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CONTINUITY AND COMPREHENSION IN INTUITIONISTIC FORMAL SYSTEMS

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Two questions which are of fundamental importance in the foundations of constructive mathematics are

(1) Are all extensional functions (say from N^N to N) continuous?

(2) What general principles for defining sets (or species) are constructively justifiable?

This paper is concerned with metamathematical results related to these questions.

Within the framework of a formal system, we can ask if one can find any necessary relations between the answers to the two questions posed above. We show that, within the language of second-order arithmetic, one cannot find any such relations; even if one includes Church's thesis, which says that every constructive function is recursive. In earlier work, we have proved independence results related to question (1) in the context of the language of arithmetic. The main tool of the present paper is an extension of our earlier methods to second-order comprehension principles.

It is fairly easy to prove the consistency of strong principles of set existence with the continuity of extensional functions, even in the presence of Church's thesis (see discussion in **[3]**). And, as mentioned, the case where one does not have strong set existence principles has been dealt with in **[1]**. The main problem, then, is the independence of the continuity of extensional functions from strong set existence principles. Of course, if all the formal axioms considered are classically true, this independence is trivial; but we are interested in the independence in an axiomatic framework including nonclassical principles. Foremost among such principles is Church's thesis, which (in a suitable formulation) will reduce members of N^N to recursive indices, and functions from N^N to N to effective operations, which compute the function value recursively from an index of the argument. Thus, under Church's thesis CT, the statement "all functions are continuous" reduces to an arithmetical proposition about effective operations. This proposition (for the case of N^N) is called KLS, after Kreisel, Lacombe, and Shoenfield, who gave a classical proof of it **[5]**. It also happens that this sentence KLS lies in a syntactic class for which CT is conservative (over all the theories we will consider; see discussion in the text). Thus the difficult part of our problem is to prove the independence of KLS from various principles of set existence.

In [1], we prove the independence of KLS from various theories formulated in the language of arithmetic HA (and thus having no set-existence principles). In order to extend our results to extensions of HA with set-existence principles, we had to take a circuitous route through the deepest part of proof theory. Namely, we were able to extend our independence proofs to those theories which admit "proof-theoretic treatments" (cut-elimination theorems) in a certain precise sense. (These theories are conservative over HA + transfinite induction on all recursive well-orderings.) The precise definition of "proof-theoretic treatment" requires formalization of the usual results, and, for instance for HAS, the author has not yet seen such a treatment carried out in detail, though he has recently heard that Leivant has done so. In this paper, we take a more direct approach.

The independence proofs in [1] proceed by introducing a "realizability" interpretation of HA called fp-realizability (for "formal-provable" realizability), under which KLS fails to be realized. Until now, we have been unable to extend this realizability to other languages. In the present paper, we show how to extend it to HAS. In addition to its use for independence proofs, this extension is of interest for its own sake, as a contribution to the general program (see [6]) of studying intuitionistic formal systems by considering what interpretations, other than the intended one, they allow.

The system HAS is of particular interest from a formal standpoint (regardless of whether one believes it to be a constructive theory) because it is "maximal" in the sense that other known constructive theories can be interpreted in HAS. In particular, this applies to the system EM (for "explicit mathematics") recently developed by Feferman [F] and its variants. The system EM and its variants were introduced in a (largely successful, we think) attempt to provide a simple formal system meeting two requirements:

(A) The primitive notions of *EM* are quite close to the fundamental notions of informal constructive mathematics.

(B) The mathematical practice of the new school of constructivists, especially as represented by Bishop's book [4], can be formalized in *EM* by a simple process of transcription.

In [3] we have made a study of some metamathematical questions concerning continuity and principles of continuous choice; the result that KLS is independent of $EM + CT + CA$ given in [3] depends on the interpretation of EM in HAS. See the last part of this paper for discussion, and [3] for details.

It should be pointed out that, given a formula X which defines a (provably) complete separable metric space, we can consider effective operations from X to the reals R and formulate the principle of continuity $\mathrm{KLS}(X, R)$. (For details see [3]). The independence of

KLS(R, R) in various arithmetic theories was proved in **[2]**. The proof carries over easily to the case of KLS(X, R), as soon as one discusses how to formalize complete separable metric spaces, as is done in **[3]**. Thus the independence results of this paper apply equally to KLS(X, R) as to KLS.

