

ON COMPLETENESS OF THE BERGMAN METRIC AND ITS SUBORDINATE METRICS, II

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Let M be a complex manifold of dimension n furnished with both the Bergman metric and the Carathéodory distance. The main result of the present paper is to prove that the Bergman metric is always greater than or equal to the Carathéodory distance on M . The case where M is a bounded domain in the space C^n was already considered by the author in Proc. Nat. Acad. Sci. (U.S.A.), 73 (1976), 4294.

1. Introduction. The main purpose of the present paper is to prove the following

THEOREM A. *Let M be a complex manifold which admits both the Bergman metric s_M and the Carathéodory differential metric α_M . For each $z \in M$ and each holomorphic tangent vector ξ ,*

$$(1) \quad \alpha_M(z, \xi) \leq s_M(z, \xi).$$

Let ρ_M and d_M denote the integrated metrics on M which are induced from α_M and s_M , respectively. Then the Carathéodory distance c_M ([2]) satisfies

$$(2) \quad c_M \leq \rho_M \leq d_M$$

and there are cases when ρ_M differs from c_M and d_M .

From this observation and Theorem A, we obtain

THEOREM B. *Let M be a complex manifold given as in Theorem A. Then the Bergman metric is complete in M whenever the Carathéodory distance is complete.*

If in particular M is a bounded domain in the complex Euclidean space C^n ($n \geq 1$), M always admits the Bergman metric and the Carathéodory differential metric.

Theorems A and B have a number of interesting consequences.

In [4], C. Earle has proved the completeness of the Carathéodory distance in the Teichmüller space $T(g)$ of a compact Riemann surface of genus $g \geq 2$. Therefore, Theorem B immediately implies the following

THEOREM C. *In the Teichmüller space $T(g)$ of any compact Riemann surface of genus $g \geq 2$, the Bergman metric is complete.*

Recently, S. Wolpert [11] and T. Chu have independently proved that the Weil–Petersson metric is not complete in $T(g)$. Therefore, we have the following

THEOREM D. *In the Teichmüller space $T(g)$ of any compact Riemann surface of genus $g \geq 2$, the Weil–Petersson metric is not uniformly equivalent to the Bergman metric.*

Finally we have

THEOREM E. *Let G be a bounded open connected subset of a separable complex Hilbert space X of finite or infinite dimension, and let M be a complex manifold of finite dimension which admits s_M . If G is homogeneous, then there exists a constant, depending only on G , such that for any holomorphic mapping $f: M \rightarrow G$*

$$(3) \quad \alpha_G(f(z), Df(z)\xi) \leq k(G)s_M(z, \xi) \quad (z \in M, \xi \in C^n),$$

where $Df(z)$ denotes the Frechét derivative of f at $z \in M$.

If in particular G is a ball, B , in X , then

$$(4) \quad \alpha_B(f(z), Df(z)\xi) \leq s_M(z, \xi).$$

Theorem E contains Theorem A as a special case when B is the unit disc in the complex plane C .

2. The kernel form and invariant metric of Bergman. The theory of the Bergman kernel function and invariant metric on a bounded domain in the space C^n has been extended to a complex manifold by S. Kobayashi [7] and also by A. Lichnerowicz [8].

Let $\mathcal{F}(M)$ be the set of holomorphic n -forms

$$\alpha = a dz_1 \wedge \cdots \wedge dz_n$$

on M such that

$$(1) \quad \left| \int_M \alpha \wedge \bar{\alpha} \right| < \infty.$$

Then $\mathcal{F}(M)$ is a separable complex Hilbert space with an inner product

given by

$$(2) \quad (\alpha, \beta) = i^{n^2} \int_M \alpha \wedge \bar{\beta} \quad (\alpha, \beta \in \mathcal{F}(M)).$$

Let $\{\varphi_0, \varphi_1, \dots\}$ be an orthonormal basis for \mathcal{F} . Then every $\alpha \in \mathcal{F}$ may be represented uniquely by the convergent series

$$(3a) \quad \alpha(z) = \sum_{\nu=0}^{\infty} c_{\nu} \varphi_{\nu}(z), \quad c_{\nu} = (\alpha, \varphi_{\nu}),$$

or

$$(3b) \quad a_U(z) = \sum_{\nu=0}^{\infty} c_{\nu} (\Phi_{\nu})_U(z),$$

where $\varphi_{\nu} = (\Phi_{\nu})_U dz_1 \wedge \dots \wedge dz_n$, in a local coordinate neighborhood U of $z \in M$.

