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# SYMMETRIZABLE-CLOSED SPACES

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**Symmetrizable-closed, semimetrizable-closed, minimal symmetrizable, and minimal semimetrizable spaces are characterized. G. M. Reed's theorem that every Moore-closed space is separable is extended to: Every Baire, semimetrizable-closed space is separable. Several examples are given.**

If $P$ is a topological property, a Hausdorff $P$-space will be called *$P$-closed* provided that it is a closed subset of every Hausdorff $P$-space in which it can be embedded. A Hausdorff $P$-space $(X, \mathcal{T})$ will be called *minimal $P$* if there exists no Hausdorff $P$-topology on $X$ strictly weaker than $\mathcal{T}$.

In **[3]** J. W. Green characterized and studied Moore-closed and minimal Moore spaces. In this paper we obtain some analogous results for semimetrizable spaces and symmetrizable spaces.

A *symmetric* for a topological space $X$ is a mapping $d: X \times X \to [0, \infty)$ such that

(1) For all $x, y \in X$, $d(x, y) = d(y, x)$, and $d(x, y) = 0$ if and only if $x = y$.

(2) A set $V \subset X$ is open if and only if for each $x \in V$ there exists $n \in N$ such that $V$ contains the set $B(n, x) = \{y \in X \mid d(x, y) < 1/n\}$.

A space $X$ which admits a symmetric is said to be *symmetrizable*, and if, in addition, each $B(n, x)$ is a neighborhood of $x$, then $X$ is said to be *semimetrizable* and $d$ is called a *semimetric* for $X$. Equivalently, $X$ is semimetrizable via $d$ provided that for $x \in X$, $A \subset X$, and $d(x, A) = \inf\{d(x, a) \mid a \in A\}$, the condition $x \in \bar{A}$ if and only if $d(x, A) = 0$ is satisfied.

A number of the techniques used here are not new; for example, see **[2]**. The terminology used is standard. One perhaps not too familiar concept is that of $\theta$-adherence. A point $p$ of a topological space is said to be a *$\theta$-adherent point* (or be in the *$\theta$-adherence*) of a filter base $\mathcal{F}$ provided that for every set $F \in \mathcal{F}$ and neighborhood $V$ of $p$, one has $F \cap \bar{V} \neq \emptyset$.

Our first two theorems are characterization theorems.

THEOREM 1. *Let $(X, \mathcal{T})$ be a symmetrizable Hausdorff space. The following are equivalent.*

(i) *The space $(X, \mathcal{T})$ is minimal symmetrizable.*

(ii) *Every countable filter base on $(X, \mathcal{T})$ which has a unique $\theta$-adherent point is convergent.*

*Proof.* (ii) implies (i). Suppose that $(X, \mathscr{S})$ is symmetrizable and Hausdorff and $\mathscr{S} \subset \mathscr{T}$. Let $d$ be a symmetric for $(X, \mathscr{S})$. For each point $p \in X$ the filter base

$$\mathscr{B}_p = \{\{x: d(x,p) < 1/n\}: n \in \boldsymbol{N}\}$$

has a unique $\theta$-adherent point in $(X, \mathscr{S})$, namely $p$, and so $\mathscr{B}_p$ also has at most one $\theta$-adherent point in $(X, \mathscr{T})$. By (ii) and the relation $p \in \cap \mathscr{B}_p$, it follows that each $\mathscr{B}_p$ must converge to $p$ in $(X, \mathscr{T})$. Thus for every $T \in \mathscr{T}$ and $p \in T$ there exists $n \in \boldsymbol{N}$ such that $T \supset \{x: d(x,p) < 1/n\}$, that is, $T \in \mathscr{S}$. Therefore, $\mathscr{T} \subset \mathscr{S}$ and $(X, \mathscr{T})$ is minimal symmetrizable.

