138] is valid without change for weak-*Dirichlet algebras.

2. Support sets. If no nonzero function $H^\infty(m)$ can vanish on a set of positive measure, then $H^\infty(m)$ is a maximal weak-*closed subalgebra (cf. [**3**]). This shows the importance of the support set of each function in $H^\infty(m)$. We shall investigate properties of support sets of functions in superalgebras of A, in particular, in the algebra $H^\infty(m)$.

DEFINITION. Let B^∞ be a weak-*closed superalgebra of A. We say that the characteristic function χ_E is *minimal for* B^∞ in case any characteristic function χ_F in B^∞ which satisfies the strict inequality $\chi_F \lneqq \chi_E$ on a set of positive measure must be zero a.e. Note that we do *not* assume that χ_E lies in B^∞. Similarly, χ_E is called *maximal for* B^∞ in case any characteristic function χ_F in B^∞ which satisfies the strict inequality $\chi_E \lneqq \chi_F$ on a set of positive measure must be 1 a.e.

LEMMA 3. *Let B^∞ be a weak-*closed superalgebra of A.*

(1) *If B^∞ contains $H^\infty(m)$ properly, there exists a nontrivial characteristic function in B^∞.*

(2) *There exists no nontrivial minimal (maximal) characteristic function for B^∞ in B^∞.*

Proof. Assertion (1) is shown in the proof of [**3**, Theorem]. We shall show assertion (2). Let χ_{E_0} be a minimal characteristic function for B^∞ in B^∞. Then, it follows that there exists no nonconstant real-valued function in $\chi_{E_0}\mathscr{L}^\infty_B$ and hence in $\chi_{E_0}B^\infty$. For if it were not the case, then $\chi_{E_0}\mathscr{L}^\infty_B$ would be a nontrivial commutative von Neumann algebra of operators on $L^2(m)$ contrary to the assumption on χ_{E_0}. On the other hand, Lemma 2 shows that there exists k in $H^\infty(m)$ such that $\chi_{E_0}k$ is a nonconstant real-valued function in $\chi_{E_0}B^\infty$. This contradiction shows that there exists no nontrivial minimal characteristic function for B^∞ in B^∞. If χ_{F_0} were a nontrivial maximal characteristic function for B^∞ in B^∞, then $1 - \chi_{F_0}$ would be a nontrivial minimal characteristic function for

B^∞ in B^∞. Since this is not possible by what was just proved, χ_{F_0} cannot be a nontrivial maximal characteristic function for B^∞ in B^∞.

LEMMA 4. *If M is a closed invariant subspace of $L^2(m)$ (invariant under multiplication by functions in A), then $M \cap L^\infty(m)$ is a weak-*closed invariant subspace. Moreover, the map $M \to M \cap L^\infty(m)$ is one-to-one and onto.*

The proof for logmodular algebras [**1**, p. 131] is valid without change for weak-*Dirichlet algebras.

LEMMA 5. *Let B^∞ be a weak-*closed superalgebra of A and suppose $D^\infty = [\chi_f B^\infty]_* + (1-\chi_f)L^\infty(m)$ for some f in I_B^∞. Then D^∞ is a weak-*closed superalgebra which contains B^∞, χ_f is in D^∞, and f lies in I_D^∞.*

Proof. It is clear that D^∞ is a weak-*closed superalgebra which contains B^∞ and χ_f. By Lemma 1 and Lemma 4, $I_B^\infty \supseteq I_D^\infty$ but it is not clear that $f \in I_D^\infty$. Since $f \in I_B^\infty$, by Lemma 2,

$$\int_X f\chi_f g\,dm = 0 \qquad g \in B^\infty$$

and hence

$$\int_X f\chi_f g\,dm = 0 \qquad g \in D^\infty.$$

Thus again by Lemma 1 and Lemma 4, it follows that $f \in I_D^\infty$.

THEOREM 1. *If f is a function in B^∞ such that $0 \lneqq \chi_f \lneqq 1$, then there exists a nonzero g in B^∞ such that $\chi_g \lneqq \chi_f$.*

Proof. Suppose $f \in I_B^\infty$. If $fh = 0$ a.e for all h in I_B^∞, then by Lemma 1 and Lemma 4, it follows that $f \in \mathcal{L}_B^\infty$. Thus $\chi_f \in \mathcal{L}_B^\infty \subset B^\infty$, so by (2) of Lemma 3, there exists a nonzero characteristic function χ_E in B^∞ such that $\chi_E \lneqq \chi_f$. Thus we may assume that $fh \neq 0$ for some h in I_B^∞. Since I_B^∞ is an ideal of B^∞, $fh \in I_B^\infty$ and $\chi_f \geqq \chi_{fh} \gneqq 0$.