Moreover,

$$(4) \quad (\alpha, \alpha) = \|\alpha\|^2 = \sum_{\nu=0}^{\infty} |c_{\nu}|^2.$$

Let V be a local coordinate neighborhood of $\zeta \in M$ in which $\varphi_{\nu}(\zeta) = (\Phi_{\nu})_V(\zeta) d\zeta_1 \wedge \dots \wedge d\zeta_n$. Then the series

$$(5) \quad i^{n^2} \sum_{\nu=0}^{\infty} \varphi_{\nu}(z) \wedge \overline{\varphi_{\nu}(\zeta)} = i^{n^2} \sum_{\nu=0}^{\infty} (\Phi_{\nu})_U(z) \overline{(\Phi_{\nu})_V(\zeta)} dz_1 \wedge \dots \wedge dz_n \wedge \overline{d\zeta_1} \wedge \dots \wedge \overline{d\zeta_n}$$

converges absolutely and uniformly on every compact subset of $M \times \bar{M}$, where \bar{M} is the complex manifold conjugate to M , and hence, represents a holomorphic $2n$ -form on $M \times \bar{M}$. Moreover, the sum (5) is independent of choice of orthonormal basis. The Bergman kernel form is defined by the sum (5) and written as

$$(5a) \quad \kappa(z, \bar{\zeta}) = \kappa_{\zeta}(z) = i^{n^2} k(z, \bar{\zeta}) dz_1 \wedge \dots \wedge dz_n \wedge \overline{d\zeta_1} \wedge \dots \wedge \overline{d\zeta_n}$$

with a locally defined Bergman kernel function:

$$(5b) \quad k(z, \bar{\zeta}) = \sum_{\nu=0}^{\infty} (\Phi_{\nu})_U(z) \overline{(\Phi_{\nu})_V(\zeta)}, \quad (z, \zeta) \in U \times V.$$

Further we define the reduced kernel form by

$$(6) \quad K_\zeta(z) = k(z, \zeta) dz_1 \wedge \cdots \wedge dz_n.$$

As in the classical case, see [1], the reduced kernel form has the reproducing property of n -forms in \mathcal{F} . More precisely,

LEMMA 1. For any $\alpha \in \mathcal{F}$ with $\alpha(z) = a_U(z) dz_1 \wedge \cdots \wedge dz_n$,

$$(7) \quad a_U(z) = (\alpha, K_z) = i^{n^2} \int_M \alpha(t) \wedge K(z, \bar{t}) \quad (z \in M).$$

Proof. First we observe that for each fixed $z \in M$, $K_z(t)$ is a holomorphic n -form in M . From the uniform convergence of the series (3a) and (5),

$$\begin{aligned} (\alpha, K_z) &= \left(\sum_\nu c_\nu \varphi_\nu, \sum_\mu \overline{\Phi_\mu(z)} \varphi_\mu \right) \\ &= \sum_\nu c_\nu \left(\varphi_\nu, \sum_\mu \overline{\Phi_\mu(z)} \varphi_\mu \right) \\ &= \sum_\nu c_\nu \sum_\mu \Phi_\mu(z) (\varphi_\nu, \varphi_\mu) = \sum_\nu c_\nu \Phi_\nu(z) = a_U(z). \end{aligned}$$

Setting in Lemma 1 $\alpha = K_\zeta$, $\zeta \in M$, we have

$$(8) \quad k_\zeta(z) = (K_\zeta, K_z) = \overline{(K_z, K_\zeta)} = \overline{k_z(\zeta)}.$$