1. Extension of fp-realizability to HAS. The theory HAS contains variables for integers and for sets (species) of integers. Induction is extended to the new language, and the comprehension axiom is included as a schema, for all formulae B,

$$\text{(CA)} \qquad \exists X \forall n (n \in X \leftrightarrow B(n, y, Z))$$

For a more detailed description of HAS, see **[6]**. The letters HAS stand for, Heyting's arithmetic with species. Whether or not one wishes to admit HAS, with its impredicative comprehension axiom, as a constructively valid theory, it remains of technical interest because every known constructively valid theory can be interpreted in it. In this section we show how to extend the definition of fp-realizability given in **[1]** for HA to HAS, thus giving a direct proof of the independence results of **[1]** and **[2]**. Familiarity with the definition just mentioned will be assumed; however, none of the other work in **[1]** and **[2]** is a prerequisite to this paper.

In arithmetic, we can write Pr("$A(\bar{y})$") to assert that $A(\bar{y})$ is provable. Here $\bar{y}$ is the numeral for y; thus $A(\bar{y})$ has no free variables; nevertheless Pr("$A(\bar{y})$") has free variable y. We are enabled to do this by the existence of a function Num(y) which produces from y the term $\bar{y}$. This "naming" function has no analog for species variables. There is no way to write Pr('$A(X)$') to express that $A(X)$ is provable for a particular species X; and the capability to do this seems essential if fp-realizability is to be properly defined. We surmount this difficulty by adding a new predicate Pr to HAS, with the intended interpretation that Pr('A', y, X) means that $A(y, X)$ is "provable" (in some sense) for this specific y, X. We then axiomatize the properties of Pr needed to make fp-realizability work; this auxiliary theory we call HAS$^+$. The trick is to carry out this axiomatization in such a way that HAS$^+$ has a model in which Pr can be interpreted as a formal provability predicate, to make the new fp-realizability coincide with the old. It is important that the auxiliary theory HAS$^+$ contain the full comprehension schema for formulae involving the new symbol Pr, because we need to form $\{x: B(x)$ is realized$\}$ in order to get CA realized. We now proceed to carry out these steps.

First, some technical details. Throughout the paper, we use X, Y, Z as abbreviations for $X_1, \cdots, X_n$, etc., where possibly $n = 0$. We shall need a pairing function to pair two species X and Y into a single species

$\langle X, Y\rangle$; it will be convenient to assume HAS to be augmented by function symbols for such a pairing function and its un-pairing functions.

We now describe an auxiliary theory HAS$^+$, which contains all the apparatus of HAS, plus a predicate symbol Pr, which takes two number and one species argument, so we can form expressions like Pr('A', y, X). This is to be thought of as saying $A(y, X)$ is provable, for these specific parameters y and X. There are a couple of technical details, in making this precise—first, A can have any number of free variables, not just y and X, and Pr has to have only a fixed number of arguments. Second, we may wish to substitute parameters y and X for only some of the free variables of A. To solve this minor problem, we Gödel number the formulae of HAS$^+$ as formulae of k free variables, of which j are parameters. Then we write Pr('A', y, X), we mean that $A(y_1, \cdots, y_m, X_1, \cdots, X_n)$ is provable, where the integers m and n are read from the Gödel number 'A' and $y = \langle y_1, \cdots, y_m\rangle$ and $X = \langle X_1, \cdots, X_n\rangle$. In case A has no parameters, Pr('A', y, X) shall mean A is provable. Note that Pr is intended to apply to all A of HAS$^+$, including A mentioning Pr.

When no confusion should result, we will abbreviate Pr('B', y, X) by Pr('$B(y, X)$'). If we think there is danger of confusion with the free variables of B which are not considered as parameters, we revert to the more formal notation. In case B has no parameters, we write Pr('B') for Pr('B', y, X). These conventions are especially convenient if, for instance, B has two set parameters and no number parameters; we can write Pr('B', X, W) instead of Pr('B', $y, \langle X, W\rangle$) where y is an irrelevant number variable.

Let us Gödel number finite sets Γ of formulae, and write Pr('Γ', y, X) as an abbreviation for the conjunction of all Pr('A', y, X) over A in Γ. Let $\mathrm{Pr}_{\mathrm{PC}}$('$\Gamma$', '$A$') be a natural formalization of, "Γ proves A by means of Heyting's predicate calculus."

