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ON L^p, L^q MULTIPLIERS OF FOURIER TRANSFORMS

RICHARD J. BAGBY

Let m be a tempered distribution on R^n. We say m is an L^p, L^q multiplier (more briefly: $m \in M_p^q$) if, for each $\phi \in \mathcal{S}$, the inverse Fourier transform of $m\hat{\phi}$ is in L^q, and there is a constant C such that $\|\mathcal{F}^{-1}(m\hat{\phi})\|_q \leqq C\|\phi\|_p$ for all such ϕ. The basic problem we shall consider is that of establishing sufficient conditions that a locally integrable function $m \in M_p^q$ in the case $1 < p < q < \infty$.

1. Introduction. The primary reference for multipliers is Hörmander [4], where he proves a number of results. Perhaps the best known of these is his convexity-symmetry theorem: For a given $m \in \mathcal{S}'$, the set $\{(x, y): m \in M_{1/x}^{1/y}\}$ is a convex subset of $[0,1]\times[0,1]$ which is symmetric about the line $x + y = 1$. He also proved that when $1 < p \leqq 2 \leqq q < \infty$ and $1/r = 1/p - 1/q$, M_p^q contains the Lorentz space $L(r,\infty)$ and that only when $p \leqq 2 \leqq q$ can a condition $|m| \leqq F$ imply $m \in M_p^q$.

Whenever m is a multiplier we have $\mathcal{F}^{-1}(m\hat{\phi}) = K * \phi$ where $K \in \mathcal{S}'$ and $\hat{K} = m$. O'Neil [6] showed that if $K \in L(r',\infty)$ then $\|K*\phi\|_q \leqq C\|K\|_{r,\infty}^*\|\phi\|_p$ for $1 < p < q < \infty$, $1/p = 1/q + 1/r$. Thus, any hypotheses which imply $\hat{m} \in L(r',\infty)$ also imply $m \in M_p^q$. Peetre [7] uses this idea to prove a multiplier theorem in terms of homogeneous Lipschitz spaces.

Our attack on the multiplier problem is based on a method due to Hahn [1, 2]. We obtain some extensions of his theorem as well as a refinement of Peetre's theorem. Our hypotheses include the condition $m \in L(r,\infty)$ but do not, in general, imply $\hat{m} \in L(r',\infty)$. Our conclusions take the form $m \in M_p^q$ for $1/p = 1/q + 1/r$ with $1/p$ and $1/q$ sufficiently close to $1/2$.

2. Preliminaries. Let x denote a point in R^n. The usual inner product in R^n is denoted $x \cdot y$. Lebesgue measure in R^n is denoted dx.

The space $\mathcal{S}$ consists of those C^∞ functions on R^n which, along with each derivative, vanish more rapidly at infinity than any rational function. Its dual, $\mathcal{S}'$, is called the space of tempered distributions.

The Fourier transform is defined on $\mathcal{S}$ by

$$\hat{\phi}(x) = \int \exp(-2\pi i x \cdot y)\phi(y)\,dy.$$

The inversion formula is

$$\phi(x) = \mathcal{F}^{-1}\hat{\phi}(x) = \int \exp(2\pi i x \cdot y)\hat{\phi}(y)\,dy.$$

The Fourier transform is extended to $\mathcal{S}'$ by the formula

$$\langle \hat{u}, \phi \rangle = \langle u, \hat{\phi} \rangle \text{ for } u \in \mathcal{S}',\ \phi \in \mathcal{S}.$$

If either u or $\hat{u}$ is in L^1, then the appropriate integral formula is also valid. For a simple exposition of Fourier analysis in $\mathcal{S}$ and $\mathcal{S}'$ see Stein and Weiss **[8]**.