(i) implies (ii). Assume that there exist a point $q \in X$ and filter base $\mathscr{F} = \{F_n: n \in \boldsymbol{N}\}$ on $X$ such that:

(a) for each $n \in \boldsymbol{N}$, $F_n \supset F_{n+1}$;

(b) $q$ is the unique $\theta$-adherent point of $\mathscr{F}$ in $(X, \mathscr{T})$;

(c) $\mathscr{F}$ fails to be convergent; and

(d) $F_1 = X$.

We will prove that $(X, \mathscr{T})$ cannot be minimal symmetrizable.

Let $\mathscr{V} = \{V \in \mathscr{T}: \text{if } q \in V \text{ then } V \text{ contains some member of } \mathscr{F}\}$. Then $\mathscr{V}$ is a topology on $X$ with $\mathscr{V} \subset \mathscr{T}$, and because $\mathscr{F}$ has no $\theta$-adherent point other than $q$, the space $(X, \mathscr{V})$ is Hausdorff. By (c), $\mathscr{V} \neq \mathscr{T}$.

Now consider any symmetric $d$ for $(X, \mathscr{T})$. Define $d^*: X \times X \to [0, \infty)$ by the rule

$$d^*(y,x) = d^*(x,y) = \begin{cases} d(x,y) & \text{if } x \neq q \neq y \\ 0 & \text{if } x = q = y \\ \min\{d(x,y), 1/n\} & \text{if } y = q \text{ and } x \in F_n \backslash F_{n+1}. \end{cases}$$

Clearly, $d^*$ is a symmetric for the space $(X, \mathscr{V})$, and so $(X, \mathscr{T})$ cannot be minimal symmetrizable.

THEOREM 2. *Let $X$ be a symmetrizable Hausdorff space. The following are equivalent.*

(i) *$X$ is symmetrizable-closed.*

(ii) *Every countable filter base on $X$ has a $\theta$-adherent point.*

*Proof.* (ii) implies (i). Suppose that there exists a symmetrizable Hausdorff space $Y$ such that $X$ is a subspace of $Y$ but $X \neq \bar{X}$. Because $\bar{X}$ is a closed subset of $Y$, $\bar{X}$ is symmetrizable (e.g., see [**5**, p. 93]). Let $d$ be a symmetric for $\bar{X}$. Since $X$ fails to be a closed subset of $\bar{X}$, there must exist a point $p \in \bar{X} \backslash X$ with $0 = \inf\{d(p,x): x \in X\}$. Thus for each $n \in \boldsymbol{N}$,

$$F_n = \{x \in X\colon d(p,x) < 1/n\}$$

is nonempty, and so $\mathcal{F} = \{F_n\colon n \in N\}$ is a countable filter base on $X$. Obviously $\mathcal{F}$ has no $\theta$-adherent point in $X$.

(i) implies (ii). Assume that there exists a filter base $\mathcal{G} = \{G_n\colon n \in N\}$ on $X$ such that $G_1 = X$, each $G_n \supset G_{n+1}$, and $\mathcal{G}$ has no $\theta$-adherent point in $X$. Choose a new point $q \notin X$, let $E = X \cup \{q\}$, and call a subset $V$ of $E$ open if and only if (a) $V \cap X$ is open in $X$ and (b) if $q \in V$ then for some $n \in N$, $V \supset G_n$. Then $E$ is a Hausdorff space in which $X$ is embedded as a proper dense subspace. $E$ is also symmetrizable, for if $d$ is any symmetric for $X$, then the function $d^*\colon E \times E \to [0,\infty)$ determined by the rule

$$d^*(x,y) = d^*(y,x) = \begin{cases} d(x,y) & \text{if } x, y \in X \\ 0 & \text{if } x = q = y \\ 1/n & \text{if } x \in G_n \backslash G_{n+1} \text{ and } y = q, \end{cases}$$

is easily seen to be a symmetric for $E$.

For many properties $P$, $P$-minimality is a sufficient condition for $P$-closedness. For $P$ = symmetrizable, the same is true.