We now can give the axioms and rules of inference of the system HAS$^+$. They are:

(i) $\mathrm{Pr}_{\mathrm{PC}}$('$\Gamma$', '$A$') & Pr('$\Gamma$', y, X) $\rightarrow$ Pr('A', y, X)

(ii) From B infer Pr('B')

(iii) From $A \rightarrow B$ infer Pr('$A(y, X)$') $\rightarrow$ Pr('$B(y, X)$')

(iv) Pr('B', y, X) $\rightarrow$ Pr('Pr('B', y, X)', y, X)

(v) the axioms and rules of HA, plus induction extended to the language of HAS$^+$, plus comprehension in the form $\exists W(Q(W, y, Z)$ & Pr('$Q(W, y, X)$')) where $Q(W, y, Z)$ is $\forall n(n \in W \leftrightarrow B(n, y, Z))$, and B is any formula of HAS$^+$.

Note that HAS$^+$ is trivially consistent, since $(\omega, \mathcal{P}(\omega))$ can be made into a model of HAS$^+$ by interpreting Pr to be universally valid (as if it were the proof-predicate of an inconsistent theory). We shall need a much more informative model, however.

We introduce the abbreviation B^P for $B(y, X)$ & Pr('B', y, X), where y and X are all the free variables of B. Thus B^P has the same free variables as B, and expresses that B is "true and provable".

If A is a formula of HAS$^+$, and ϕ is a formula of HAS$^+$ with no free species variables and exactly one free numerical variable, we write $A(\phi)$ for

$$\forall Z((\forall n(n \in Z \leftrightarrow \phi(n)))^P \rightarrow A(Z)).$$

Thus if A doesn't contain Pr, $A(\phi)$ is provably equivalent to the result of replacing $n \in Z$ in A by $\phi(n)$.

We now describe a model $\mathcal{M}$ for HAS$^+$. The integers of $\mathcal{M}$ are standard. The sets of $\mathcal{M}$ are all pairs (X, ϕ) such that X is a subset of ω and ϕ is a formula of HAS$^+$ with one free numerical variable and no free species variables. The interpretation of Pr is the predicate *Pr* defined by,

$$Pr(\text{`}A\text{'}, y, (X, \phi)) \text{ iff } \mathrm{HAS}^+ \vdash A(y, \phi)$$

(making the natural conventions of notation in case A has more than one species variable.)

LEMMA 1. *$\mathcal{M}$ is a model of* HAS$^+$.

Proof. We check the axioms and rules (i) through (v), verifying by induction on the length of proof that every theorem of HAS$^+$ is true in $\mathcal{M}$.

(i) Suppose $\mathcal{M} \models \mathrm{Pr}_{\mathrm{PC}}(\text{`}\Gamma\text{'}, \text{`}A\text{'})$ & $\mathrm{Pr}(\text{`}\Gamma((X, \phi))\text{'})$. Since the integers of $\mathcal{M}$ are standard, in fact $\Gamma$ proves $A$ by means of predicate calculus. Also, for each $B$ in $\Gamma$, we have $Pr(\text{`}B((X, \phi))\text{'})$, that is, $\mathrm{HAS}^+ \vdash B(\phi)$. One sees easily that $\mathrm{HAS}^+ \vdash A(\phi)$. Hence $Pr(\text{`}A((X, \phi))\text{'})$.

(ii) Suppose $\mathrm{HAS}^+ \vdash B$ and $\mathcal{M} \models B$. We have to show $\mathcal{M} \models \mathrm{Pr}(\text{`}B\text{'})$. Since $B$ has no parameters, $Pr(\text{`}B\text{'})$ iff $\mathrm{HAS}^+ \vdash B$, so we are done with (ii).

(iii) Suppose $\mathrm{HAS}^+ \vdash A \rightarrow B$, and $Pr(\text{`}A((X, \phi))\text{'})$; we want to show $Pr(\text{`}B((X, \phi))\text{'})$; we have $\mathrm{HAS}^+ \vdash A(\phi)$; one checks easily that then $\mathrm{HAS}^+ \vdash B(\phi)$, hence $\mathrm{Pr}(\text{`}B((X, \phi))\text{'})$.

