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REPRESENTING CODIMENSION-ONE HOMOLOGY CLASSES BY EMBEDDED SUBMANIFOLDS

WILLIAM H. MEEKS III AND JULIE PATRUSKY

In this paper we prove a general theorem on representing codimension-one homology classes on compact manifolds. Our theorem gives an elementary proof of the classical result that a one-dimensional homology class on a compact orientable surface can be represented by an embedded circle precisely when the class is primitive. We will call a homology class primitive if it is the zero class or if it is not a nontrivial multiple of another class.

Our representation theorem is motivated by our earlier work [2] on the classical situation. For compact orientable surfaces we developed a simple algorithm for representing primitive homology classes by embedded circles. This algorithm is also useful for proving other results related to two and three dimensional topology which are not implied by the general result here. Recently Mark Meyerson [3] has given another proof of the two dimensional case by applying Lickorish "twist" homeomorphisms to a fixed homology class.

The following theorem deals with the decomposition of a n-dimensional manifold M^n by an embedded submanifold N^{n-1} which represents a homology class $\delta \in H_{n-1}(M, Z)$. We will say that such a representation N is minimal if there is no other representation N' of δ having fewer path components.

REPRESENTATION THEOREM. *Suppose M is a compact orientable piecewise linear n-dimensional manifold and $\gamma \in H_{n-1}(M, Z)$ is a primitive nonzero class.*

1. *The class $\delta = \kappa\gamma$ has a minimal representation by a submanifold.*
2. *If N is a minimal representation for $\delta = \kappa\gamma$ then each path component of $M - N$ has two ends and the number of path components of $N = |N|$ is κ.*

Proof. The Poincaré dual to $\delta = \kappa\gamma$ can be represented by a piecewise linear mapping $P(\delta): M \to S^1$. The standard duality theorems imply that wherever $r \in S^1$ is a regular value for $P(\delta)$, then $P^{-1}(r) \subset M$ is an embedded submanifold representing δ. Hence any $\delta \in H_{n-1}(M, Z)$ has a minimal representative.

Suppose now that N is an oriented path-connected submanifold of M with $[N] \neq 0 \in H_{n-1}(M, Z)$. Since N is path connected and nontrivial on homology then $M - N$ is path connected. Hence there is an embedded circle $\sigma \subset M$ with $[\sigma] \cap [N] = +1$, where $\cap$ is the intersection pairing on homology. This implies that $[N]$ is primitive.

DEFINITION. If $N \subset M$ is an embedded submanifold, and U is a path component of $M - N$, then T = *end closure* of U is formed by placing a compact boundary on each end of U.

Let N be an embedded submanifold representing $\delta \in H_{n-1}(M, Z)$. Suppose T is the end closure of an oriented path component of $M - N$ with at least three ends, E_1, E_2, E_3, coming from cuts along distinct path components N_1, N_2, N_3 of N. Orient the end E_i as the respective component N_i is oriented. Let γ_1 be a path joining $p_1 \in E_1$ to $p_2 \in E_2$, and let γ_2 be a path joining $p_2 \in E_2$ to $p_3 \in E_3$ with $\gamma_1 \cap \gamma_2 = \{p_2\}$. If $\operatorname{sgn}(\gamma_1 \cap E_1) \neq \operatorname{sgn}(\gamma_1 \cap E_2)$, then we can take the connected sum of N_1 and N_2 in M along γ_1, and similarly, if $\operatorname{sgn}(\gamma_2 \cap E_2) \neq \operatorname{sgn}(\gamma_2 \cap E_3)$, we can connect N_2 and N_3 in M along γ_2. If neither of these cases hold, form the composite path $\gamma_3 = \gamma_1 \circ \gamma_2^{-1}$. Clearly, $\operatorname{sgn}(\gamma_3 \cap E_1) \neq \operatorname{sgn}(\gamma_3 \cap E_3)$. Since the normal bundle to E_2 is trivial, we may push γ_3 off of E_2 and take the connected sum of N_1 and N_3 in M along this variation of γ_3.

A slight change in the above argument shows that the condition that E_1, E_2, E_3 come from cuts along distinct path components of N is not needed. Hence, if N is a minimal representative of δ, then each path component of $M - N$ has two ends. This implies that $[N] = |N| \cdot [N_1]$, where N_1 is a path component of N. Since N_1 is primitive, the theorem is proved.

REFERENCES

1. D. R. J. Chillingworth, *Winding numbers on surfaces*, II., Math. Ann., **199** (1972), 142–153.

2. W. Meeks and J. Patrusky, *Representing homology classes by embedded circles on a compact surface.* To be published in Illinois J. Math.

3. M. D. Meyerson, *Representing homology classes of closed orientable surfaces.* To be published in Proc. A.M.S.

4. B. L. Reinhart, *Algorithms for Jordan curves on compact surfaces*, Ann. Math., **75** (1962), 209–222.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES
AND
UNIVERSITY OF CALIFORNIA, BERKELEY