COROLLARY 3. *Every minimal symmetrizable Hausdorff space* $(X, \mathcal{T})$ *is symmetrizable-closed.*

*Proof.* If $d$ is a symmetric for $(X, \mathcal{T})$ and $\mathcal{F}$ is a descending sequence of nonempty sets having no $\theta$-adherent point in $(X, \mathcal{T})$, with $X \in \mathcal{F}$, then for any point $q \in X$, the function $d^*$ defined in the proof of Theorem 1 is a symmetric for a strictly weaker symmetrizable Hausdorff space $(X, \mathcal{V})$.

COROLLARY 4. *Every regular, symmetrizable-closed space is compact.*

*Proof.* In a regular space $\theta$-adherence and adherence are equivalent concepts, so by Theorem 2, every regular symmetrizable-closed space is countably compact. By a result of Nedev [7] every countably compact symmetrizable Hausdorff space is compact.

For various properties $P$, topologists have often been interested in the question as to whether or not there exists a non-compact $P$-space in which every closed subset is $P$-closed. If $P$ = Hausdorff or completely Hausdorff, the answer is known to be no, but if $P$ = regular, the question is open. For $P$ = symmetrizable, the following result holds.

COROLLARY 5. *Let $X$ be a symmetrizable Hausdorff space in which every closed subset is symmetrizable-closed. Then $X$ is compact.*

*Proof.* Obviously no infinite discrete space can be symmetrizable-closed, so every infinite closed subset of $X$ must have a limit point, that is, $X$ must be countably compact.

Let us now give some examples of these concepts.

EXAMPLE 6. In **[1]** N. Bourbaki pointed out that a certain space $X$ due to Urysohn is a minimal Hausdorff space that fails to be compact. We will describe this space and show that it is also semimetrizable, in order to show that there exist noncompact, Hausdorff minimal symmetrizable spaces.

Let

$$X = \boldsymbol{N} \cup \{n \pm 1/m : n, m \in \boldsymbol{N}, m > 2\} \cup \{\pm \pi\}.$$

Define $d: X \times X \to [0, \infty)$ by the rule

$$d(x, y) = d(y, x) = \begin{cases} 0 & \text{if } x = y; \\ |x - y| & \text{if } x, y \notin \{\pm \pi\}; \\ 1 & \text{if } x \in \boldsymbol{N} \text{ and } y \in \{\pm \pi\}, \text{ or} \\ & \text{if } x = n + 1/m \text{ and } y = -\pi, \text{ or} \\ & \text{if } x = n - 1/m \text{ and } y = \pi, \text{ or} \\ & \text{if } x = \pi \text{ and } y = -\pi, \text{ where} \\ & m, n \in \boldsymbol{N} \text{ and } m > 2; \\ 1/n & \text{if } x = n + 1/m \text{ and } y = \pi, \text{ or} \\ & \text{if } x = n - 1/m \text{ and } y = -\pi, \\ & \text{where } m, n \in \boldsymbol{N} \text{ and } m > 2. \end{cases}$$

Call a subset $V$ of $X$ open if and only if for each point $v \in V$ there exists $e > 0$ with $\{x : d(x, v) < e\} \subset V$. Then $d$ is a semimetric for the space $X$, and $X$ is homeomorphic with the space in **[1]** ($X$ is also described in [**2**, p. 101]).

EXAMPLE 7. If $X$ is as in Example 6, then its subspace

$$Y = \boldsymbol{N} \cup \{n + 1/m : n, m \in \boldsymbol{N}, m > 2\} \cup \{\pi\}$$

is well known to be Hausdorff-closed but not minimal Hausdorff. Since $Y$ is a subspace of $X$, it is also semimetrizable. If $Y'$ denotes the space which has the same points as those of $Y$ but which is topologized so that it is the one-point compactification of the space $Y \backslash \{\pi\}$, then $Y'$ is metrizable, and so one sees that $Y$ is not minimal semimetrizable. Thus

$Y$ is an example of a Hausdorff semimetrizable, symmetrizable-closed space that is not minimal semimetrizable.