(iv) Note first that if R is any formula of HAS$^+$, then $\mathrm{HAS}^+ \vdash R^P(Z) \rightarrow \mathrm{Pr}(\text{`}R^P(Z)\text{'})$, by axioms (i) and (iv). We have to verify $\mathcal{M} \models \mathrm{Pr}(\text{`}B(X)\text{'}) \rightarrow \mathrm{Pr}(\text{`}\mathrm{Pr}(\text{`}B(X)\text{'})\text{'})$. (Any possible ambiguity in the abbreviated notation is cleared up in the precise statement of axiom (iv).) Suppose $Pr(\text{`}B((X, \phi))\text{'})$. Let $R(Z)$ be

$$\forall n(n \in Z \leftrightarrow \phi(n)).$$

Then we have, from the definition of *Pr*,

$$(*) \qquad \mathrm{HAS}^{+} \vdash R^{P}(Z) \rightarrow B(Z)$$

What we want is, $Pr(\text{`Pr(`}B((X,\phi))\text{')')}$; that is, we want $\mathrm{HAS}^{+} \vdash R^{P}(Z) \rightarrow \mathrm{Pr}(\text{`}B(Z)\text{'})$. To get it, argue in $\mathrm{HAS}^{+}$ as follows: Assume $R^{P}(Z)$; then $\mathrm{Pr}(\text{`}R^{P}(Z)\text{'})$ as noted above; so by axiom (iii) applied to (*), we get $\mathrm{Pr}(\text{`}B(Z)\text{'})$, still within $\mathrm{HAS}^{+}$. This shows $\mathrm{HAS}^{+} \vdash R^{P}(Z) \rightarrow \mathrm{Pr}(\text{`}B(Z)\text{'})$, i.e. $Pr(\text{`Pr(`}B((X,\phi))\text{')')}$, verifying (iv).

(v) The induction and number-theoretical axioms hold because the integers of $\mathcal{M}$ are standard.

We now verify comprehension. Let B be given, and fix (Z,ϕ) in $\mathcal{M}$, and fix y. For simplicity we suppress mention of y. Take $W = \{k : \mathcal{M} \models B(k,(Z,\phi))\}$ and take ψ to be the formula $B(k,\phi)$ with free variable k. We will show that $\mathcal{M} \models Q((W,\psi),(Z,\phi))$ and $\mathcal{M} \models \mathrm{Pr}(\text{`}Q((W,\psi),(Z,\phi))\text{'})$ where $Q(X,Z)$ is $\forall k(k \in X \leftrightarrow B(k,Z))$. (This will verify the comprehension axiom of HAS^{+} in $\mathcal{M}$.) First, $\mathcal{M} \models Q((W,\psi),(Z,\phi))$ since

$$\mathcal{M} \models k \in (W,\psi) \text{ iff } k \in W \text{ iff } \mathcal{M} \models B(k,(Z,\phi)).$$

Second, observe that $Q(\psi,\phi)$ is

$$\forall X \forall Z((\forall k(k \in X \leftrightarrow B(k,\phi)))^{P} \;\&\; (\forall k(k \in Z \leftrightarrow \phi(k)))^{P} \\ \rightarrow \forall k(k \in X \leftrightarrow B(k,Z)))$$

which follows in HAS^{+} from

$$\forall Z((\forall k(k \in Z \leftrightarrow \phi(k)))^{P} \rightarrow (B(k,\phi) \leftrightarrow B(k,Z))),$$

which is equivalent to

$$\forall Z \forall U((\forall k(k \in Z \leftrightarrow \phi(k)))^{P} \;\& \\ (\forall k(k \in U \leftrightarrow \phi(k)))^{P} \rightarrow (B(Z) \leftrightarrow B(U))).$$

This latter follows in HAS^{+} from $\forall Z \forall U(\forall k(k \in Z \leftrightarrow k \in U) \rightarrow (B(Z) \leftrightarrow B(U)))$. This, however, is a theorem of HAS^{+}, for each fixed B, as is easily seen by induction on the complexity of B. We have therefore shown that $\mathrm{HAS}^{+} \vdash Q(\psi,\phi)$; hence $Pr(\text{`}Q((W,\psi),(Z,\phi))\text{'})$, verifying axiom (v) and completing the proof of Lemma 1.

We are now in a position to define fp-realizability for HAS. The definition will associate to each formula A of HAS another formula A^{R}, with the same free variables. A^{R} will be a formula of HAS^{+}. $A^{R}(y,Z)$

is read, "$A(y, Z)$ is fp-realized." The use of the abbreviation B^P introduced above, in case B is A^R, leads to the notation $A^{RP}(y, Z)$, which is read "$A(y, Z)$ is fp-realized, and provably so." The definition is given by the following inductive clauses, which are exactly the same as for HA except for the added clauses dealing with species quantification, and of course the new interpretation of Pr. The hard (new) work has been done in the construction and modeling of HAS$^+$.

$$A^R \text{ is } A \text{ for } A \text{ prime}$$
$$(A \,\&\, B)^R \text{ is } A^R \,\&\, B^R$$
$$(A \vee B)^R \text{ is } A^{RP} \vee B^{RP}$$
$$(A \to B)^R \text{ is } A^{RP} \to B^R$$
$$(\neg A)^R \text{ is } (A \to 0 = 1)^R$$
$$(\exists x A)^R \text{ is } \exists x A^{RP}\text{; similarly for } (\exists X A)^R$$
$$(\forall x A)^R \text{ is } \forall x A^R\text{; similarly for } (\forall X A)^R$$

Before proceeding to the soundness proof for fp-realizability, we need a few lemmas.