By taking fh if necessary we may assume that $f \in I_B^\infty$. Suppose $D^\infty = [\chi_f B^\infty]_* + (1-\chi_f)L^\infty(m)$, then by Lemma 5, it follows that $f \in I_D^\infty$ and $\chi_f \in D^\infty$. By (2) of Lemma 3, there exists a nonzero χ_E in D^∞ such that $\chi_f \gneqq \chi_E$. Since I_D^∞ is an ideal of D^∞, $\chi_E f \in I_D^\infty$ and hence $\chi_E f \in B^\infty$. Suppose $g = \chi_E f$, then g is a nonzero function in B^∞ and $\chi_f \gneqq \chi_g$.

It is natural to ask if whenever there is a function f in B^∞ such that $0 \lneqq \chi_f \lneqq 1$, there also exists a function g in B^∞ such that

$\chi_f \lneqq \chi_g \lneqq 1$. However, the third example of §6 shows that in general such a g need not exist.

3. Non-maximality. Muhly [3] showed that if $H^\infty(m)$ is an integral domain, then $H^\infty(m)$ is a maximal weak-*closed subalgebra of $L^\infty(m)$. In this section, we shall show that if $H^\infty(m)$ is not an integral domain, there is no maximal proper weak-*closed superalgebra of A.

LEMMA 6. *Let B^∞ be a weak-*closed superalgebra of A. Then B^∞ has the form $B^\infty = \chi_{E_0}B^\infty + (1-\chi_{E_0})L^\infty(m)$, where $(1-\chi_{E_0})L^\infty(m)$ is the largest subspsace of B^∞ reducing $L^\infty(m)$. χ_{E_0} is called the essential function of B^∞.*

THEOREM 2. *If $H^\infty(m)$ is not maximal as a weak-*closed subalgebra of $L^\infty(m)$, then there is no algebra which contains $H^\infty(m)$ and is maximal among the proper weak-*closed subalgebra of $L^\infty(m)$.*

Proof. Suppose B^∞ contains $H^\infty(m)$ and is maximal among the proper weak-*closed subalgebras of $L^\infty(m)$. Then by assumption $B^\infty \neq H^\infty(m)$. Since $B^\infty \neq L^\infty(m)$, Lemma 6 implies that we can find a nonzero χ_{E_0} in B^∞ such that $B^\infty = \chi_{E_0}B^\infty + (1-\chi_{E_0})L^\infty(m)$ and the algebra $(1-\chi_{E_0})L^\infty(m)$ is the largest subspace of B^∞ reducing $L^\infty(m)$. By Lemma 3, there exists $\chi_F \in B^\infty$ such that $0 \lneqq \chi_F \lneqq \chi_{E_0}$. For such a χ_F in B^∞, set $D^\infty = \chi_F B^\infty + (1-\chi_F)L^\infty(m)$. Then D^∞ is a weak-*closed subalgebra which contains B^∞. Since $\chi_F \lneqq \chi_{E_0}$ and $(1-\chi_{E_0})L^\infty(m)$ is the largest subspace of B^∞ reducing $L^\infty(m)$, it follows that D^∞ contains B^∞ properly and $D^\infty \neq L^\infty(m)$. This contradiction proves theorem.

4. Relation between two superalgebras. In this section, we shall investigate the relation between two superalgebras. Let B_1^∞ and B_2^∞ be weak-*closed superalgebras of A such that $\chi_F B_1^\infty \subseteq \chi_F B_2^\infty$ for some χ_F in B_1^∞. If $\chi_E \cdot \chi_F B_1^\infty \neq \chi_E \cdot \chi_F B_2^\infty$ for all χ_E in B_1^∞ with $\chi_E \cdot \chi_F \neq 0$, then we write $\chi_F B_1^\infty < \chi_F B_2^\infty$. For a weak-*closed superalgebra B^∞ of A, we define $B_{\min}^\infty$ to be the intersection of all weak-*closed superalgebras $\{B_\alpha^\infty\}$ such that $B^\infty \subseteq B_\alpha^\infty$ and $\chi_{E_0}B^\infty < \chi_{E_0}B^\infty$, χ_{E_0} being the essential function of B^∞.

LEMMA 7. *Let B^∞ be a weak-*closed superalgebra of A.*

(1) *Each weak-*closed superalgebra D^∞ such that $B^\infty \subseteq D^\infty \subseteq B_{\min}^\infty$ has the form*

$$D^\infty = \chi_E B^\infty + (1-\chi_E)B_{\min}^\infty$$

for some χ_E in B^∞.

(2) *If f is a function in I_B^∞ and χ_f ($\neq 1$) is minimal for B^∞, then f lies in $I_{B_{\min}}^\infty$.*

Proof. (1) Let $\alpha = \sup\{m(F); \chi_F D^\infty = \chi_F B^\infty (\chi_F \in B^\infty)\}$. Choose χ_{E_n} in B^∞ with $m(E_n) \to \alpha$ and $\chi_{E_1} \leqq \chi_{E_2} \leqq \cdots$. Set $E = \bigcup_{n=1}^\infty E_n$, then $\chi_E \in B^\infty$, $\chi_E D^\infty = \chi_E B^\infty$ and $(1-\chi_E)D^\infty > (1-\chi_E)B^\infty$. By the definition of $B_{\min}^\infty$, it follows that $(1-\chi_E)D^\infty = (1-\chi_E)B_{\min}^\infty$ and hence $D^\infty = \chi_E B^\infty + (1-\chi_E)B_{\min}^\infty$.

