CONVERGENCE THEOREMS IN BANACH ALGEBRAS

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It is shown that several convergence theorems for linear operators, usually established by Hilbert space techniques are consequences of the general relations between convergence, monotonicity and order units in ordered Banach spaces.

1. Introduction. Hilbert space techniques have been used to investigate the convergence properties of matrix and linear operator iterative methods by Reich [11], Stein [14], [15], Ostrowski [7], John [6], Householder [4], [5], Petryshyn [8], [9], [10] and de Pillis [2], among others. It is the purpose of the present note to show that these results also follow from the general properties of nonnegative operators on ordered Banach spaces with normal cone and order units. In order to cover most of the above mentioned results without excessive technical preliminaries, we shall present our technique of proof in the framework of Banach algebras.

A direct proof of the relations between convergence, monotonicity and order units, not going through generalizations of the Perron–Frobenius theory, is proposed in §2. Needed properties of the hermitian elements of complex unital Banach algebras and of B^* algebras are established in §3. Convergence properties are considered in §4, mainly in B^* algebras.

General notations and terminology are taken from Schaefer [13] with the following exceptions: an element x of an ordered vector space E with cone C is called nonnegative (resp. positive) and denoted $x \geqq 0$ (resp. $x > 0$) if $x \in C$ (resp. if x is an order unit); $A \in L(E, F)$ where E and F are ordered vector spaces is called nonnegative (resp. positive) if $x \geqq 0$ implies $Ax \geqq 0$ (if $x \geqq 0$, $x \neq 0$ implies $Ax > 0$); it is called monotone if $Ax \geqq 0$ implies $x \geqq 0$.

Specific notations and terminology relative to Banach algebras are taken from Bonsall and Duncan [1].

2. Monotonicity and convergence. Convergence properties of nonnegative operators on an ordered (real) Banach space with normal cone and order units are established in the present section.

THEOREM 2.1. *Let E be an ordered* (real) *Banach space with normal cone and order units*; *let $T \in \mathcal{L}(E)$, $T \geqq 0$; then*

$$(2.1) \qquad r(T) = \inf_{x>0} \inf_{\alpha>0} \{\alpha \mid Tx \leqq \alpha x\}$$

where $x \in E$ and $\alpha \in R$.

Proof. The relation (2.1) may be written

$$(2.2) \qquad r(T) = \inf_{x>0} \| T \|_x$$

where $\| T \|_x$ denotes the norm induced on $\mathcal{L}(E)$ by the Minkowski functional of the order interval $[-x, x]$; the latter functional is a norm on E, equivalent to the original one since the cone of E is normal (cf. Corollary 2 of Theorem 6.2, in Chap. V of Schaefer [13]). Therefore $r(T) \leqq \| T \|_x$ for any $x \in E$, $x > 0$; thus

$$(2.3) \qquad r(T) \leqq \inf_{x>0} \| T \|_x.$$

Let $\alpha = r(T) + \epsilon > r(T)$, let $y \in E$, $y > 0$ and set

$$(2.4) \qquad x = \left(I - \frac{1}{\alpha} T\right)^{-1} y = \sum_{k=0}^{\infty} \frac{1}{\alpha^k} T^k y$$

the convergence of the series being guaranteed by $r((1/\alpha)T) = (1/\alpha)r(T) < 1$; on the other hand

$$(2.5) \qquad x = y + \sum_{k=1}^{\infty} \frac{1}{\alpha^k} T^k y \geqq y > 0$$

thus $x > 0$; moreover

$$(2.6) \qquad (I - \frac{1}{\alpha} T)x = y > 0$$

thus

$$(2.7) \qquad Tx < \alpha x$$

and, therefore $\| T \|_x < \alpha$; since α may be chosen arbitrarily close to $r(T)$, we have

(2.8)
$$\inf_{x>0} \| T \|_x \leqq r(T).$$

The conclusion follows.