The symbol $\|f\|_p$ always denotes the L^p norm for functions on R^n, $1 \leqq p \leqq \infty$. If $m \in M_p^q$, we set

$$M_p^q(m) = \sup\{\|\mathcal{F}^{-1}(m\hat{\phi})\|_q \colon \phi \in \mathcal{S}, \|\phi\|_p = 1\}.$$

$L(p, q)$ denotes the Lorentz space of measurable functions f on R^n whose nonincreasing rearrangement f^* on $(0, \infty)$ satisfies

$$\|f\|^*_{pq} = \left\{(q/p)\int_0^\infty [t^{1/p}f^*(t)]^q t^{-1}\,dt\right\}^{1/q} < \infty$$

when p and q are finite and

$$\|f\|^*_{p\infty} = \sup t^{1/p}f^*(t) < \infty$$

when $q = \infty$.

These spaces are treated in Hunt **[5]** and Stein and Weiss **[8]**.

For f a function on R^n and $h \in R^n$, the difference operator is given by

$$\Delta_h f(x) = f(x + h) - f(x).$$

Higher differences are given by

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1}f)(x) = \sum_{j=0}^{k} C_{k,j}(-1)^{k-j}f(x + jh).$$

With our definition of Fourier transform,

$$(\Delta_h^k f)^\wedge = [\exp(2\pi i x \cdot h) - 1]^k \hat{f}.$$

Let k be a fixed positive integer and suppose f is a function such that $\Delta_h^k f \in L^p$ for a.e. $h \in R^n$. For $\lambda > 0$, set

$$\omega(\lambda, f, p) = \sup_{|h|\leqq\lambda} \|\Delta_h^k f\|_p.$$

For $0 < \alpha < k$, the homogeneous Lipschitz space $\dot{W}_p^{\alpha,q}$ consists of those functions f (modulo polynomials of degree at most some sufficiently large N) such that $\lambda^{-\alpha}\omega(\lambda, f, p)$ is in $L^q((0,\infty), \lambda^{-1}d\lambda)$. These are essentially the same spaces treated in Herz **[3]**; interpolation properties of these spaces are given in Peetre **[7]**.

3. Convolution products as multipliers. In this section we offer some improvements and variants of Hahn's results **[1, 2]**.

THEOREM 3.1. *Suppose* $m = f*g$, *where* $f \in L^s$ *and* $g \in L^t$ *with* $1 \leqq s \leqq t$, $1/s + 1/t \geqq 1$. *Then* $m \in M_p^q$ *and* $M_p^q(m) \leqq c\|f\|_s\|g\|_t$ *for all* p, q *satisfying*

(i) $1 \leqq p \leqq q \leqq \infty$
(ii) $1/p - 1/q = 1/s + 1/t - 1$
(iii) $1/2 - 1/t \leqq 1/q \leqq 1/p \leqq 1/2 + 1/t$.

Proof. When $1/s + 1/t = 1$ this is the basic theorem in **[1]**; we shall use this to prove other cases. Note that $1/s + 1/t > 1$, $s \leqq t$ implies $s < 2$.

First let us suppose $t \leqq 2$. By the Hausdorff–Young inequality we have $\|\mathcal{F}^{-1}f\|_{s'} \leqq \|f\|_s$ and $\|\mathcal{F}^{-1}g\|_{t'} \leqq \|g\|_t$. Setting

$$1 - 1/r = 1/r' = 1/s' + 1/t' = 2 - 1/s - 1/t$$

we have $\mathcal{F}^{-1}(f*g) = (\mathcal{F}^{-1}f)(\mathcal{F}^{-1}g) \in L^{r'}$ by Hölder's inequality. Young's inequality gives

$$\begin{aligned}\|\mathcal{F}^{-1}((f*g)\hat{\phi})\|_q &= \|\phi * \mathcal{F}^{-1}(f*g)\|_q \\ &\leqq \|\phi\|_p\|\mathcal{F}^{-1}(f*g)\|_{r'}\end{aligned}$$

for $1 \leqq p \leqq q \leqq \infty$, $1/p + 1/r' = 1/q + 1$. The last equation reduces to (ii).

Now we suppose $t > 2$. Fix $g \in L^t$, $\phi \in \mathcal{S}$, and set $1/p_0 = 1/2 + 1/t$. Define

$$Tf = \mathcal{F}^{-1}((f*g)\hat{\phi}).$$

First we note $\|Tf\|_2 \leqq C\|f\|_1\|g\|_t\|\phi\|_{p_0}$. Since $\|f*g\|_t \leqq \|f\|_1\|g\|_t$, this follows by applying Plancherel's theorem, Hölder's inequality, and the Hausdorff–Young inequality.