(2) Let f be in I_B^∞ and let χ_f ($\neq 1$) be minimal for B^∞. Suppose $D^\infty = [\chi_f B^\infty]_* + (1-\chi_f)L^\infty(m)$. By Lemma 5, $f \in I_D^\infty$, $\chi_f \in D^\infty$ and hence in order to prove assertion (2), it is sufficient to prove that $I_D^\infty \subseteq I_{B_{\min}}^\infty$. If there existed a nonzero χ_E in B^∞ such that $\chi_E \leqq \chi_{E_0}$ and $\chi_E D^\infty = \chi_E B^\infty$, where χ_{E_0} is the essential function of B^∞, then $\chi_E \cdot \chi_f \in B^\infty$ because $\chi_f \in D^\infty$. Since χ_f ($\neq 1$) is minimal for B^∞, it follows that $\chi_E \cdot \chi_f = 0$ a.e. and hence $\chi_E < 1 - \chi_f$. By the definition of D^∞, $\chi_E B^\infty = \chi_E L^\infty(m)$ and hence $\chi_E \leqq 1 - \chi_{E_0}$. This contradiction shows that $\chi_{E_0} B^\infty < \chi_{E_0} D^\infty$, hence $D^\infty \supseteq B_{\min}^\infty$. By Lemma 1 and Lemma 4, it follows that $I_D^\infty \subseteq I_{B_{\min}}^\infty$.

LEMMA 8. *Let B_1^∞ and B_2^∞ be weak-*closed superalgebras of A. If B_2^∞ contains B_1^∞ properly, there exists a nontrivial minimal characteristic function for B_1^∞ in B_2^∞.*

Proof. Suppose there exists no nontrivial minimal characteristic function for B_1^∞ in B_2^∞. Then if χ_E is in B_2^∞, then χ_E lies in B_1^∞. For given $\chi_E \in B_2^\infty$, let $\alpha = \sup\{m(F); \chi_F \leqq \chi_E\ (\chi_F \in B_1^\infty)\}$. Then, as in the proof of (1) in Lemma 7, there is χ_{F_0} in B_1^∞ such that $\chi_{F_0} \leqq \chi_E$ and $m(F_0) = \alpha$. If $m(E) > \alpha$, then $(1-\chi_{F_0})\chi_E$ would be a minimal characteristic function for B_1^∞ in B_2^∞ contrary to the assumption on B_2^∞. Hence $m(E) = \alpha$ and hence $\chi_E = \chi_{F_0} \in B_1^\infty$. On the other hand, as in the proof of (1) of Lemma 3 we can show that there exists at least one characteristic function χ_S in B_2^∞ with $\chi_S \notin B_1^\infty$. This contradiction implies that there exists a nontrivial minimal characteristic function for B_1^∞ in B_2^∞.

LEMMA 9. *Let B_1^∞ and B_2^∞ be weak-*closed superalgebras of A such that $B_1^\infty \subseteq B_2^\infty$. Let $\bar{K} = B_2^2 \ominus B_1^2$, where '$\ominus$' denotes the orthogonal complement of B_1^2 in B_2^2. If $\chi_f \in B_1^\infty$ for every $f \in K$, then each weak-*closed superalgebra B^∞ such that $B_1^\infty \subseteq B^\infty \subseteq B_2^\infty$ has the form $B^\infty = \chi_E B_1^\infty + (1-\chi_E)B_2^\infty$ for some χ_E in B_1^∞.*

Proof. Suppose $\bar{S} = B_2^2 \ominus B^2$, then $\bar{S} \subseteq \bar{K}$. Hence the hypothesis shows that $\chi_f \in B_1^\infty$ for every $f \in S$. Let $\alpha = \sup\{m(E_f); f \in S\}$. If $f, g \in S$, there exists h in S with $E_h = E_f \cup E_g$. For let $h = f + (1-\chi_f)g$, since $\mathscr{L}_B^\infty S \subseteq S$ and hence $\mathscr{L}_{B_1}^\infty S \subseteq S$, then h lies in S. Choose $f_n \in S$ with $m(E_{f_n}) \to \alpha$ and $E_{f_1} \subseteq E_{f_2} \subseteq \cdots$. Alter the function f_n by the technique above so that their supports are disjoint. Suppose $f_0 = \sum_{n=1}^{\infty} 2^{-n} f_n$, then $f_0 \in S$, $m(E_{f_0}) = \alpha$ and hence $\chi_{f_0} = \chi_E$, where E is the support set of S. Thus $\chi_E \in B_1^\infty$. Since $(1-\chi_E)B_2^2$ is orthogonal to $\bar{S}$ and is contained in B_2^2, the set $(1-\chi_E)B_2^2$ is contained in B^2. Thus by Lemma 4, it follows that $B^\infty \supseteq \chi_E B_1^\infty + (1-\chi_E)B_2^\infty$ and $\chi_E B_1^\infty + (1-\chi_E)B_2^\infty$ is a weak-*closed superalgebra. If the two superalgebras above did not coincide, by Lemma 8, there would exist at least one nontrivial minimal χ_{F_0} for $\chi_E B_1^\infty + (1-\chi_E)B_2^\infty$ in B^∞. Then it may be assumed that $\chi_{F_0} \leqq \chi_E$. For if it were not so, the set $\chi_{F_0}(1-\chi_E)B_2^\infty$ would be contained in $\chi_E B_1^\infty + (1-\chi_E)B_2^\infty$ since χ_E lies in B_2^∞. By (2) of Lemma 3, there exists a nonzero χ_{E_1} in $\chi_E B_1^\infty + (1-\chi_E)B_2^\infty$ such that $\chi_{F_0}(1-\chi_E) \gneqq \chi_{E_1}$. This contradicts minimality of χ_{F_0} for $\chi_E B_1^\infty + (1-\chi_E)B_2^\infty$.