For $P$ = semimetrizable, the results one can obtain concerning the concepts $P$-closed and $P$-minimal are much more similar to those in **[3]**. Since the proofs are not too different from some of the ones above and in **[3]** and **[9]**, the details are omitted. First two definitions are needed.

A topological space is called *feebly compact* if every countable open filter base has an adherent point. A space is called *semiregular* if it has a base consisting of regular open sets, i.e., sets having the form $V = (\bar{V})°$.

THEOREM 8. *Let $X$ be a semimetrizable Hausdorff space. The following are equivalent.*

(i) *$X$ is semimetrizable-closed.*

(ii) *$X$ is feebly compact.*

THEOREM 9. *Let $X$ be a semimetrizable Hausdorff space. The following are equivalent.*

(i) *$X$ is minimal semimetrizable.*

(ii) *Every countable open filter base on $X$ with a unique adherent point is convergent.*

(iii) *$X$ is semiregular and semimetrizable-closed.*

For semimetrizable spaces, it is easy to show that the concepts semimetrizable-closed and symmetrizable-closed are distinct. For example, let $X$ be any noncompact, regular, semimetrizable-closed space (such as one of the spaces discussed in **[3]**). By Corollary 4, $X$ cannot be symmetrizable-closed.

Not too much is known concerning the density character and cardinality of semimetrizable-closed and symmetrizable-closed spaces. G. M. Reed **[8]** has proved that every Moore-closed space is separable, but I do not know if an analogous result holds for all semimetrizable or symmetrizable spaces. (A proof is given in **[10]** that a feebly compact symmetrizable space is separable if it has a dense set of isolated points.) In our final theorem it is shown that Reed's condition Moore-closed space, or, equivalently, feebly compact Moore space (see **[3]**), can be weakened.

We recall that a topological space $X$ is said to be a *Baire* space provided that for every countable family $\mathscr{C}$ of dense open subsets of $X$, the set $\cap \mathscr{C}$ is also dense. It is known **[6]** that every regular, feebly compact space is a Baire space.

THEOREM 10. *Every Baire, feebly compact, semimetrizable space $X$ is separable.*

*Proof.* The proof will consist of two parts. We will first prove that $(*)$ every family of pairwise disjoint nonempty open subsets of $X$ is countable. Next, using $(*)$, we will construct a countable dense subset for $X$.

Let $d$ be a semimetric for $X$. For $x \in X$ and $n \in \boldsymbol{N}$, $\{y \in X: d(x,y) < 1/n\}$ will be denoted by $B(n,x)$, and the interior of $B(n,x)$ will be denoted by $I(n,x)$.

*Proof* of $(*)$: Suppose that there exists an uncountable family $\mathscr{V}$ of pairwise disjoint nonempty open subsets of $X$. For each $V \in \mathscr{V}$ and $m \in \boldsymbol{N}$ let

$$V_m = \{x \in V: B(m,x) \subset V\}^-,$$

and note that since $V = \cup\{V_m: m \in \boldsymbol{N}\}$, it follows from the Baireness of $X$ that one can select an integer $m(V)$ for which $V_{m(V)}$ has nonempty interior. Choose $i \in \boldsymbol{N}$ such that $\mathscr{W} = \{V \in \mathscr{V}: m(V) = i\}$ is uncountable, and for each $W \in \mathscr{W}$ let $J_W$ denote the interior of $W_i$. By the feeble compactness of $X$, there must exist a point $p \in X$ at which $\mathscr{J} = \{J_W: W \in \mathscr{W}\}$ fails to be locally finite. But consider any set $J_W \in \mathscr{J}$ with $\phi \neq K = J_W \cap I(i,p)$. Because $K$ is a nonempty open subset of $W_i$, there must exist a point $q \in W$ with $B(i,q) \subset W$ and with $q \in K$. Then $d(p,q) < 1/i$ and so $p \in B(i,q) \subset W$. This latter relation, however, shows that $\mathscr{J}$ must be locally finite at $p$, for given any $J_V \in \mathscr{J}$ with $V \neq W$, we have $W \cap J_V = \emptyset$. Thus we have obtained a contradiction, and the proof of $(*)$ is complete.