LEMMA 2.1.   *Let E be an ordered* (real) *vector space with order units and let $A \in L(E)$ be monotone and surjective. Then*

(2.9)
$$x > 0 \implies A^{-1}x > 0.$$

Proof.   Let $x > 0$, $y \in E$ and set $u = Ay$.   Since $x > 0$, there exists $\alpha \in R$, $\alpha > 0$ such that $\alpha u \leqq x$, thus such that $\alpha y \leqq A^{-1}x$.   Thus $A^{-1}x > 0$.

THEOREM 2.2.   *Let E be an ordered Banach space with normal cone and order units; let $T \in \mathscr{L}(E)$, $T \geqq 0$.   Then the following conditions are equivalent:*
  (1) $r(T) < 1$;
  (2) $I - T$ *is monotone and surjective*;
  (3) *there exists $x > 0$ such that $Tx < x$.*

Proof.
  (1) $\implies$ (2) is obvious since the cone of $\mathscr{L}(E)$ is closed;
  (2) $\implies$ (3): let $y \in E$, $y > 0$; by Lemma 2.1, we have $x = (I - T)^{-1}y > 0$, thus $(I - T)x = y > 0$, whence $Tx < x$;
  (3) $\implies$ (1):   by Theorem 2.1.

REMARKS.   These results generalize well known theorems of matrix iterative analysis, classically obtained as consequences of the Perron–Frobenius theory of nonnegative matrices.   For an intrinsic summary of the theory in finite dimensional spaces, we refer to Rheinboldt and Vandergraft [12].   Note that our attempt here was neither at the most general nor at the most complete possible setting but at the most direct proofs of the few general properties needed in §4.

3.   Hermitian elements in Banach algebras.

3.1.   *Hermitian elements in complex unital Banach algebras.*   Let A be a complex unital Banach algebra with unit e; by $p(a)$ we denote the functional

(3.1)
$$p(a) = \sup \{|\lambda| ; \lambda \in V(a)\} \qquad a \in A$$

where $V(a)$ is the numerical range of a in A; i.e.

(3.2)
$$V(a) = \{f(a); f \in D(e)\}$$

with

(3.3) $D(e) = \{f \in S(A'); f(e) = 1\}$

where $S(A')$ is the unit sphere in the strong dual A' of A; by Proposition 10.4 and Theorem 10.14 of Bonsall and Duncan [1], $p(a)$ is a norm on A, equivalent to the original one.

An element $a \in A$ is called hermitian if $V(a) \subset R$; the space $E = \mathrm{Her}(A)$ of hermitian elements of A is a real closed subspace of A; the set $C = \mathrm{Pos}(A)$ of the elements $a \in A$ such that $V(a) \subset R_+$ is a proper closed cone in E and we consider here E as an ordered (real) Banach space with nonnegative cone C. In particular, we consider that $\mathscr{L}(E)$ is ordered by the induced order (i.e. by the cone of the bounded nonnegative endomorphisms of E).

LEMMA 3.1. *Let A be a complex unital Banach algebra, $E = \mathrm{Her}(A)$, $a \in E$, $a \geqq 0$ and $0 \notin V(a)$; then a is an interior point of $C = \mathrm{Pos}(A)$ in E.*

Proof. Set $\lambda = \inf\{\alpha \mid \alpha \in V(a)\}$; by Proposition 10.4 of Bonsall and Duncan [1], $\lambda > 0$ and

(3.4) $V(a + b) \subset V(a) + V(b) \subset V(a) + [-\|b\|, \|b\|]$

for any $b \in E$. Thus, if $\|b\| < \lambda$, we have $V(a + b) \subset R_+$, and the proposition follows.

REMARK. When the cone of an ordered topological vector space has nonempty interior, the notions of interior point and of order unit coincide; accordingly for any $a \in \mathrm{Her}(A)$, we have $a > 0$ if and only if $f(a) > 0$ for any $f \in D(e)$; we also recall here that a cone with order units is generating.