It is clear that $\chi_{F_0} S \subseteq S$. If $\chi_{F_0} S \neq \{0\}$, since $\chi_f \in B_1^\infty$ for every $f \in S$, χ_{F_0} may not be minimal. If $\chi_{F_0} S = \{0\}$, the set E may not be the support set of S. Thus $B^\infty = \chi_E B_1^\infty + (1-\chi_E)B_2^\infty$.

THEOREM 3. *Let B_1^∞ and B_2^∞ be weak-*closed superalgebras of A such that $B_1^\infty \subseteq B_2^\infty$ and hence $I_{B_1}^\infty \supseteq I_{B_2}^\infty$. If $f \in I_{B_2}^\infty$ for every $f \in I_{B_1}^\infty$ such that χ_f is minimal for B_1^∞, then each weak-*closed superalgebra B^∞ such that $B_1^\infty \subseteq B^\infty \subseteq B_2^\infty$ has the form*

$$B^\infty = \chi_E B_1^\infty + (1-\chi_E)B_2^\infty$$

for some χ_E in B_1^∞.

Proof. Suppose $K = B_2^2 \ominus B_1^2$, $\bar{K} = I_{B_2}^2 \ominus I_{B_2}^2$ by Lemma 1. If $k = \min(1/|f|, 1)$ for $f \in K$, then k is in $L^\infty(m)$ and $\log k$ is in $L^1(m)$. Consequently, by [**7**, Theorem 2.5.9] there is an outer function g in $H^\infty(m)$ such that $k = |g|$. Then, by Lemma 4 $fg \in I_{B_1}^2 \cap L^\infty(m) = I_{B_1}^\infty$. However, fg does not lie in $I_{B_2}^\infty$. For since g is the outer function, there exist g_n in $H^\infty(m)$ such that $g_n fg \to f (n \to \infty)$ weakly in $L^2(m)$. If $fg \in I_{B_2}^\infty$, by $g_n fg \in I_{B_2}^\infty$, it follows that $f \in I_{B_2}^\infty$ contrary to the assumption on f. Thus $fg \notin I_{B_2}^\infty$ and $\chi_f = \chi_{fg}$. By the hypothesis, χ_f is not minimal for B_1^∞ and hence there exists nonzero χ_E in B_1^∞ such that $\chi_f \gneqq \chi_E$. If $\chi_f \neq \chi_E$, let $h = (1-\chi_E)f$, then h lies in K again. We can repeat the above argument for $g = (1-\chi_E)f$ and hence we can show that $\chi_f \in B_1^\infty$ as in the proof of Lemma 8. Now Lemma 9 proves theorem.

THEOREM 4. *Let B_1^∞ and B_2^∞ be weak-*closed superalgebras of A such that $B_1^\infty \subseteq B_2^\infty$ (so $I_{B_1}^\infty \supseteq I_{B_2}^\infty$). Suppose $\chi_{E_0} B_1^\infty < \chi_{E_0} B_{1\min}^\infty$ for the essential function χ_{E_0} of B_1^∞. Then the following are equivalent.*

(1) *If f is in $I_{B_1}^\infty$ and χ_f $(\neq 1)$ is minimal for B_1^∞, then f lies in $I_{B_2}^\infty$.*

(2) *If f and g are in $I_{B_1}^\infty$, if both χ_f and χ_g are minimal for B_1^∞, and if $fg = 0$, a.e., then either f or g lies in $I_{B_2}^\infty$.*

(3) *Each weak-*closed superalgebra* B^∞ *such that* $B_1^\infty \subseteq B^\infty \subseteq B_2^\infty$ *has the form*

$$B^\infty = \chi_E B_1^\infty + (1-\chi_E)B_2^\infty$$

for some χ_E *in* B_1^∞.

Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (1). Take $f \in I_{B_2}^\infty$ such that χ_f $(\neq 1)$ is minimal for B_1^∞. Suppose $D^\infty = [\chi_f B_1^\infty] + (1-\chi_f)L^\infty(m)$, then by Lemma 5, D^∞ is a weak-*closed superalgebra such that $B_1^\infty \subseteq D^\infty$, $f \in I_D^\infty$, and $\chi_f \in D^\infty$. By (2) of Lemma 3, there exists at least one χ_E in D^∞ such that both $\chi_E f$ and $(1-\chi_E)f$ are nonzero functions in I_D^∞ (so in $I_{B_1}^\infty$). Since χ_f is minimal for B_1^∞, it follows that both $\chi_E\chi_f$ and $(1-\chi_E)\chi_f$ are minimal for B_1^∞. (2) implies that $\chi_E f \in I_{B_2}^\infty$ or $(1-\chi_E)f \in I_{B_2}^\infty$. Thus we have proved that, for $f \in I_{B_1}^\infty$ such that χ_f is minimal, there exists $\chi_F \in B_2^\infty$ such that $\chi_F f \neq 0$ and $\chi_F f \in I_{B_2}^\infty$. Thus we can show that $f \in I_{B_2}^\infty$ as in the proof of Lemma 8.

Assertion (1) implies (3) by Theorem 3. We will show that assertion (3) implies (1). If we can show that $B_2^\infty \subseteq B_{1\min}^\infty$ and hence $I_{B_{1\min}}^\infty \subseteq I_{B_2}^\infty$, then by (2) of Lemma 7, it follows that if $f \in I_{B_1}^\infty$ and χ_f is minimal for B_1^∞, then $f \in I_{B_2}^\infty$, and the proof is complete. As in the proof of Lemma 7 there is χ_{F_0} in B_1^∞ such that $\chi_{F_0}B_1^\infty = \chi_{F_0}B_2^\infty$, $(1-\chi_{F_0})B_1^\infty < (1-\chi_{F_0})B_2^\infty$, and $(1-\chi_{F_0}) \leqq \chi_{E_0}$. It is clear that $(1-\chi_{F_0})B_2^\infty \supseteq (1-\chi_{F_0})B_{1\min}^\infty$. Suppose $(1-\chi_{F_0})B_2^\infty \neq (1-\chi_{F_0})B_{1\min}^\infty$, and let $D^\infty = (1-\chi_{F_0})B_{1\min}^\infty + \chi_{F_0}B^\infty$. Then $B_1^\infty \subseteq D^\infty \subsetneq B_2^\infty$. By hypothesis, we can write $D^\infty = \chi_F B_1^\infty + (1-\chi_F)B_2^\infty$ for some χ_F in B_1^∞. $D^\infty = (\chi_F + \chi_{F_0} - \chi_F \cdot \chi_{F_0})B_1^\infty + (1-\chi_F)(1-\chi_{F_0})B_2^\infty$ because $B_2^\infty = \chi_{F_0}B_1^\infty + (1-\chi_{F_0})B_2^\infty$. If $\chi_F(1-\chi_{F_0}) = 0$ a.e., then $D^\infty = B_2^\infty$. Hence $\chi_F(1-\chi_{F_0}) \neq 0$ and $\chi_F(1-\chi_{F_0})B_{1\min}^\infty \subseteq \chi_F D^\infty = \chi_F B_1^\infty$. Thus $\chi_F(1-\chi_{F_0}) \leqq \chi_F \cdot \chi_{E_0} \leqq \chi_{E_0}$. This contradicts that $\chi_{E_0}B_1^\infty < \chi_{E_0}B_{1\min}^\infty$. Thus $B_2^\infty = \chi_{F_0}B_1^\infty + (1-\chi_{F_0})B_{1\min}^\infty \subseteq B_{1\min}^\infty$.

5. Two canonical superalgebras. As corollaries of the results in §4, we shall show that there are two canonical superalgebras of A. We define $H_{\max}^\infty$ to be the weak-*closed superalgebra of A generated by $H^\infty(m)$ and χ_f for all f in $H^\infty(m)$. This superalgebra was considered by the author **[5]**. If no nonzero function in $H^\infty(m)$ can vanish on a set of positive measure, then $H_{\max}^\infty = H^\infty(m)$.

COROLLARY 1. *Each weak-*closed superalgebra* B^∞ *of* A *which contains* $H_{\max}^\infty$ *has the form* $B^\infty = \chi_E H_{\max}^\infty + (1-\chi_E)L^\infty(m)$ *for some* χ_E *in* $H_{\max}^\infty$.

Proof. Apply Theorem 4 with $B_1^\infty = H_{\max}^\infty$ and $B_2^\infty = L^\infty(m)$. By definition of $H_{\max}^\infty$, $\chi_f \in H_{\max}^\infty$ for every $f \in I_{H_{\max}}^\infty$ and hence if χ_f $(\neq 1)$ is minimal for $H_{\max}^\infty$, then by (2) of Lemma 3, $f = 0$ a.e.

If $\chi_{E_0}H^\infty_{\max} < \chi_{E_0}B^\infty$ for the essential function χ_{E_0} of $H^\infty_{\max}$, by Corollary 1 it follows that $B^\infty = L^\infty(m)$. Hence $(H^\infty_{\max})_{\min} = L^\infty(m)$ and if $\chi_{E_0} \neq 0$, then $\chi_{E_0}H^\infty_{\max} < \chi_{E_0}(H^\infty_{\max})_{\min}$.