Pick s_1 so that $1/s_1 + 1/t = 1$. Then by the result mentioned above,

$$\|Tf\|_{p_0} \leqq C\|f\|_{s_1}\|g\|_t\|\phi\|_{p_0}.$$

Now we invoke the Riesz–Thorin theorem **[10]**. Set $1/s = (1-\theta)/1 + \theta/s_1$ and $1/q_0 = (1-\theta)/2 + \theta/p_0$. We obtain

$$\| Tf \|_{q_0} \leqq C \| f \|_s \| g \|_t \| \phi \|_p,$$

with $1/q_0 = 1/p_0 - (1/s + 1/t - 1)$, so that the theorem has been proved for the smallest value of p and q satisfying (i), (ii), and (iii). Hörmander's convexity-symmetry theorem **[4]** completes the proof.

Hahn **[2]** proved a similar theorem by a more complicated interpolation argument; however, his choice of end points did not yield the full range of p and q obtained above.

Below we offer a useful variant involving Lorentz spaces.

THEOREM 3.2. *Suppose $m = f * g$ where $f \in L(s, \infty)$ and $g \in L(t, \infty)$ with $1 < s \leqq t$ and $1/s + 1/t > 1$. Then $m \in M_p^q$ and $M_p^q(m) \leqq C \| f \|_{s,\infty}^* \| g \|_{t,\infty}^*$ for all p, q satisfying*

(i) $1 < p < q < \infty$

(ii) $1/p - 1/q = 1/t + 1/s - 1$

(iii) $1/2 - 1/t < 1/q < 1/p < 1/2 + 1/t$.

Proof. We apply the Marcinkiewicz interpolation theorem **[5]** to each of the three linear operators obtained by fixing two of the arguments in

$$T(f, g, \phi) = \mathscr{F}^{-1}((f * g)\hat{\phi}).$$

For technical reasons it is convenient to set $\alpha = \min(1/s, 1/t)$, drop the condition $s \leqq t$ and replace $1/t$ by α in (iii) in both Theorems 3.1 and 3.2.

Because of the strict inequalities in Theorem 3.2, we may select $s_0 < s < s_1$ so that if $1/q_i = 1/p - 1/s_i - 1/t + 1$, then all the hypotheses of Theorem 3.1 are satisfied when s and q are replaced by s_i and q_i respectively. Regarding p, t, g, and ϕ as fixed, we have

$$\| T(f, g, \phi) \|_{q_i} \leqq C_i \| f \|_{s_i} \| g \|_t \| \phi \|_p \qquad i = 1, 2.$$

Thus interpolation yields

$$\| T(f, g, \phi) \|_{q,\infty}^* \leqq C \| f \|_{s,\infty}^* \| g \|_t \| \phi \|_p.$$

Fixing p, s, f, and ϕ, a similar argument yields

$$\| T(f, g, \phi) \|_{q,\infty}^* \leqq C \| f \|_{s,\infty}^* \| g \|_{t,\infty}^* \| \phi \|_p.$$

Finally we fix s, t, f and g and perform one last interpolation to obtain

$$\begin{aligned}\| T(f, g, \phi)\|_q &\leqq \| T(f, g, \phi)\|^*_{q,p} \\ &\leqq C\|f\|^*_{s,\infty}\| g\|^*_{t,\infty}\|\phi\|^*_{p,p} \\ &= C\|f\|^*_{s,\infty}\| g\|^*_{t,\infty}\|\phi\|_p.\end{aligned}$$

The most obvious application of Theorem 3.2 is to Riesz potentials; i.e., convolutions with the kernel $c_{\alpha,n}|x|^{\alpha-n} \in L(n/(n-\alpha),\infty)$.

When $1/s + 1/t = 1$ so that $p = q$ the analogue of Theorem 3.2 fails; clearly $f * g$ is not a multiplier for $f(x) = g(x) = |x|^{-n/2}$. A paper of Strichartz **[9]** discusses the failure of the methods above in such a situation. However, we have a simple proof of a related theorem which we give below.