In particular, $k_z(z) \geq 0$. $k_z(z) > 0$ holds whenever M satisfies

(A1) For any z in M , there is an $\alpha \in \mathcal{F}(M)$ such that $\alpha(z) \neq 0$. In this case,

$$(9) \quad s^2(z, \xi) = \sum_{\alpha, \beta=1}^n \frac{\partial^2 \log k(z, \bar{z})}{\partial z_\alpha \partial \bar{z}_\beta} \xi_\alpha \bar{\xi}_\beta \quad (z \in M, \xi \in C^n)$$

is a well-defined positive semidefinite hermitian form which is invariant under biholomorphic mappings of M . In fact $s^2(z, \xi)$ is positive definite if and only if M satisfies

(A2) For every holomorphic tangent vector ξ at $z \in M$, there is an $\alpha \in \mathcal{F}(M)$ such that $\alpha(z) = 0$ and

$$da \cdot \xi = \sum_{\mu=1}^n \frac{\partial a}{\partial z_\mu}(z) \xi_\mu \neq 0,$$

where $\alpha = a dz_1 \wedge \cdots \wedge dz_n$.

Therefore, any complex manifold M with properties (A1) and (A2) is entitled to an invariant Kähler metric s_M of Bergman.

3. An extension of Schwarz inequality. Let $\mathcal{M}(\Omega)$ be the set of square integrable n -forms defined on a measurable subset Ω of a complex manifold M of dimension n . Then $\mathcal{M}(\Omega)$ is a separable complex Hilbert space with respect to the inner product:

$$(1) \quad (\alpha, \beta)_\Omega = i^{n^2} \int_\Omega \alpha \wedge \bar{\beta} \quad (\alpha, \beta \in \mathcal{M}(\Omega)).$$

We need the following extension of the Schwarz inequality.

LEMMA 2. Let $\{\alpha_\nu\}$ and $\{\beta_\nu\}$ be two sequences (finite or infinite) from $\mathcal{M}(\Omega)$ such that

$$(2) \quad \sum_\nu (\alpha_\nu, \alpha_\nu)_\Omega < \infty, \quad \sum_\nu (\beta_\nu, \beta_\nu)_\Omega < \infty.$$

Then

$$(3) \quad M^*M \leq N \cdot \sum_\nu (\alpha_\nu, \alpha_\nu)_\Omega$$

where “ \leq ” denotes the matrix inequality, i.e., $A \leq B$ if and only if $B - A$ is positive semidefinite, M and N the matrices whose entries are $M_{\mu\nu} = (\alpha_\mu, \beta_\nu)_\Omega$ and $N_{\mu\nu} = (\beta_\mu, \beta_\nu)_\Omega$ ($\mu, \nu = 0, 1, 2, \dots$), respectively, and M^* the adjoint of M .

Proof. It is enough to prove the case where $\{\alpha_\nu\}$ and $\{\beta_\nu\}$ are infinite sequences. The other cases can be proved in the same way. Let $u = (u_0, u_1, \dots)$ be any non-zero constant vector in $\ell^2(C)$. Then

$$(4) \quad \begin{aligned} u^* M^* M u &= \sum_{\mu=0}^{\infty} \left(\sum_{\nu=0}^{\infty} M_{\mu\nu} u_\nu \right)^* \left(\sum_{\nu=0}^{\infty} M_{\mu\nu} u_\nu \right) \\ &= \sum_{\mu=0}^{\infty} \left| \left(\alpha_\mu, \sum_{\nu=0}^{\infty} \beta_\nu \bar{u}_\nu \right)_\Omega \right|^2 \end{aligned}$$

By the Schwarz inequality in $\mathcal{M}(\Omega)$, (4) becomes

$$\begin{aligned}
 (5) \quad u^* M^* M u &\leq \sum_{\mu=0}^{\infty} (\alpha_{\mu}, \alpha_{\mu})_{\Omega} \left(\sum_{\nu=0}^{\infty} \beta_{\nu} \bar{u}_{\nu}, \sum_{\tau=0}^{\infty} \beta_{\tau} \bar{u}_{\tau} \right)_{\Omega} \\
 &\leq u^* N u \sum_{\mu=0}^{\infty} (\alpha_{\mu}, \alpha_{\mu})_{\Omega}
 \end{aligned}$$

from which (3) follows, since u was arbitrary.