COROLLARY 2. *Let B^∞ be a weak-*closed superalgebra of A. If each weak-*closed superalgebra D^∞ of A which contains B^∞ has the form $D^\infty = \chi_E B^\infty + (1-\chi_E)L^\infty(m)$ for some χ_E in B^∞, then $B^\infty \supseteq H^\infty_{\max}$.*

Proof. We may assume that $B^\infty \neq L^\infty(m)$. It is easy to show that $B^\infty_{\min} = L^\infty(m)$ and hence $I^\infty_{B_{\min}} = \{0\}$. Applying Lemma 7, if $f \in I^\infty_B$ and χ_f ($\neq 1$) is minimal for B^∞, then $f = 0$ a.e. Hence if $f \in I^\infty_B$ with $0 \lneqq \chi_f \lneqq 1$, then there exists nonzero χ_E in B^∞ such that $\chi_f \gneqq \chi_E$. If $f \in B^\infty$, $f \neq 0$, then $f \in \mathscr{L}^\infty_B$ or there exists a function g in I^∞_B such that $gf \neq 0$. Thus if $f \in B^\infty$ and $f \neq 0$, then there exists nonzero χ_F in B^∞ such that $\chi_f \gneqq \chi_F$. As in the proof of Lemma 8, we can show that $\chi_f \in B^\infty$. Thus $B^\infty \supseteq H^\infty_{\max}$.

The second canonical superalgebra of A is $H^\infty_{\min}$. If $\chi_E \in H^\infty(m)$, then $\chi_E = 0$ a.e. or $\chi_E = 1$ a.e. So $H^\infty_{\min}$ is an intersection of all weak-*closed superalgebras $\{B^\infty_\alpha\}$ which contains $H^\infty(m)$ properly. Then $H^\infty_{\min}$ may coincide with or may be different from $H^\infty(m)$. If $H^\infty_{\min} \neq H^\infty(m)$, then $H^\infty_{\min}$ is the minimum weak-*closed superalgebra which contains $H^\infty(m)$ properly.

COROLLARY 3. *Let B^∞ be a weak-*closed superalgebra of A which contains $H^\infty(m)$ properly. Suppose $H^\infty_{\min} \neq H^\infty(m)$. Then the following are equivalent.*

(1) *If f in $H^\infty(m)$ vanishes on a set of positive measure, then f lies in I^∞_B.*

(2) *If f and g in $H^\infty(m)$ and $fg = 0$ a.e., then f lies in I^∞_B or g lies in I^∞_B.*

(3) *Each weak-*closed superalgebra D^∞ such that $H^\infty(m) \subseteq D^\infty \subseteq B^\infty$ coincides with $H^\infty(m)$ or B^∞.*

(4) *B^∞ is a minimum weak-*closed superalgebra which contains $H^\infty(m)$ properly,* i.e. $B^\infty = H^\infty_{\min}$.

Proof. Since $H^\infty_{\min} \neq H^\infty(m)$, assertions (3) and (4) are equivalent. Apply Theorem 4 with $B^\infty_1 = H^\infty(m)$ and $B^\infty_2 = B^\infty$, then $I^\infty_{B_1} = H^\infty_0$ and $I^\infty_{B_2} = I^\infty_B$. If $f \in H^\infty(m)$ vanishes on a set of positive measure, then by Jensen's inequality, $f \in H^\infty_0$. For any nonzero function f in $H^\infty_0(m)$, χ_f is minimal for $H^\infty(m)$.

As a corollary of Corollary 3, Muhly's theorem [3] follows.

COROLLARY 4. (Muhly) *The following properties for $H^\infty(m)$ are equivalent.*

(1) *No nonzero function in* $H^\infty(m)$ *can vanish on a set of positive measure.*

(2) $H^\infty(m)$ *is an integral domain.*

(3) $H^\infty(m)$ *is a maximal weak-*closed subalgebra of* $L^\infty(m)$, i.e. $H^\infty_{\min} = L^\infty(m)$.

Proof. Apply Corollary 3 with $B^\infty = L^\infty(m)$ remarking $I^\infty_B = \{0\}$.

We can show the next result which was shown by the author [**5**, Theorem 1] as a slight modification of Hoffman [**2**, p. 194].

COROLLARY 5. *Suppose* $H^\infty_0 = ZH^\infty(m)$ *for some inner function* Z *in* $H^\infty(m)$ *and let* B^∞ *be the weak-*closure of* $\bigcup_{n=0}^\infty \bar{Z}H^\infty(m)$. *Then* B^∞ *is the minimum of all weak-*closed superalgebras of* A *which contains* $H^\infty(m)$ *properly*, i.e. $B^\infty = H^\infty_{\min}$ ($\neq H^\infty(m)$).