We shall call *pencil* of elements of an algebra A the set of linear combinations $a - \lambda b$ where $a \in A$, $b \in A$ and λ is a scalar parameter.

DEFINITION 3.1. Let a and b be elements of a complex unital Banach algebra, the set

(3.5) $V_b(a) = \{\lambda \mid (\exists f)\, (f \in D(e) \text{ and } f(a - \lambda b) = 0)\}$

is called the numerical range of the pencil $a - \lambda b$. This notion will be used here under the additional assumption that $b \in \mathrm{Her}(A)$ with $b > 0$; in the latter case it is readily seen that

$$V_b(a) = \left\{ \frac{f(a)}{f(b)} ; f \in D(e) \right\}.$$

Definition 3.1 generalizes (3.2) since $V(a) = V_e(a)$ where e is the unit of A; in other words $V(a)$ is the numerical range of the pencil $a - \lambda e$. Generalizing (3.1), we also introduce

$$(3.6) \qquad p_b(a) = \sup\{|\lambda| ; \lambda \in V_b(a)\}$$

and we have, in particular, $p(a) = p_e(a)$.

LEMMA 3.2. *Let A be a complex unital Banach algebra with unit e, and $b \in E = \mathrm{Her}(A)$, $b > 0$; then*

$$(3.7) \qquad p_b(a) = \inf_{\lambda > 0} \{-\lambda b < a < \lambda b\}.$$

Proof. For any $a \in E$ and $\lambda > 0$, the relation

$$(3.8) \qquad -\lambda b < a < \lambda b$$

is successively equivalent to

$$(3.9) \qquad \lambda b + a > 0 \text{ and } \lambda b - a > 0$$

$$(3.10) \qquad f(\lambda b + a) > 0 \text{ and } f(\lambda b - a) > 0 \text{ for any } f \in D(e)$$

$$(3.11) \qquad \lambda + \frac{f(a)}{f(b)} > 0 \text{ and } \lambda - \frac{f(a)}{f(b)} > 0 \text{ for any } f \in D(e)$$

$$(3.12) \qquad \left| \frac{f(a)}{f(b)} \right| < \lambda \text{ for any } f \in D(e).$$

Thus (3.7) is equivalent to

$$(3.13) \quad p_b(a) = \inf\left\{ \lambda ; (\forall f)\left(f \in D(e) \Rightarrow \left| \frac{f(a)}{f(b)} \right| < \lambda \right) \right\} \equiv \sup\{|\lambda| ; \lambda \in V_b(a)\}$$

which completes the proof.

REMARK. In other words $p_b(a)$ is the Minkowski functional of the order interval $[-b, b]$. Since, as stated previously, $p_e(a)$ is a norm on E, equivalent to $\|a\|$, it follows from the Corollary 2 of V.6.2 in Schaefer [13] that $C = \mathrm{Pos}(A)$ is a normal cone and therefore that $p_b(a)$ is also an equivalent norm for arbitrary $b \in E$, $b > 0$.

It must be remembered that E is considered here as an ordered Banach space; in particular $p_b(a)$ is not an algebra norm. Nevertheless, we have

LEMMA 3.3. *Let A be a complex unital Banach algebra, $E = \text{Her}(A)$, $a \in A$, $b \in E$, $b > 0$. Then*

$$(3.14) \qquad r(b^{-1}a) \leqq p_b(a) \leqq \|b^{-1}\| \|a\|.$$

Proof. If $\lambda \in \sigma(b^{-1}a)$, then $a - \lambda b$ is singular, thus $0 \in \sigma(a - \lambda b) \subset V(a - \lambda b)$; therefore, for some $f \in D(e)$, we have $f(a - \lambda b) = 0$, thus also $\lambda = f(a)/f(b)$, i.e. $\lambda \in V_b(a)$. The left inequality follows.