In the case where $M = C^n$ and Ω is a measurable subset of C^n , we define $\mathcal{M}(\Omega)$ to be the set of square integrable functions on Ω . Lemma 2 then holds in this case. We shall state it separately for the future use.

COROLLARY 1. *Let $\{a_{\nu}\}$ and $\{b_{\nu}\}$ be two sequences (finite or infinite) from $\mathcal{M}(\Omega)$, $\Omega \subset C^n$, such that*

$$(6) \quad \sum_{\nu} (a_{\nu}, a_{\nu})_{\Omega} < \infty, \quad \sum_{\nu} (b_{\nu}, b_{\nu})_{\Omega} < \infty$$

Then

$$(7) \quad M^* M \leq N \cdot \sum_{\nu} (a_{\nu}, a_{\nu})_{\Omega},$$

where M and N are matrices whose entries are $(a_{\mu}, b_{\nu})_{\Omega}$ and $(b_{\mu}, b_{\nu})_{\Omega}$ ($\mu, \nu = 0, 1, 2, \dots$), respectively.

4. The main theorems.

THEOREM 1. *Let $f = (f_0, f_1, \dots)$ be a holomorphic mapping from a complex manifold M satisfying properties (A1) and (A2) of §2 into a separable complex Hilbert space X of finite or infinite dimension such that*

$$(1) \quad \|f(z)\|_X \leq Q \quad \text{for some } Q > 0.$$

Then

$$(2) \quad \|Df(z)\xi\|_X \leq Q s_M(z, \xi) \quad (z \in M, \xi \in C^n),$$

where $\|\cdot\|_X$ denotes the usual norm in X .

Proof. For each $z \in M$, let

$$(3) \quad \alpha_{\mu}(t) = f_{\mu}(t) K_z(t) = f_{\mu}(t) k_z(t) dt_1 \wedge \dots \wedge dt_n \quad (\mu = 0, 1, 2, \dots)$$

$$(4) \quad \beta_{\nu}(t) = \frac{\partial}{\partial \bar{z}_{\nu}} \left(\frac{K_z(t)}{k_z(z)} \right) = \frac{1}{k_z^2(z)} \left(k_z(z) \frac{\partial K_z(t)}{\partial \bar{z}_{\nu}} - K_z(t) \frac{\partial k_z(z)}{\partial \bar{z}_{\nu}} \right)$$

($\nu = 1, 2, \dots, n$)

where

$$(5) \quad \frac{\partial K_z(t)}{\partial \bar{z}_\nu} = \frac{\partial k(t, \bar{z})}{\partial \bar{z}_\nu} dt_1 \wedge \cdots \wedge dt_n.$$

In view of the reproducing property of the kernel form, see Lemma 1, we obtain

$$(6) \quad \sum_\mu (\alpha_\mu, \alpha_\mu) = \sum_\mu (f_\mu K_z, f_\mu K_z) = \left(\sum_\mu f_\mu \bar{f}_\mu K_z, K_z \right) \leq Q^2(K_z, K_z) = Q^2 k(z, \bar{z}),$$

$$(7a) \quad (\beta_\mu, \beta_\nu) = \frac{1}{k^4(z)} \left\{ k_z^2(z) \left(\frac{\partial K_z}{\partial \bar{z}_\mu}, \frac{\partial K_z}{\partial \bar{z}_\nu} \right) - k_z(z) \frac{\partial k_z(z)}{\partial z_\nu} \left(\frac{\partial K_z}{\partial \bar{z}_\mu}, K_z \right) - k_z(z) \frac{\partial k_z(z)}{\partial \bar{z}_\mu} \left(K_z, \frac{\partial K_z}{\partial \bar{z}_\nu} \right) + \frac{\partial k_z(z)}{\partial \bar{z}_\mu} \frac{\partial k_z(z)}{\partial z_\nu} (K_z, K_z) \right\}.$$