Proof. By Theorem 5 of [**6**] and the proof of Corollary 3 of [**6**], it follows that $H^\infty(m) = \mathscr{H}^\infty \oplus I^\infty_B$ where $\mathscr{H}^\infty$ is the weak-*closure of polynomials of Z. By Jensen's inequality and $Z\mathscr{H}^\infty = \left\{f \in \mathscr{H}^\infty; \int_X fdm = 0\right\}$, it follows that if $g \in H^\infty(m)$ and $g \in I^\infty_B$, then $\log|g| \in L^1(m)$ and hence $|g| > 0$ a.e. Apply Corollary 3.

If $H^\infty(m)$ is an integral domain, then $H^\infty(m) = H^\infty_{\max} \subseteq H^\infty_{\min} = L^\infty(m)$. If $H^\infty(m)$ is not an integral domain, then $H^\infty(m) \subseteq H^\infty_{\min} \subseteq H^\infty_{\max} \subseteq L^\infty(m)$. We are interested in case $H^\infty(m)$ is not an integral domain. If $H^\infty_0 = ZH^\infty(m)$ for some inner function Z, then $H^\infty(m) \neq H^\infty_{\min}$ by Corollary 5. In general, $H^\infty_{\min}$ may coincide with or be different from $H^\infty(m)$. In the second example in §6 $H^\infty_{\min}$ coincides with $H^\infty(m)$. In general, $H^\infty_{\max}$ may coincide with or be different from $L^\infty(m)$. In the first example in §6 $H^\infty_{\max}$ coincides with $L^\infty(m)$. In general, $H^\infty_{\min}$ may coincide with or be different from $H^\infty_{\max}$.

Since $H^\infty(m)$ has no nonconstant real-valued function, $H^\infty(m)$ has not a subspace reducing $L^\infty(m)$, i.e. the essential function of $H^\infty(m)$ is constant. But when $H^\infty(m)$ is not an integral domain, it is not clear whether $H^\infty_{\min}$ has a subspace reducing $L^\infty(m)$. For in case which $H^\infty_{\min} \neq H^\infty(m)$, $H^\infty_{\min}$ has nonconstant real-valued functions. Many natural examples show that $H^\infty_{\min}$ has no subspace reducing $L^\infty(m)$. The third example in §6 shows that in general $H^\infty_{\min}$ need not have a subspace reducing $L^\infty(m)$.

6. Examples. First example. Let A be the algebra of continuous complex-valued functions on the infinite torus T^∞, the countable product of circles, which are uniform limits of polynomials in $z_1^{\ell_1} z_2^{\ell_2} \cdots z_n^{\ell_n}$ where $(\ell_1, \ell_2, \cdots, \ell_n, 0, 0, \cdots) \in \Gamma$ and Γ is the set of $(\ell_1, \ell_2, \cdots) \in Z^\infty$, the countable direct sum of the integers, whose last nonzero entry is positive, together with 0. Denote by m the normalized Haar measure on T^∞, then A is the weak-*Dirichlet algebra of $L^\infty(m)$.

We shall show that $H^\infty_{\max} = L^\infty(m)$. Let B^∞_n be the weak-*closure of $\bigcup_{i=0}^\infty \bar{z}_n^i H^\infty(m)$. Then

$$H^\infty(m) \subsetneqq B^\infty_1 \subsetneqq B^\infty_2 \cdots \subsetneqq B^\infty_n \cdots \subseteqq L^\infty(m).$$

It is sufficient to show that $H^\infty_{\max}$ contains $\bar{z}_n$ for any n. Let $\mathscr{L}^\infty_{B_n}$ be the self-adjoint part of B^∞_n, then we can show that there exists f in $H^\infty(m)$ such that $\chi_f = \chi_E$ for every χ_E in $\mathscr{L}^\infty_{B_n}$ and $\mathscr{L}^\infty_{B_n}$ is generated by characteristic functions in $\mathscr{L}^\infty_{B_n}$. Since $\chi_f \in H^\infty_{\max}$ for every f in $H^\infty(m)$, $H^\infty_{\max}$ contains $\mathscr{L}^\infty_{B_n}$ and hence contains $\bar{z}_n$. Thus $H^\infty_{\max} = L^\infty(m)$.

Second example. Let A be the algebra of continuous complex-valued functions on the infinite torus T^∞ which are uniform limits of polynomials in $z_1^{\ell_1}, z_2^{\ell_2} \cdots z_n^{\ell_n}$ where $(\ell_1, \ell_2, \cdots, \ell_n, 0, 0, \cdots) \in \Gamma$ and Γ is the set of $(\ell_1, \ell_2, \cdots) \in Z^\infty$ whose first non-zero entry is positive, together with 0. Denote by m the normalized Haar measure on T^∞, then A is the weak-*Dirichlet algebra of $L^\infty(m)$.

We shall show that $H^\infty_{\min} = H^\infty(m)$. Let B^∞_n be the weak-*closure of $\bigcup_{i=0}^\infty \bar{z}_n^i H^\infty(m)$, then

$$L^\infty(m) = B^\infty_1 \supsetneqq H^\infty_{\max} = B^\infty_2 \supsetneqq B^\infty_3 \supsetneqq \cdots H^\infty(m).$$

It is easy to show that $\bigcap_{n=1}^\infty B^\infty_n = H^\infty(m)$.