THEOREM 3.3. *Suppose* $m = f * g$ *where* $f \in L(s, 1)$ *and* $g \in L(t,\infty)$ *with* $1/s + 1/t = 1$. *Then* $m \in M^p_p$ *and* $M^p_p(m) \leqq C\|f\|^*_{s,1}\|g\|^*_{t,\infty}$ *for* $|1/p - 1/2| < \min(1/s, 1/t)$.

Proof. Set $T(f, g, \phi) = \mathscr{F}^{-1}((f * g)\hat{\phi})$. Let E be a measurable set of finite measure $|E|$ and let χ_E denote its characteristic function. Fix $\phi \in \mathscr{S}$ and consider the mapping

$$Ag = T(\chi_E, g, \phi)$$

We take $t_0 < t < t_1$, $1/s_0 + 1/t_0 = 1$, and $1/s_1 + 1/t_1 = 1$ with t_0 and t_1 sufficiently close to t to give $|1/p - 1/2| \leqq \min(1/s_0, 1/s_1, 1/t_0, 1/t_1)$. Then by Hahn's theorem **[1]** we have

$$\begin{aligned}\| Ag\|_p &\leqq C\|\chi_E\|_{s_i}\| g\|_{t_i}\|\phi\|_p \\ &= C|E|^{1/s_i}\| g\|_{t_i}\|\phi\|_p \quad \text{for } i = 1, 2.\end{aligned}$$

Now for $g \in L(t,\infty)$ we write $g = g_0 + g_1$ where $g_0(x) = g(x)$ if $|g(x)| > \lambda$ and vanishes otherwise. Then $g_0 \in L^{t_0}$ and $g_1 \in L^{t_1}$; moreover $\| g_0\|_{t_0} \leqq C\lambda^{1-t/t_0}\| g\|^{*t/t_0}_{t,\infty}$ and $\| g_1\|_{t_1} \leqq C\lambda^{1-t/t_1}\| g\|^{*t/t_1}_{t,\infty}$. We thus obtain

$$\begin{aligned}\| Ag\|_p &\leqq \| Ag_0\|_p + \| Ag_1\|_p \\ &\leqq C|E|^{1/s_0}\lambda^{1-t/t_0}\| g\|^{*t/t_0}_{t,\infty}\|\phi\|_p \\ &\quad + C|E|^{1/s_1}\lambda^{1-t/t_1}\| g\|^{*t/t_1}_{t,\infty}\|\phi\|_p.\end{aligned}$$

Choosing $\lambda = |E|^{-1/t} \|g\|_{t,\infty}^*$ yields

$$\|Ag\|_p \leqq C|E|^{1/s} \|g\|_{t,\infty}^* \|\phi\|_p$$

Now we regard g and ϕ as fixed and consider the operator

$$Bf = T(f, g, \phi).$$

When $f = \chi_E$ we have $\|Bf\|_p \leqq C|E|^{1/s} \|g\|_{t,\infty}^* \|\phi\|_p$; hence by a result of Stein and Weiss [**8**, Chap. V, Th. 3.13] we have

$$\|Bf\|_p \leqq C\|f\|_{s,1}^* \|g\|_{t,\infty}^* \|\phi\|_p$$

for all $f \in L(s, 1)$.

4. Lipschitz functions as multipliers. We begin by deriving a representation theorem for Lipschitz functions which is a simple variant of a theorem of Herz. This theorem expresses Lipschitz functions in terms of convolution products and consequently allows us to use the results of the previous section.