From Lemma 1, we also have

$$(8a) \quad \left(\frac{\partial K_z}{\partial \bar{z}_\mu}, \frac{\partial K_z}{\partial \bar{z}_\nu} \right) = \frac{\partial^2}{\partial z_\nu \partial \bar{z}_\mu} (K_z, K_z) = \frac{\partial^2 k(z, \bar{z})}{\partial z_\nu \partial \bar{z}_\mu}$$

$$(8b) \quad \left(\frac{\partial K_z}{\partial \bar{z}_\mu}, K_z \right) = \frac{\partial}{\partial \bar{z}_\mu} (K_z, K_z) = \frac{\partial}{\partial \bar{z}_\mu} k(z, \bar{z}).$$

Therefore, (7a) becomes

$$(7b) \quad (\beta_\mu, \beta_\nu) = \frac{1}{k^3(z, \bar{z})} \left[k(z, \bar{z}) \frac{\partial^2 k(z, \bar{z})}{\partial z_\nu \partial \bar{z}_\mu} - \frac{\partial k(z, \bar{z})}{\partial z_\nu} \frac{\partial k(z, \bar{z})}{\partial \bar{z}_\mu} \right] = \frac{1}{k(z, \bar{z})} \frac{\partial^2}{\partial z_\nu \partial \bar{z}_\mu} \log k(z, \bar{z}).$$

$$(9) \quad (\alpha_\mu, \beta_\nu) = \left(\alpha_\mu, \frac{\partial}{\partial \bar{z}_\nu} \left(\frac{K_z}{k_z(z)} \right) \right) = \frac{\partial}{\partial z_\nu} \left(\alpha_\mu, \frac{K_z}{k_z(z)} \right) = \left(\frac{\partial f_\mu}{\partial z_\nu} \right) (z).$$

From Lemma 2 applied to $\mathcal{F}(M)$, together with (6), (7b), (9) and (9) of §2, Theorem 1 follows.

Let $\mathcal{H}(M, B^m)$ be the set of all holomorphic mappings f of a complex manifold M into the unit ball B^m in the space C^m ($1 \leq m \leq \omega$). Following H. Reiffen [10] we define

$$(10) \quad \alpha_M^{(m)}(z, \xi) = \sup \{ \|Df(z)\xi\|_m : f \in \mathcal{H}(M, B^m) \}$$

for $(z, \xi) \in M \times C^m$, where B^ω denotes the unit ball in the Hilbert space $C^\omega = \ell^2(C)$ with the usual ℓ^2 -norm.

It is easy to see that $\alpha_M^{(m)}$ is a pseudo differential metric in the sense of Grauert and Reckziegel [5], and that $\alpha_M^{(m)}$ becomes a differential metric whenever M satisfies the properties (A1) and (A2) of §2 by bounded mappings in the class $\mathcal{H}(M, B^m)$. We note that $\alpha_M^{(1)} = \alpha_M$ is the Carathéodory differential metric of H. Reiffen [10]. However, it turns out that for all $m, 1 \leq m \leq \omega$, $\alpha_M^{(m)}$ coincide with α_M , as it is seen in the following.

LEMMA 3. *Let M be a complex manifold of dimension n . For each $z \in M$ and each $\xi \in C^n$,*

$$(11) \quad \alpha_M^{(m)}(z, \xi) = \alpha_M^{(\omega)}(z, \xi) \quad \text{for all } m \geq 1.$$

Proof. Suppose that $f = (f_1, f_2, \dots, f_m) \in \mathcal{H}(M, B^m)$. Then $\tilde{f} = (f, 0) = (f_1, \dots, f_m, 0, 0, \dots)$ is a holomorphic mapping of M into B^ω . Let

$$\tilde{\mathcal{H}}(M, B^\omega) = \{\tilde{f}: \tilde{f} = (f, 0), f \in \mathcal{H}(M, B^m)\}.$$

Then

$$\tilde{\mathcal{H}}(M, B^\omega) \subset \mathcal{H}(M, B^\omega) \quad \text{and} \quad \|Df(z)\xi\|_m = \|D\tilde{f}(z)\xi\|_\omega.$$