The right inequality follows from

$$(3.15) \qquad p_b(a) \leqq \frac{\sup\{|\lambda| ; \lambda \in V(a)\}}{\inf\{\lambda ; \lambda \in V(b)\}}$$

since

$$(3.16) \qquad \sup\{|\lambda| ; \lambda \in V(a)\} \leqq \|a\|$$

and

$$(3.17) \qquad \inf(V(b)) = \inf(\sigma(b)) = \frac{1}{\sup(\sigma(b^{-1}))} = \frac{1}{\|b^{-1}\|}$$

by Theorems 10.4, 10.5, 10.17 and 7.4 of Bonsall and Duncan [1].

3.2. *The Stein transformation in B^* algebra.* We recall that a B^* algebra is a complex unital Banach algebra A with a linear involution $a \to a^*$ such that $\|a^*a\| = \|a\|^2$ for any $a \in A$. If A is a B^* algebra, $a \in A$ is hermitian if and only if it is symmetric, i.e. $a^* = a$.

DEFINITION 3.1. Let A be a B^* algebra and let $a \in A$; then the linear transformation T_a on $E = \text{Her}(A)$ into A defined by $T_a: x \to a^*xa$ for each $x \in E$ is called the Stein transformation associated with a.

LEMMA 3.4. *Let A be a B^* algebra, $E = \text{Her}(A)$ and T_a be the Stein transformation associated with $a \in A$; then $T_a \in \mathcal{L}(E)$ and $T_a \geqq 0$.*

Proof. Let $x \in E$, $x \geqq 0$ and $y = T_a(x) = a^*xa$; by Lemma 38.7 of Bonsall and Duncan [1], there exists $u \in E$, $u \geqq 0$ such that $u^2 = x$; since $E = \text{Sym}(A)$ we have $u^* = u$, thus $y = (a^*u^*)(ua)$ which entails $y \geqq 0$

by Lemma 38.9 of Bonsall and Duncan [1]. On the other hand, it is clear that T_a is bounded.

LEMMA 3.5. *Let A be a B^* algebra with unit e, $E = \mathrm{Her}(A)$, and T_a be the Stein transformation associated with $a \in A$. Then*
(1) $\|T_a\| = \|a\|^2$
(2) $r(T_a) = (r(a))^2$.

Proof. We have

$$(3.18) \qquad \|T_a\| = \sup_{\substack{x \in E \\ x \neq 0}} \frac{\|a^* x a\|}{\|x\|} \leq \|a^*\| \cdot \|a\| = \|a\|^2$$

Moreover, for $x = e$,

$$(3.19) \qquad \|T_a\| \geq \|a^* a\| = \|a\|^2$$

and the first assertion follows.
Applying (1) to a^n, we get

$$(3.20) \qquad \|T_a^n\| = \|T_{a^n}\| = \|a^n\|^2.$$

Therefore

$$(3.21) \qquad r(T_a) = \lim_{n \to \infty} \|T_a^n\|^{1/n} = \lim_{n \to \infty} \|a^n\|^{2/n} = (r(a))^2.$$

4. Convergence theorems. New proofs, based on the relations between monotonicity, convergence and order units are proposed in the present section for several convergence theorems usually established in the framework of Hilbert spaces.
The following theorem is a generalization of a result by Householder [4]:

THEOREM 4.1. *Let A be a complex unital Banach algebra with unit e, $E = \mathrm{Her}(A)$ and $t \in E$; then $r(t) < 1$ if and only if $-e < t < e$.*

Proof. By Lemma 3.3 and by Theorem 10.17 of Bonsall and Duncan [1], we have $r(t) = p(t)$. The conclusion follows by Lemma 3.2.

We now consider the application of Theorem 2.2 in B^* algebras; thus we have

THEOREM 4.2. *Let A be a B^* algebra, $E = \mathrm{Her}(A)$ and T_t be the*

Stein transformation associated with $t \in A$; *then the following conditions are equivalent*:

(1) $r(t) < 1$;

(2) $I - T_t$ *is monotone and surjective*;

(3)   *there exists* $x \in E$, $x > 0$ *such that* $(I - T_t)x > 0$.