LEMMA 4.1. *Let k be a fixed positive integer. There is a function $\phi \in C_0^\infty(0, \infty)$ such that*

$$\int [\exp(2\pi i x_1) - 1]^k \phi(|x|)|x|^{-n} dx = 1.$$

Proof. It suffices to produce $\phi \in C_0^\infty(0, \infty)$ such that the above integral is nonzero. Since we can construct $\phi_m \in C_0^\infty(0, \infty)$ such that

$$\lim_{m \to \infty} \int [\exp(2\pi i x_1) - 1]^k \phi_m(|x|)|x|^{-n} dx = \int [\exp(2\pi i x_1) - 1]^k \exp(-\lambda\pi|x|^2)\, dx,$$

it suffices to show that this last integral is nonzero. If we expand $[\exp(2\pi i x_1) - 1]^k$ and recognize some well-known integrals we obtain

$$(-1)^k \lambda^{n/2} \int [\exp(2\pi i x_1) - 1]^k \exp(-\lambda\pi|x|^2)\, dx = \sum_{j=0}^{k} (-1)^j C_{k,j} \exp(-\pi j^2/\lambda)$$

which approaches 1 as $\lambda \downarrow 0$.

LEMMA 4.2. *For ϕ as in Lemma 4.1, set*

$$K(x, y) = \int \exp(2\pi i x \cdot z)\phi(|y||z|)\, dz.$$

Then

(i) $K(\cdot, y) \in \mathcal{S}$ *for* $y \neq 0$ *and* $\|K(\cdot, y)\|_p = C_p |y|^{-n(1-1/p)}$, $1 \leqq p \leqq \infty$.

(ii) $\int [\exp(2\pi i x \cdot y) - 1]^k \hat{K}(x, y)|y|^{-n} dy = 1$

for $x \neq 0$, where $\hat{K}$ denotes the Fourier transform with respect to x.

Proof. For each $y \in R^n$, the function $x \to \phi(|y||x|)$ defines a function in $\mathcal{S}$; $K(\cdot, y)$ is its inverse Fourier transform. A change of variable yields

$$K(x, y) = |y|^{-n} K(|y|^{-1}x, e), \quad \text{with } |e| = 1,$$

hence

$$\|K(\cdot, y)\|_p = |y|^{-n(1-1/p)} \|K(\cdot, e)\|_p.$$

The integral in (ii) is

$$\int [\exp(2\pi i x \cdot y) - 1]^k \phi(|y||x|)|y|^{-n} dy.$$

Substituting $y = |x|^{-1}y'$ and then rotating coordinates to make y_1 parallel to x, we obtain the integral of Lemma 4.1.

THEOREM 4.3. *Suppose $f \in L^1 + L^r$ where $1 < r < \infty$ and also $f \in \dot{W}_p^{\alpha,q}$ where $0 < \alpha < n/p$. Then*

$$f(x) = \int\int K(x - z, y)\Delta_y^k f(z)|y|^{-n} dy dz$$

with the integral converging absolutely for a.e. x.

Proof. First we establish the absolute convergence of the integral. Set

$$F(x) = \int\int |K(x - z, y)|\, |\Delta_y^k f(z)|\, |y|^{-n} dy dz.$$

The simplest case occurs when $q = 1$. Choose s so that $n(1 - 1/s) = \alpha$ and t so that $1/p + 1/s = 1/t + 1$. By Minkowski's and Young's inequalities we have

$$\begin{aligned}\|F\|_t &\leqq \int \left\| \int |K(\cdot - z, y)|\, |\Delta_y^k f(z)|\, dz \right\|_t |y|^{-n} dy \\ &\leqq \int \|K(\cdot, y)\|_s \|\Delta_y^k f(\cdot)\|_p |y|^{-n} dy \\ &\leqq C \int |y|^{-n(1-1/s)} \omega(|y|, f, p) |y|^{-n} dy \\ &= \int_0^\infty \lambda^{-\alpha} \omega(\lambda, f, p) \lambda^{-1} d\lambda.\end{aligned}$$

By definition of $\dot{W}_p^{\alpha,1}$, the last integral is finite. Thus $|F(x)| < \infty$ a.e. and the original integral defines a function in L^t. Note $p < t < \infty$.