LEMMA 2. $\mathrm{HAS}^+ \vdash A^{RP}(y, Y) \to \mathrm{Pr}(`A^{RP}(y, Y)\text{'})$

Proof. Argue in HAS$^+$ as follows: Suppose $A^{RP}(y, Y)$, i.e. $A^R(y, Y)$ & $\mathrm{Pr}(`A^R(y, Y)\text{'})$. By axiom (iv), $\mathrm{Pr}(`\mathrm{Pr}(`A^R(y, Y)\text{'})\text{'})$. Then by axiom (i), $\mathrm{Pr}(`A^{RP}(y, Y)\text{'})$.

LEMMA 3. *Let A be a formula with free variables $x = x_1, \cdots, x_n$, some of which may be species variables. Let t be a term in (possibly) x_1 (which may be a species or number variable). Then* $\mathrm{HAS}^+ \vdash A^R(t, x_2, \cdots, x_n) \leftrightarrow (At)^R(x_2, \cdots, x_n)$; *and*

$$\mathrm{HAS}^+ \vdash A^{RP}(t, x_2, \cdots, x_n) \leftrightarrow (At)^{RP}(x_2, \cdots, x_n).$$

Proof. The two assertions can be proved simultaneously by induction on the complexity of A. If A is prime there is nothing to prove. We leave the reader to use the induction hypothesis to prove that $\mathrm{HAS}^+ \vdash A^R(t) \leftrightarrow (At)^R$ (suppressing mention of $x_2, \cdots, x_n$). Then, applying axiom (iii) of HAS$^+$, we conclude that $\mathrm{HAS}^+ \vdash A^{RP}(t) \leftrightarrow (At)^{RP}$.

LEMMA 4. $\mathrm{HAS}^+ \vdash A^{RP} \,\&\, (A \to B)^{RP} \to B^{RP}$

Proof. Let y, Y include the free variables of A and B. Argue in HAS$^+$ as follows: Suppose A^{RP} & $(A \to B)^{RP}$. By Lemma 2, we have

Pr('$A^{RP}(y, Y)$'). Since $(A \to B)^R$ is $A^{RP} \to B^R$, and we have $(A \to B)^{RP}$, we have Pr('$A^{RP}(y, Y) \to B^R(y, Y)$'). Combining this with Pr('$A^{RP}(y, Y)$'), and applying axiom (i), we obtain Pr('$B^R(\bar{y}, Y)$'). Combining this with B^R, we have B^{RP}.

LEMMA 5. *Let B have only x free. Then* $\mathrm{HAS}^+ \vdash (\forall x B)^{RP} \to \forall x B^{RP}$. *Similarly if B has only Y free.*

Proof. Argue in HAS^+. $(\forall x B)^{RP}$ is $\forall x B^R$ & Pr('$\forall x B^R$'). $\forall x B^{RP}$ is $\forall x(B^R$ & Pr('$B^R(x)$')). Thus it suffices to show Pr('$\forall x D$') $\to \forall x$ Pr('$D(x)$'), and in the case of a species variable Y, Pr('$\forall Y D$') $\to \forall Y$ Pr('$D(Y)$'); but these are special cases of axiom (i).

THEOREM 1. (Soundness of fp-realizability for HAS). *If* $\mathrm{HAS} \vdash A$, *then* $\mathrm{HAS}^+ \vdash A^R$.