Third example. Let $\mathscr{A}$ be the σ-algebra of all Borel sets on the torus T^2. Let $\mathscr{A}_0$ be the σ-subalgebra of $\mathscr{A}$ consisting of Borel sets of the form $E_1 \times T$ where E_1 is a Borel set on the circle T. Suppose $\mathscr{B}$ be the σ-subalgebra which consists of all Borel sets such that $\{(E_0^c \times T) \cap F; F \in \mathscr{A}_0\} \cup \{(E_0 \times T) \cap F'; F' \in \mathscr{A}\}$ for some fixed Borel set E_0 on T such that $\theta(E_0) < 1$, where θ is the normalized Haar measure on T.

Denote by m the normalized Haar measure on T^2 and denote by m_0 the restriction to $\mathscr{B}$. Let A be the algebra of complex-valued Borel function on T^2 which are polynomials in $z^n q^m$ where

$$(n, m) \in \Gamma = \{(n, m); m > 0\} \cup \{(n, m); n \geqq 0\}$$

and $q = \chi_{E_0 \times T} \cdot w$ and both z and w are coordinate functions on T^2. Then A is a weak-*Dirichlet algebra of $L^\infty(m_0)$. For it is clear that m_0 is multiplicative on A. To show that $A + \bar{A}$ is weak-*dense in $L^\infty(m_0)$ it is sufficient to show that the characteristic functions for the Borel sets of T^2 of the form of $(E_1 \times T) \cup \{(E_0 \times T) \cap F\}$, where F is any

Borel set of T^2, are in the weak-*closure of $A + \bar{A}$. However it is not difficult to show this.

By Corollary 5, the minimal superalgebra $H^\infty_{\min} = H^\infty_{\max}$ is a weak-*closure of $\bigcup_{n=0}^\infty \bar{z}^n H^\infty(m_0)$ which contains $H^\infty(m_0)$ properly. Then $I^\infty_{H_{\min}}$ is $\bigcap_{n=0}^\infty z^n H^\infty(m_0)$ and the support set of $I^\infty_{H_{\min}}$ is $E_0 \times T$. Since $H^2_{\min} \oplus \bar{I}^2_{H_{\min}} = L^2(m_0)$ by Lemma 1, $H^\infty_{\min}$ has a subspace reducing $L^\infty(m_0)$. For $q = \chi_{E_0 \times T} \cdot w$ in $H^\infty(m_0)$, χ_q satisfies that if $\chi_q \lneqq \chi_f$ for $f \in H^\infty(m_0)$, then $\chi_f = 1$, a.e. For if $\chi_f \lneqq 1$, by Corollary 3, it follows that $f \in I^\infty_{H_{\min}}$.

Fourth example. Let A be the algebra of continuous complex-valued functions on the polydisc $T^3 = \{(z_1, z_2, z_3) \in C^3; |z_1| = |z_2| = |z_3| = 1\}$ which are uniform limit of polynomials in $z_1^{\ell_1} z_2^{\ell_2} z_3^{\ell_3}$ where

$$(\ell_1, \ell_2, \ell_3) \in \Gamma = \{(\ell_1, \ell_2, \ell_3); \ell_3 > 0\} \cup \{(\ell_1, \ell_2, 0); \ell_2 > 0\} \cup \{(\ell_1, 0, 0); \ell_1 > 0\}.$$

Denote by m the normalized Haar measure on T^3, then A is a weak-*Dirichlet algebra of $L^\infty(m)$. $H^\infty_{\min}$ is the weak-*closure of $\bigcup_{n=0}^\infty \bar{z}_1^n H^\infty(m)$. $H^\infty_{\max}$ is the weak-*closure of $\bigcup_{n=0}^\infty \bar{z}_2^n H^\infty(m)$. Theorem 3 can be applied each weak-*closed superalgebra B^∞ such that $H^\infty_{\min} \subseteq B^\infty \subseteq H^\infty_{\max}$ has form $B^\infty = \chi_E H^\infty_{\min} + (1 - \chi_E) H^\infty_{\max}$ for some $\chi_E \in H^\infty_{\min}$. For it is sufficient to show that if $f \in I^\infty_{H_{\min}}$ and χ_f is minimal for $H^\infty_{\min}$, then $f \in I^\infty_{H_{\max}}$. By [**6**, Theorem 4], $H^\infty(m) = H^\infty(m) \cap \bar{H}^\infty_{\max} \oplus I^\infty_{H_{\max}}$ and hence if $f \in I^\infty_{H_{\min}}$, then $f = u + f_0$ for some $u \in H^\infty(m) \cap \bar{H}^\infty_{\max}$ and for some $f_0 \in I^\infty_{H_{\max}}$. It is not difficult to show that if $u \neq 0$, then χ_f is not minimal for $H^\infty_{\min}$. Moreover $H^\infty_{\max} = (H^\infty_{\min})_{\min}$.

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