The general case is only a little harder. Write $F(x) = G(x) + H(x)$, where

$$G(x) = \int \int_{|y| \leqq 1} |K(x - z, y)|\, |\Delta_y^k f(z)|\, |y|^{-n} dy dz.$$

Choose s_0 and s_1 so that $0 < 1 - 1/s_0 < \alpha/n < 1 - 1/s_1 < 1/p$. Setting $1/p + 1/s_i = 1/t_i + 1$, the arguments of the previous paragraph yield

$$\|G\|_{t_0} \leqq C \int_0^1 \lambda^{-n(1-1/s_0)} \omega(\lambda, f, p) \lambda^{-1} d\lambda$$

and

$$\|H\|_{t_1} \leqq C \int_1^\infty \lambda^{-n(1-1/s_1)} \omega(\lambda, f, p) \lambda^{-1} d\lambda.$$

Since $n(1 - 1/s_0) < \alpha < n(1 - 1/s_1)$, Hölder's inequality bounds both of these by

$$C \left\{ \int_0^\infty [\lambda^{-\alpha} \omega(\lambda, f, p)]^q \lambda^{-1} d\lambda \right\}^{1/q} \text{ if } q < \infty$$

and by $C \sup \lambda^{-\alpha} \omega(\lambda, f, p)$ if $q = \infty$. Set

$$g(x) = \int \int K(x - z, y) \Delta_y^k f(z) |y|^{-n} dy dz.$$

We show $f = g$ by showing the tempered distributions $\hat{f}$ and $\hat{g}$ can differ only by a distribution supported at the origin, and hence $f - g$ is a polynomial. The integrability properties of f and g require such a polynomial to be zero identically.

Let $\psi \in \mathcal{O}$; i.e., let ψ be a C^∞ function on R^n with compact support not containing the origin. By absolute convergence,

$$\langle \hat{g}, \psi \rangle = \langle g, \hat{\psi} \rangle = \int \langle K(\cdot, y) * \Delta_y^k f, \hat{\psi} \rangle | y |^{-n} dy.$$

We have

$$\begin{aligned}\langle K(\cdot, y) * \Delta_y^k f, \hat{\psi} \rangle &= \langle \hat{K}(\cdot, y)(\Delta_y^k f)^\wedge, \psi \rangle \\ &= \langle \hat{f}, [\exp(2\pi i x \cdot y) - 1]^k \hat{K}(\cdot, y)\psi \rangle.\end{aligned}$$

From the properties of $\hat{K}$ and ψ we see easily that the mapping

$$y \to [\exp(2\pi i x \cdot y) - 1]^k \hat{K}(x, y)\psi(x) | y |^{-n}$$

defines a continuous mapping from R^n into $\mathcal{S}$ with compact support. Thus we may write

$$\begin{aligned}\langle \hat{g}, \psi \rangle &= \langle \hat{f}, \int [\exp(2\pi i x \cdot y) - 1]^k \hat{K}(\cdot, y)\psi(\cdot) | y |^{-n} dy \rangle \\ &= \langle \hat{f}, \psi \rangle \text{ by Lemma 4.2.}\end{aligned}$$

Hence $\hat{g} - \hat{f}$ is supported at the origin as desired.

At this stage we should remark that the hypotheses of Theorem 4.3 appear to be slightly redundant. However, this redundancy resolves any ambiguity which might appear in the statement $f \in \dot{W}_p^{\alpha,q}$, elements of which are equivalence classes modulo polynomials in Peetre's definition.

THEOREM 4.4. *Suppose* $m \in L^1 + L^r$ *where* $1 < r < \infty$ *and also* $m \in \dot{W}_t^{\alpha,1}$ *with* $0 < \alpha/n < 1/t \leqq 1/2$. *Then* $m \in M_p^q$ *for all* p, q *satisfying*

(i) $1/p - 1/q = 1/t - \alpha/n$

(ii) $1/2 - 1/t \leqq 1/q < 1/p \leqq 1/2 + 1/t$.