Proof. By Lemma 3.4, $T_t \in \mathscr{L}(E)$ with $T_t \geqq 0$; by Lemma 3.5, $r(T_t) = r(t)^2$; the conclusion follows by Theorem 2.2.

As a consequence, we have the following generalization of a theorem by Stein [14], [15].

THEOREM 4.3.   *Let A be a B^* algebra, $E = \mathrm{Her}(A)$, $a \in E$ and T_t be the Stein transformation associated with $t \in A$.   If $(I - T_t)a > 0$, then $r(t) < 1$ if and only if $a > 0$.*

Proof. Let $b = (I - T_t)a > 0$.   If $r(t) < 1$, $I - T_t$ is monotone and surjective, thus $a > 0$ by Lemma 2.1.   If, on the other hand, we have $a > 0$, then $r(t) < 1$ by Theorem 4.2, [3].

The preceding results apply in practice when, in a given Banach space V, one solves a linear system

$$ax = y, \qquad x \in V, \qquad y \in V, \qquad a \in \mathscr{L}(v)$$

by an iterative scheme of the form

$$bx^{(j+1)} = cx^{(j)} + y \qquad\qquad j = 0, 1, 2 \cdots$$

with $a = b - c$; assuming that $b^{-1} \in \mathscr{L}(V)$, one introduces the iteration operator $t = b^{-1}c$ and the condition $r(t) < 1$ is equivalent to the norm-convergence of the Neumann expansion for $(e - t)^{-1}$ where e is the identity operator of $\mathscr{L}(V)$.

In this context, we have the following generalization of Theorem 4.1 (cf. Petryshyn [9]):

THEOREM 4.4.   *Let A be a B^* algebra and $E = \mathrm{Her}(A)$; let $a, b, c \in E$ with $a = b - c$, $b > 0$ and set $t = b^{-1}c$.   Then $r(t) < 1$ if and only if $-b < c < b$.*

Proof. Since $b \in E$, $b > 0$, there exists $u \in E$, $u > 0$ such that $u^2 = b$.   Let $T_{u^{-1}}$ be the Stein transformation associated with u^{-1} and $S_{u^{-1}}$ be the similarity transformation defined by $S_{u^{-1}}x = uxu^{-1}$ for all $x \in A$.   We have

$$(4.1) \qquad\qquad T_{u^{-1}}(b \pm c) = S_{u^{-1}}(e \pm t)$$

thus, by Lemma 3.4, $S_{u^{-1}}t \in E$ and, by Theorem 4.1, $r(t) = r(S_{u^{-1}}t) < 1$ if and only if

$$(4.2) \qquad\qquad -e < S_{u^{-1}}t < e$$

or

$$(4.3) \qquad\qquad S_{u^{-1}}(e \pm t) > 0.$$

Since, on the other hand, $T_{u^{-1}}$ is both nonnegative and monotone with $T_u = (T_{u^{-1}})^{-1}$, it follows from Lemma 2.1 that the preceding relation is equivalent to

$$(4.4) \qquad\qquad b \pm c = T_u S_{u^{-1}}(e \pm t) > 0$$

or

$$(4.5) \qquad\qquad -b < c < b.$$

As an immediate consequence, we have the following result in B^* algebras.

COROLLARY. *Let A be a B^* algebra, $E = \text{Her}(A)$, $a, b \in E$, $b > 0$; then $r(b^{-1}a) = p_b(a)$.*

REMARK. It follows from Lemma 3.3 that the sufficient condition of Theorem 4.4 still holds in a complex unital Banach algebra; whether or not the necessary condition also extends in the latter case remains an open question. In the same framework, practical application of Theorem 4.3 is made easier by the following result which generalizes a theorem by Feingold and Spohn [3].

