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GAUGE GROUPS AND CLASSIFICATION OF BUNDLES WITH SIMPLE STRUCTURAL GROUP

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Suppose π_i $i = 1, 2$ are principal K-bundles which are C^r-isomorphic in the sense that there exists a K-equivariant C^r-diffeomorphism $f: \mathcal{P}_1 \to \mathcal{P}_2$. If h belongs to the gauge group H_2 of $\mathcal{P}_2$ then $h \circ f$ lies in H_1 and we have a group isomorphism $H_2 \to H_1$ which is C^∞. It is the purpose of this paper to investigate the converse in the case where K is a simple Lie group. (If K is abelian the gauge group of every K bundle over X is $C^r(X, K)$ so there is no hope of a converse. However for simple groups the situation is much better).

0. Introduction. Let K be a compact connected Lie group with Lie algebra $\mathcal{K}$. Let $\pi: \mathcal{P} \to X$ be a principal K-bundle of class C^∞ where X is a compact, connected C^∞-manifold.

Throughout this paper r will be a positive integer which is chosen at this time and remains unchanged from here on.

We denote by H the subgroup of $C^r(\mathcal{P}, K)$ consisting of all those h for which $h(pk) = k^{-1}h(p)k$ for all p in $\mathcal{P}$ and $k \in K$. H is naturally isomorphic to the group of all C^r-bundle automorphisms of $\mathcal{P}$ which cover the identity on X **[1, 2]**. The group H will be called the gauge group of π the terminology being motivated by current usage in theoretical physics. $C^r(\mathcal{P}, K)$ is a Banach Lie group and H is a sub-manifold and so H is a Banach Lie group **[2]**. The Lie algebra of H can be identified as $\mathcal{H} = \{h: \mathcal{P} \to \mathcal{K} \mid h \text{ is } C^r \text{ and } h(pk) = Ad(k^{-1})h(p) \text{ for } p \in \mathcal{P},\ k \in K\}$.

The bracket in $\mathcal{H}$ and the exponential map exp: $\mathcal{H} \to H$ are the natural pointwise operations.

1. Ideals in $\mathcal{H}$. Suppose $\mathcal{I} \subset \mathcal{H}$ is an ideal. For $p \in \mathcal{P}$ $e_p: \mathcal{H} \to \mathcal{K}$ is defined by $e_p(h) = h(p)$ for $h \in \mathcal{H}$. e_p is a Lie algebra epimorphism so $e_p(\mathcal{I})$ is an ideal in $\mathcal{K}$.

LEMMA 1.1. *If $p \in \mathcal{P}$ and $k \in K$ then $e_p(\mathcal{I}) = e_{pk}(\mathcal{I})$.*

Proof. $e_{pk}(h) = h(pk) = Ad(k^{-1})h(p) = Ad(k^{-1})e_p(h)$. Thus $e_{pk}(\mathcal{I}) = Ad(k^{-1})e_p(\mathcal{I})$. But $e_p(\mathcal{I})$ is an ideal in $\mathcal{K}$ so $Ad(k^{-1})e_p(\mathcal{I}) = e_p(\mathcal{I})$.

DEFINITION 1.2. If $x \in X$ let $\mathcal{K}_x = e_p(\mathcal{I})$ where $p \in \pi^{-1}(x)$.

DEFINITION 1.3. If $\mathcal{I}$ is an ideal in $\mathcal{H}$ we say $\mathcal{I}$ has property s if $[\mathcal{I}, \mathcal{H}] = \mathcal{I}$.

We recall that $[\mathcal{I}, \mathcal{H}]$ is the Lie subalgebra of $\mathcal{H}$ generated by all elements of the form $[a, b]$ where $a \in \mathcal{I}$, $b \in \mathcal{H}$. $[\mathcal{I}, \mathcal{H}]$ consists exactly of all finite sums $\Sigma_i [a_i, b_i]$, $a_i \in \mathcal{I}$, $b_i \in \mathcal{H}$.

We denote by $\mathcal{F}(X)$ the algebra of C^r, real valued functions on X. $\mathcal{H}$ is a module over $\mathcal{F}(X)$ for if $f \in \mathcal{F}(X)$ and $h \in \mathcal{H}$ define $fh: \mathcal{P} \rightarrow \mathcal{K}$ by $(fh)(p) = f(\pi(p))h(p)$. One easily sees fh lies in $\mathcal{H}$ so we have a module.