Therefore,

$$(12) \quad \begin{aligned} \alpha_M^{(m)}(z, \xi) &= \sup\{\|Df(z)\xi\|_m : f \in \mathcal{H}(M, B^m)\} \\ &= \sup\{\|D\tilde{f}(z)\xi\|_\omega : \tilde{f} \in \tilde{\mathcal{H}}(M, B^\omega)\} \\ &\leq \alpha_M^{(\omega)}(z, \xi). \end{aligned}$$

The opposite inequality follows from the following observation.

$$(13) \quad \begin{aligned} \|Df(z) \cdot \xi\|_\omega &= \sup\{|\ell(Df(z) \cdot \xi)| : \ell \in \ell^2(C)^*, \|\ell\| = 1\} \\ &= \sup\{|\ell(D(\ell \cdot f)(z) \cdot \xi)| : \ell \in \ell^2(C)^*, \|\ell\| = 1\} \\ &\leq \alpha_M(z, \xi), \end{aligned}$$

where $\ell^2(C)^*$ denotes the dual of $\ell^2(C)$.

The second half of Lemma 3 is due to Clifford Earle (by communication) to whom the author is indebted.

It should be pointed out that the method of the proof of Theorem 1 is essentially due to K. H. Look [9]. In fact, he has proved Theorem 1 for the case when M is a bounded domain in C^n and $X = C^n$. However,

K. H. Look did not seem to realize Lemma 3 which enabled us to relate Theorem 1 to the Carathéodory distance.

Theorem A is now an immediate corollary of Theorem 1, or rather a special case of Theorem 1.

Proof of Theorem A. Set $X = C$ and $Q = 1$ in Theorem 1. Then (2) becomes

$$(14) \quad |Df(z)\xi| \leq s(z, \xi) \quad (z \in M, \xi \in C)$$

for all $f \in \mathcal{H}(M, B^1)$, and Theorem A follows.

Proof of Theorem E. Let x_0 be any fixed point in G and let $\gamma: G \rightarrow G$ be a holomorphic automorphism of G such that $\gamma(x) = x_0$, where $x = f(z)$, $z \in M$. Then $\gamma \cdot f$ is a holomorphic mapping of M into G such that $(\gamma \cdot f)(z) = x_0$. Let Q be the radius of the smallest ball in X which contains G . We may assume that the center of this ball lies at the origin. By Theorem 1,

$$(15a) \quad \|D(\gamma \cdot f)(z)\xi\|_X \leq Qs(z, \xi), \quad (z \in M, \xi \in C^n).$$

It is known [3] that if G is bounded then there are two positive continuous functions λ and Λ in G such that

$$(16) \quad \lambda(x)\|\xi\|_X \leq \alpha_G(x, \xi) \leq \Lambda(x)\|\xi\|_X \quad (x \in G),$$

for each $\xi \in X$. Set $\eta = Df(z)\xi$. Then (15a) becomes

$$(15b) \quad \|D\gamma(x)\eta\|_X \leq Qs(z, \xi), \quad x = f(z).$$

By the invariant property of the Carathéodory differential metric α_G under biholomorphic mappings of G , see [3],

$$(17) \quad \alpha_G(x, \eta) = \alpha_G(\gamma(x), D\gamma(x)\eta) = \alpha_G(x_0, D\gamma(x)\eta).$$

From the second half of (16), (17), and (15b),

$$(15c) \quad \alpha_G(f(z), Df(z)\xi) \leq \Lambda(x_0)Qs(z, \xi).$$

The first half of Theorem E follows from (15c) when we set

$$(18) \quad k(G) = Q \inf_{x \in G} \Lambda(x).$$

If in particular G is a ball, say $B = \{x \in X: \|x\|_x < R\}$, $R > 0$, then $Q = R$ and inequalities (16) may be reduced to

$$(19) \quad \frac{\|\xi\|_x}{\sqrt{R^2 - \|x\|_x^2}} \cong \alpha_B(x, \xi) \cong \frac{R\|\xi\|_x}{R^2 - \|x\|_x^2} \quad (x \in B, \xi \in X),$$

see [3]. Therefore, $k(G) = 1$ in (18) which proves the rest of Theorem E.

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