THEOREM 4.5. *Let A be a B^* algebra and $E = \text{Her}(A)$; let $a \in E$, $b, c \in A$ with $a = b - c$ and $b \in \text{Inv}(A)$; set $t = b^{-1}c$ and let T_t be the Stein transformation associated with t. Then $(b^* + c) \in E$; moreover $(I - T_t)a > 0$ if and only if $(b^* + c) > 0$.*

Proof. We have

$$(4.6) \qquad (b^* + c)^* = b + c^* = a + c + c^* = a^* + c + c^* = b^* + c$$

thus $(b^* + c) \in E$.

On the other hand, the following identity is readily established

(4.7) $(I - T_t)a = ab^{*-1}(b^* + c)b^{-1}a$

and the conclusion follows by Lemma 3.4 and Lemma 3.1.

As a consequence, we have the following generalization of a theorem of matrix iterative analysis, sometimes attributed to Reich [11] and Ostrowski [7] who considered the particular case of the successive overrelaxation method; its general form seems however to appear first in John [6] for the sufficient condition, and in Householder [5].

THEOREM 4.6. *Let A be a B* algebra and E = Her(A); let $a \in E$, $b, c \in A$ with $a = b - c$ and $b \in \mathrm{Inv}(A)$, and set $t = b^{-1}c$. If $(b^* + c) > 0$, then $r(t) < 1$ if and only if $a > 0$.*

Proof. In virtue of Theorem 4.5, the proposition is a restatement of Theorem 4.3.

With slight additional notations, these results also cover generalizations of Petryshyn's theorems for bounded operators.

THEOREM 4.7. *Let A be a B* algebra and E = Her(A); let $a, b, c, k \in A$ with $a = b - c$, $k \in \mathrm{Inv}(A)$, ka and kb $\in E$, $kb > 0$, and set $t = b^{-1}c$. Then $r(t) < 1$ if and only if $-kb < kc < kb$.*

Proof. Using the notations of the proof of Theorem 4.4, but with $u^2 = kb$, we have

(4.8) $T_{u^{-1}}(kb \pm kc) = S_{u^{-1}}(e \pm t)$.

The conclusion follows, as precedingly.

REMARK. When $k \in E$ with $k > 0$, an element $a \in A$ such that $ka \in E$ is termed *k-symmetric* by Petryshyn; it is termed *k-positive-definite* if, in addition, $ka > 0$. In the case of finite matrices, alternate characterizations of k-positive-definite elements are considered in [8].

On the other hand, Theorems 4.5 and 4.6 admit the following generalizations.

THEOREM 4.8. *Let A be a B* algebra and E = Her(A); let $a, b, c, k, l \in A$, $a = b - c$, $kal \in E$, $b, k, l \in \mathrm{Inv}(A)$; set $u = l^{-1}b^{-1}cl$ and let T_u be the Stein transformation associated with u. Then $l^*b^*k^* + kcl \in E$ and $(I - T_u)kal > 0$ if and only if $l^*b^*k^* + kcl > 0$.*

Proof. We have

$$(l^*b^*k^* + kcl)^* = kbl + l^*c^*k^*$$
$$= kal + kcl + l^*c^*k^*$$

(4.9)
$$= l^*a^*k^* + kcl + l^*c^*k^*$$

$$= l^*b^*k^* + kcl$$

thus $l^*b^*k^* + kcl \in E$.
On the other hand, we have

(4.10) $\qquad (I - T_u)kal = l^*a^*b^{*-1}l^{*-1}(l^*b^*k^* + kcl)l^{-1}b^{-1}al$

and the conclusion follows by Lemma 3.4 and Lemma 3.1.

THEOREM 4.9. *Let A be a B^* algebra and $E = \mathrm{Her}(A)$; let $a, b, c, k, l \in A$, $a = b - c$, $kal \in E$, $b, k, l \in \mathrm{Inv}(A)$ and set $t = b^{-1}c$. If $l^*b^*k^* + kcl > 0$, then $r(t) < 1$ if and only if $kal > 0$.*

Proof. If $u = l^{-1}tl$, we have $r(t) = r(u)$; therefore, in virtue of Theorem 4.8, the proposition is a restatement of Theorem 4.3.

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