Proof. We prove that the universal closures of the axioms of HAS are provably fp-realized, and that if the universal closures of the premises of a rule of HAS are provably fp-realized, the same is true of the universal closure of the conclusion. The logical rules and axioms and the number-theoretical axioms, including induction for the extended language, are handled in exactly the same way as for arithmetical fp-realizability; the lemmas above allow the proofs in **[1]** to be read verbatim, except for changing the numbers of the lemmas referred to. This leaves only the comprehension axiom to check. Recall the form of

$$\text{(CA)} \qquad \forall y, Z \exists W \forall n (n \in W \leftrightarrow B(n, y, Z))$$

Thus, $(\mathrm{CA})^R$ is $\forall y, Z \exists W (\forall n (n \in W \leftrightarrow B(n, y, Z)))^{RP}$. Let us write $Q(W)$ for $\forall n(n \in W \leftrightarrow B^R(n, y, Z))$. Then, according to the comprehension axiom in HAS^+, we can find W such that $Q(W)$ & Pr('$Q(W)$'). We will show

$$(\forall n (n \in W \leftrightarrow B(n, y, Z)))^{RP}.$$

We have $n \in W \leftrightarrow B^R(n, y, Z)$; hence we have $(\forall n (n \in W \leftrightarrow B(n, y, Z)))^R$. Let $D(W)$ be $\forall n (n \in W \leftrightarrow B(n, y, Z))$; then we have shown $\mathrm{HAS}^+ \vdash Q(W) \to D^R(W)$, with free variable W. Since applying rule (iii) of HAS^+, we have, in HAS^+, Pr('$Q(W)$') $\to$ Pr('$D^R(W)$'). Since we have Pr('$Q(W)$'), we have Pr('$D^R(W)$'); combining this with $D^R(W)$ obtained above, we have $D^{RP}(W)$, as desired, which completes the proof of the soundness theorem.

Having completed our extension of fp-realizability to HAS, we now wish to point out the independence results which follow. As discussed in the introduction, the applications of fp-realizability are to certain statements expressing the continuity of effective operations, namely Myhill–Shepherdson's theorem MS, and various versions of Kreisel–Lacombe–Shoenfield's theorem, for instance KLS = KLS(N^N), KLS(2^N), KLS(R, R), and more generally KLS(X, R), for X a provably complete separable metric space. For the sake of completeness, we shall take the space to give these statements precisely; the reader familiar with them can skip the next paragraph.

Let TOT(y) express that y is total, i.e. $\forall x \exists n T(y, x, n)$, where T is (a formula expressing) Kleene's T-predicate. Let EXT(e) express that e is extensional, i.e.

$$\forall n, k, y, z\,(\mathrm{TOT}(y)\ \&\ \mathrm{TOT}(z)\ \&\ \mathrm{Teyn}\ \&\ \mathrm{Tezk} \rightarrow U(k) = U(n)),$$

where U is Kleene's result-extracting function. Then EO(e), which expresses that e is an effective operation, is given by

$$\forall y(\mathrm{TOT}(y) \rightarrow \exists k\ \mathrm{Tey}k)\ \&\ \mathrm{EXT}(e).$$

Let CONT(e) express the continuity of e:

$$\begin{aligned}\forall y(\mathrm{TOT}(y) \rightarrow \exists m \forall z(\mathrm{TOT}(z)\ \&\ &\overline{\{z\}}(m)\\ &= \overline{\{y\}}(m) \rightarrow \{e\}(y) = \{e\}(z)),\end{aligned}$$

where as usual $\overline{\{z\}}(m)$ denotes the sequence of the first m values of z. Then KLS is $\forall e(\mathrm{EO}(e) \rightarrow \mathrm{CONT}(e))$. Some variants of KLS are discussed in **[1]**. KLS(2^N) is similar to KLS, except that in KLS(2^N), TOT(y) is replaced by $\forall x \exists m(\mathrm{Tyxm}\ \&\ U(m) \leqq 1)$. KLS($R$, R), or more generally KLS(X, R), can be explained as follows: an element of X is a (recursive) sequence of integers (thought of as coding a sequence of elements of the countable dense base of X), satisfying the "convergence condition" $\rho(y_n, y_m) \leqq 1/n + 1/m$, where ρ is a certain (recursive) function giving the metric on the countable base of X. For instance, in the case $X = R$, the integers are thought of as coding rationals and ρ gives the metric on the rationals. (This way of looking at the reals is worked out in detail in Bishop's book **[4]**.) The "convergence condition" becomes a precise formula CC(y) if we regard y_n as an abbreviation for $\{y\}(n-1)$ and quantify over all $n, m \geqq 1$ (thus taking care of the minor problem that the indices in y_n start from 1 while those in $\{y\}(n)$ start from 0.) Then "$y \in X$" is the formula TOT(y) & CC(y). The relation of (extensional) equality in X is given by $y \sim z$ iff