Proof. It suffices to show $|\langle \mathcal{F}^{-1}(m\hat{\phi}), \hat{\psi} \rangle| \leqq C \| \phi \|_p \| \hat{\psi} \|_{q'}$ for all $\phi, \psi \in \mathcal{S}$, where $1/q' = 1 - 1/q$. We have

$$\begin{aligned}\langle \mathcal{F}^{-1}(m\hat{\phi}), \hat{\psi} \rangle &= \langle m\hat{\phi}, \psi \rangle \\ &= \int m(x)\hat{\phi}(x)\psi(x)\, dx \\ &= \int \left[\int K(\cdot, y) * \Delta_y^k m(x) | y |^{-n} dy \right] \hat{\phi}(x)\psi(x)\, dx\end{aligned}$$

by Theorem 4.3. We also have absolute convergence of the above integral; hence

$$\langle \mathcal{F}^{-1}(m\hat{\phi}), \hat{\psi}\rangle = \int \langle (K(\cdot, y)*\Delta_y^k m)\hat{\phi}, \psi\rangle |y|^{-n} dy$$

$$= \int \langle \mathcal{F}^{-1}[(K(\cdot, y)*\Delta_y^k m)\hat{\phi}], \hat{\psi}\rangle |y|^{-n} dy.$$

Hence by Hölder's inequality

$$|\langle \mathcal{F}^{-1}(m\hat{\phi}), \hat{\psi}\rangle| \leqq \int \|\mathcal{F}^{-1}[(K(\cdot, y)*\Delta_y^k m)\hat{\phi}]\|_q \|\hat{\psi}\|_{q'} |y|^{-n} dy.$$

If we set $1 - 1/s = \alpha/n$ then by Lemma 4.2 $\|K(\cdot, y)\|_s = C|y|^{-\alpha}$; hence by Theorem 3.1

$$\|\mathcal{F}^{-1}[(K(\cdot, y)*\Delta_y^k m)\hat{\phi}]\|_q \leqq C|y|^{-\alpha} \|\Delta_y^k m\|_t \|\phi\|_p.$$

Thus

$$|\langle \mathcal{F}^{-1}(m\hat{\phi}), \hat{\psi}\rangle| \leqq C\|\phi\|_p \|\hat{\psi}\|_{q'} \int |y|^{-\alpha} \|\Delta_y^k m\|_t |y|^{-n} dy.$$

Since $\|\Delta_y^k m\|_t \leqq \omega(|y|, m, t)$, we have

$$\int |y|^{-\alpha} \|\Delta_y^k m\|_t |y|^{-n} dy \leqq C\int_0^\infty \lambda^{-\alpha}\omega(\lambda, m, t)\lambda^{-1} d\lambda < \infty$$

for $m \in \dot{W}_t^{\alpha,1}$.

THEOREM 4.5. *Suppose* $m \in L^1 + L^r$ *where* $1 < r < \infty$ *and also* $m \in \dot{W}_t^{\alpha,\infty}$ *with* $0 < \alpha/n < 1/t \leqq 1/2$. *Then* $m \in M_p^q$ *for all* p, q *satisfying*

(i) $1/p - 1/q = 1/t - \alpha/n$

(ii) $1/2 - 1/t < 1/q < 1/p < 1/2 + 1/t$.

Proof. We shall see below that

$$\|\mathcal{F}^{-1}(m\hat{\phi})\|_{q,\infty}^* \leqq C_{p,q} \|\phi\|_p$$

whenever p and q satisfy (i) with $1/2 \leqq 1/p \leqq 1/2 + 1/t$. Hence the Marcinkiewicz interpolation theorem **[5]** yields

$$\|\mathcal{F}^{-1}(m\hat{\phi})\|_q \leqq C_{p,q} \|\phi\|_p$$

for all p and q satisfying (i) and $1/2 < 1/p < 1/2 + 1/t$; Hörmander's convexity-symmetry theorem **[4]** will then conclude the proof.

We now offer two proofs of the assertion $\|\mathcal{F}^{-1}(m\hat{\phi})\|^*_{q,\infty} \leqq C_{p,q}\|\phi\|_p$. The first uses real interpolation as found in Peetre **[7]**; the second is elementary.

Set $Tm = \mathcal{F}^{-1}(m\hat{\phi})$ for fixed $\phi \in \mathcal{S}$. Then $\|Tm\|_{q_0} \leqq C\|m\|_t\|\phi\|_p$ for $1/p - 1/q_0 = 1/t$, $1/2 \leqq 1/p \leqq 1/2 + 1/t$. If we choose β with $\alpha/n < \beta/n < 1/t$, Theorem 4.4 implies

$$T\colon \dot{W}_t^{\beta,1} \to L^{q_1} \text{ for } 1/q_1 = 1/p + \beta/n - 1/t,\ 1/2 \leqq 1/p \leqq 1/2 + 1/t,$$

with norm bounded by $C\|\phi\|_p$.