For the remainder of the proof, if $n \in \boldsymbol{N}$ let

$$\mathscr{B}_n = \{I(k,x): x \in X, k \in \boldsymbol{N}, \text{ and } k \geqq n\},$$

and let $\mathscr{D}_n$ be a maximal family of pairwise disjoint members of $\mathscr{B}_n$. Once the sequence $\{\mathscr{D}_n: n \in \boldsymbol{N}\}$ has been determined, choose, for each $n \in \boldsymbol{N}$ and $D \in \mathscr{D}_n$, one point $np_D$ such that $D = I(k, np_D)$ for some $k \in \boldsymbol{N}$ with $k \geqq n$, and let

$$C_n = \{np_D: D \in \mathscr{D}_n\}.$$

Then $C = \cup\{C_n: n \in \boldsymbol{N}\}$ is a countable subset of $X$, because by $(*)$, each $\mathscr{D}_n$ is countable. We will conclude the proof by proving that $C$ is also dense in $X$.

Because each $\cup\mathscr{D}_n$ is an open dense subset of $X$, the set $E = \cap\{\cup\mathscr{D}_n: n \in \boldsymbol{N}\}$ is also a dense subset of $X$.

Now consider an arbitrary point $e \in E$. For each $n \in \boldsymbol{N}$ there exists a set $I(k, np_D) \in \mathscr{D}_n$ which contains $e$. Thus each $d(e, np_D) < 1/n$, which shows that $e \in \bar{C}$.

Therefore, $E \subset \bar{C}$ and so $X = \bar{E} = \bar{C}$.

While not every Baire semimetrizable-closed space is regular (e.g., Example 6), R. W. Heath has informed the author that he can prove every regular, semimetrizable-closed space is a Moore space — to verify Heath's result, appeal to the characterizations $A$ and $B'$ in **[4]** and the well known fact that in a regular feebly compact space any countable open filter base with a unique adherent point is convergent.

Since every separable first countable Hausdorff space has cardinality $\leqq c$, it follows from Theorem 10 that every Baire semimetrizable-closed space has cardinality $\leqq c$. We will conclude by showing that if the conditions "Hausdorff, semimetrizable, and Baire" are deleted, then the bound $c$ may be exceeded.

EXAMPLE 11. Let $m$ be an arbitrary infinite cardinal number, let $\mathcal{M}_m$ be a maximal family of countably infinite subsets of $m$ such that the intersection of any two members is finite. Denote by $\{p_M : M \in \mathcal{M}_m\}$ a set of distinct point not in $m$, and let $X_m = m \cup \{p_M : M \in \mathcal{M}_m\}$. For each $M \in \mathcal{M}_m$ let $g_M : M \to N$ be one-to-one. Define $d : X_m \times X_m \to [0, \infty)$ by the rule

$$d(x,y) = d(y,x) = \begin{cases} 1 & \text{if } x, y \in m \text{ and } x \neq y; \\ 1 & \text{if } x = p_M \text{ and } y \notin \{p_M\} \cup M; \\ 1/g_M(y) & \text{if } x = p_M \text{ and } y \in M; \text{ and} \\ 0 & \text{if } x = y. \end{cases}$$

Topologize $X_m$ by declaring a set $V$ to be open if and only if for each point $v \in V$ there exists $e > 0$ with $\{x \in X_m : d(x, v) < e\} \subset V$. *Then the space $X_m$ is a feebly compact symmetrizable space of cardinality $\geqq m$.*

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