$$y \in X \ \& \ z \in X \ \& \ \forall k(\rho(y_k, z_k) \leqq 2/k).$$

We then can say $e \in R^X$ iff

$$\forall y(y \in X \rightarrow \exists k\,(\mathrm{Teyk} \,\&\, U(k) \in R) \,\&\, \forall y, z(y \sim z \rightarrow \{e\}(y) \sim \{e\}(z)),$$

where the last $\sim$ is extensional equality in R, not in X. We let $e \in \mathrm{CONT}(X, R)$ be the formula expressing straightforwardly that e determines a continuous function from X to R, using rational ε and δ. Then $\mathrm{KLS}(X, R)$ is $\forall e(e \in R^X \rightarrow e \in \mathrm{CONT}(X, R))$. Myhill–Shepherdson's continuity theorem MS concerns partial effective operations on partial functions; its exact formulation can be found in **[1]**.

The independence results of this paper apply not only to HAS^+ but also to HAS^+ augmented by "extended Church's thesis" ECT and the schema TI of transfinite induction on all recursive well-orderings. ECT, which is extensively studied in **[6]**, is the following principle:

$$\text{ECT: } \forall x\,(A(x) \rightarrow \exists y B(x, y) \rightarrow \exists e\, \forall x\,(A(x) \rightarrow \exists n\,(\mathrm{Texn} \ \& \ B(x, U(n))),$$
$$A \text{ almost negative.}$$

(A is called almost negative if it contains no disjunction, and has $\exists$ only immediately preceding primitive-recursive formulae.) Thus Church's thesis CT is obtained by taking A to be some trivially true sentence; it is easy to see that some restriction on A is needed. The principle $\mathrm{TI}(<)$ of transfinite induction on a recursive relation $<$ is the schema $\forall y\,(\forall z < yA(z) \rightarrow A(y)) \rightarrow \forall z A(z)$. The schema TI is $\mathrm{TI}(<)$ taken for all provably linear recursive orderings $<$ which determine actual well-orderings. (Thus of course TI is not recursively axiomatizable.) One reason TI is interesting is that, when added to classical arithmetic, it proves all classically true arithmetic theorems; this theorem of Kreisel, Shoenfield, and Wang is discussed in **[1]**.

THEOREM 2. KLS, $\mathrm{KLS}(2^N)$, $\mathrm{KLS}(R, R)$, $\mathrm{KLS}(X, R)$, *and* MS *are all underivable in* $\mathrm{HAS} + \mathrm{ECT} + \mathrm{TI}$.

Proof. First we show the underivability in HAS. Let T be (a natural theory giving) the arithmetical consequences of HAS^+. Use the provability predicate of T in the definition of fp-realizability given for arithmetic in **[1]**; let the resulting notion of realizability be written A^r (as distinct from A^R). We claim that for arithmetic A, and with $\mathcal{M}$ as in Lemma 1, $\mathcal{M} \models A^R$ iff A^r is true. Suppose this for the moment; we show how to finish the proof. Since T is a true theory (by Lemma 1), the main theorem of **[1]** shows that KLS^r is false. Suppose

HAS $\vdash$ KLS. Then by Theorem 1, we would have $\mathrm{HAS}^+ \vdash \mathrm{KLS}^R$; hence, by Lemma 1, $\mathcal{M} \models \mathrm{KLS}^R$; hence KLS' is true, contradiction. Hence HAS does not prove KLS. MS and the other versions of KLS are treated similarly, appealing to other theorems of **[1]**, **[2]**, and **[3]** to know they are not fp-realized. It remains to check that $\mathcal{M} \models A^R$ iff A' is true. To do this, let S be the system formed from HAS^+ by adding the axiom schema $\mathrm{Pr}_T(`B') \leftrightarrow \mathrm{Pr}(`B')$ for all arithmetic B. Then $\mathcal{M}$ is a model of S, since $\mathcal{M} \models \mathrm{Pr}_T(`B')$ iff $T \vdash B$ iff $\mathrm{HAS}^+ \vdash B$ iff $\mathcal{M} \vdash \mathrm{Pr}(`B')$. But, $S \vdash A' \leftrightarrow A^R$ and $S \vdash A^{RP} \leftrightarrow A^{\prime P}$ (where $A^{\prime P}$ means $A' \,\&\, \mathrm{Pr}_T(`A'')$), as is easily checked by simultaneous induction on the complexity of $A$. For instance, to go from $S \vdash A' \leftrightarrow A^R$ to $S \vdash A^{\prime P} \leftrightarrow A^{RP}$, note that by axiom (iii), $S \vdash \mathrm{Pr}(`A'') \leftrightarrow \mathrm{Pr}(`A^R')$; but $A'$ is arithmetic, so $S \vdash \mathrm{Pr}_T(`A'') \leftrightarrow \mathrm{Pr}(`A^R')$; so $S \vdash A^{\prime P} \leftrightarrow A^{RP}$. Combining $S \vdash A' \leftrightarrow A^R$ and the fact that $\mathcal{M}$ is a model of S, we obtain $\mathcal{M} \models A^R$ iff $\mathcal{M} \models A'$; but since A' is arithmetic, we obtain $\mathcal{M} \models A^R$ iff A' is true, as desired.