Hence $T\colon [L^t, \dot{W}_t^{\beta,1}]_{\theta,\infty} \to [L^{q_0}, L^{q_1}]_{\theta,\infty}$ continuously with norm bounded by $C\|\phi\|_p$; choosing $\theta = \alpha/\beta$ gives $[L^t, \dot{W}_t^{\beta,1}]_{\theta,\infty} = \dot{W}_t^{\alpha,\infty}$ and $[L^{q_0}, L^{q_1}]_{\theta,\infty} = L(q,\infty)$.

The second proof uses a combination of the techniques used in proving 4.3 and 4.4. Fix $r > 0$, and write $m = m_0 + m_1$ where

$$m_0(x) = \int_{|y|\leqq r} K(\cdot, y) * \Delta_y^k m(x)|y|^{-n}\,dy.$$

Choosing s_0 and s_1 so that $n(1 - 1/s_0) < \alpha < n(1 - 1/s_1)$ and setting $1/q_i = 1/p + 1 - 1/s_i - 1/t$ we obtain

$$\|Tm_0\|_{q_0} \leqq C\|\phi\|_p \int_0^r \lambda^{-n(1-1/s_0)}\omega(\lambda, m, t)\lambda^{-1}\,d\lambda$$

and

$$\|Tm_1\|_{q_1} \leqq C\|\phi\|_p \int_r^\infty \lambda^{-n(1-1/s_1)}\omega(\lambda, m, t)\lambda^{-1}\,d\lambda.$$

Since $m \in \dot{W}_t^{\alpha,\infty}$, $\omega(\lambda, m, t) \leqq C\lambda^\alpha$; hence

$$\|Tm_0\|_{q_0} \leqq C\|\phi\|_p r^{\alpha - n(1-1/s_0)}$$

and

$$\|Tm_1\|_{q_1} \leqq C\|\phi\|_p r^{\alpha - n(1-1/s_1)}.$$

By the usual properties of rearrangements and Lorentz space norms **[5]**, we have

$$\begin{aligned}\tau^{1/q}(Tm)^*(\tau) &\leqq \tau^{1/q}(Tm_0)^*(\tau/2) + \tau^{1/q}(Tm_1)^*(\tau/2)\\ &\leqq C\tau^{1/q - 1/q_0}\|Tm_0\|_{q_0} + C\tau^{1/q-1/q_1}\|Tm_1\|_{q_1}\\ &\leqq C\|\phi\|_p(\tau^{1/q-1/q_0}r^{\alpha-n(1-1/s_0)} + \tau^{1/q-1/q_1}r^{\alpha-n(1-1/s_1)}).\end{aligned}$$

Choosing $r = \tau^{-1/n}$ and simplifying the exponents yields

$$\tau^{1/q}(Tm)^*(\tau) \leqq C\|\phi\|_p$$

as desired.

By the Sobolev embedding theorem **[3, 7]**, the hypotheses of Theorem 4.5 imply $m \in L(r, \infty)$ where $1/r = 1/t + \alpha/n$. When $t = 2$ Bernstein's theorem **[3, 7]** implies $\hat{m} \in L(r', \infty)$; this is the case considered by Peetre **[7]**. It should be noted that for $t > 2$ and $\beta/n = 1/2 - 1/t + \alpha/n$, $\dot{W}_2^{\beta,q} \subset \dot{W}_t^{\alpha,q}$ and the inclusion is proper; thus we are offering a genuine refinement of Peetre's result.

ADDENDUM. After this paper had been accepted for publication, the author learned that Y. Uno [Lipschitz Functions and Convolution, Proc. Japan Acad. **50** (1974), 785–788] has published a result similar to Theorem 3.1.

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NEW MEXICO STATE UNIVERSITY
LAS CRUCES, NM 88003