Next we discuss the extension to TI. Suppose HAS + TI proved KLS. Then some finite number of instances of TI are involved in the proof. Let T be the theory formed from HAS by adding the schema of transfinite induction only for the finitely many recursive well-orderings involved in the proof. Form T^+ from T as HAS^+ was formed from HAS. Then all the results of this paper apply with HAS and HAS^+ replaced by T and T^+. In particular, the axioms of TI hold in $\mathcal{M}$ since the integers of $\mathcal{M}$ are standard; and $T^+ \vdash A^R$ for each instance A of TI included in T, just as in arithmetic. Hence, just as above for HAS, KLS is not derivable in T, contradiction. MS and the other versions of KLS are handled similarly.

As in arithmetic, we use a conservative extension result of Troelstra's to extend the independence results to ECT. Namely, KLS (in all versions) and MS belong to the class Γ_0 **[6**, p. 250**]** for which ECT is conservative. This is proved as on p. 252 of **[6]**, using the realizability of **[6**, p. 202–203**]** to extend the result to HAS. This completes the proof.

2. Remarks. There is an alternative approach to the extension of fp-realizability which is worth discussion. In this approach, we also form an auxiliary theory, called HAS°. This theory HAS° is quite different from HAS^+; it is based on the idea that in HAS° the structure of the universe of species should be completely determined—every species is defined by a formula of HAS°, just as every integer is given by a numeral in HA. To arrange this, we form HAS° by adding to HAS a function symbol F which takes species arguments and produces a number. The value $F(X)$ will be a term defining X (or technically a Gödel number of such a term). That is, HAS° also contains, for each formula B of (at least) HAS, a function symbol c_B and the axiom

$$(\mathrm{CA}^\circ) \qquad \forall n(n \in c_B(y,Z) \leftrightarrow B(n,y,Z))$$

Note that if B has only n free, then c_B is a 0-ary function, in other words a constant term. We add to HAS° an axiom asserting that $F(X)$ is the Gödel number of such a term, and $F(c_B) = {}`c_B\text{'}$; more generally $F(c_B(y,Z)) = {}`c_{B(\bar{y},F(Z))}\text{'}$ (the expression on the right can be rigorously explained.) Then B^P can be defined as $B(y,Z)$ & $\mathrm{Pr}(\bar{y}, F(Z))$, where Pr is a provability predicate in HAS; in other words, there is now no need to axiomatize Pr.

This approach is closer in spirit to the treatment of arithmetic; the function F plays the role for species that $\mathrm{Num}(y) = {}`\bar{y}\text{'}$ plays for integers. Unfortunately, if fp-realizability is defined in HAS°, we get soundness not for HAS but only for the arithmetical comprehension axiom. The reason for this is as follows: in verifying that CA is fp-realized, we have to be able to form $\{n: B^R(n,y,Z)\}$, in HAS°. Now B^R is generally a formula of HAS°, not of HAS. Therefore we need to extend the comprehension axiom in HAS° to formulae of HAS°. This can be consistently done, for formulae not involving species quantifiers; but if we allow species quantifiers in the comprehension axiom, HAS° will founder on a Cantor–Russell paradox: we will be able to prove the existence of $\{F(X): F(X) \notin X\}$, in other words, $\{y: \exists X(y = F(X) \;\&\; y \notin X)\}$.

Nevertheless, an analogous approach to fp-realizability does work for Feferman's system EM discussed in the introduction, since this system does not include second-order comprehension. Also, an approach analogous to that taken in the first part of this paper for HAS works for EM. Thus there are two different ways to extend fp-realizability to EM. However, we give details of neither, since EM can be quite simply interpreted in HAS. This interpretation (which is more carefully discussed in **[3]**) allows us to extend our independence results to EM.

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