LEMMA 1.4. *If the ideal $\mathcal{I} \subset \mathcal{H}$ has property s then $\mathcal{I}$ is a $\mathcal{F}(X)$-submodule of $\mathcal{H}$.*

Proof. Let $h \in \mathcal{I}$, $\phi \in \mathcal{F}(X)$. We show $\phi h \in \mathcal{I}$. $\mathcal{I}$ has property s so we may write $h = \Sigma_i [h_i, f_i]$ where $h_i \in \mathcal{I}$ and $f_i \in \mathcal{H}$. Then $\phi h = \Sigma_i \phi [h_i, f_i] = \Sigma_i [h_i, \phi f_i] \in \mathcal{I}$ where we used the pointwise nature of the bracket to get the last equation.

LEMMA 1.5. *If $\mathcal{H}_1$ and $\mathcal{H}_2$ correspond to bundles π_1 and π_2 and $\psi: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a Lie algebra isomorphism then if $\mathcal{I}$ has property s in $\mathcal{H}_1$ then $\psi(\mathcal{I})$ has property s in $\mathcal{H}_2$.*

Before proving the final lemma of this section we make a preliminary construction. Suppose U is open in X and ξ is a section of π over U. Suppose $h \in \mathcal{H}$ and h has support in $\pi^{-1}(U)$. Define $\bar{h}: X \rightarrow \mathcal{K}$ by,

$$\bar{h}(x) = \begin{cases} h(\xi(x)) & x \in U \\ 0 & x \notin U. \end{cases}$$

$\bar{h} \in C^r(X, \mathcal{K})$ has support in U. Conversely if we start with $\bar{h}: X \rightarrow \mathcal{K}$ having support in U we can define $h \in \mathcal{H}$ as follows. There is a unique C^∞-map $\theta: \pi^{-1}(U) \rightarrow \mathcal{K}$ such that $\xi(\pi(p))\theta(p) = p$ for $p \in \pi^{-1}(U)$. We define

$$h(p) = \begin{cases} Ad(\theta(p)^{-1})\bar{h}(\pi(p)) & p \in \pi^{-1}(U) \\ 0 & p \notin \pi^{-1}(U). \end{cases}$$

It is easily checked that $h \in \mathcal{H}$.

If $x_0 \in X$ we have:

$$H_{x_0} = \{f \in H \mid f(p) = e \quad \text{for all} \quad p \in \pi^{-1}(x_0)\}.$$

$$\mathcal{H}_{x_0} = \{h \in \mathcal{H} \mid h(p) = 0 \quad \text{for all} \quad p \in \pi^{-1}(x_0)\}.$$

LEMMA 1.6. *Assume $\mathcal{K}$ is semisimple. Then $\mathcal{H}_{x_0}$ has property s.*

Proof. Let $(\phi_i)_i$ be a finite partition of unity on X subordinate to an open cover $(U_i)_i$ such that π is trivial over each U_i. Then if $h \in \mathcal{H}_{x_0}$ we have $h = \Sigma_i \phi_i h$ and each $\phi_i h \in \mathcal{H}_{x_0}$. Therefore the problem is reduced to proving the following: If $U \subset X$ is open such that π has a local section ξ defined on U and if $h \in \mathcal{H}_{x_0}$ has support in $\pi^{-1}(U)$ then h can be written as $h = \Sigma_\nu [g_\nu, \phi_\nu]$ where $g_\nu \in \mathcal{H}_{x_0}$, $\phi_\nu \in \mathcal{H}$.

Let $\bar{h}: X \to \mathcal{K}$ correspond to h using the section ξ as above. Let $(E_i)_i$ be a basis for $\mathcal{K}$. Write $\bar{h} = \Sigma_i \bar{h}^i E_i$ where $\bar{h}^i$ are real valued. Since $\mathcal{K}$ is semisimple we may write $E_i = \Sigma_j [F_{ij}, G_{ij}]$ where F_{ij}, G_{ij} are in $\mathcal{K}$. Therefore $h = \Sigma_{i,j} \bar{h}^i [F_{ij}, G_{ij}] = \Sigma_{i,j} [\bar{h}^i F_{ij}, G_{ij}] = \Sigma_\nu [\bar{g}_\nu, \bar{\phi}_\nu]$ where $\bar{g}_\nu$ and $\bar{\phi}_\nu: X \to \mathcal{K}$ are C^r with $\bar{g}_\nu(x_0) = 0$. We can easily arrange that $\bar{g}_\nu$ and $\bar{\phi}_\nu$ have support in U. Then let g_ν, ϕ_ν be the corresponding functions on $\mathcal{P}$. Then if $p \in \mathcal{P}$ with $\pi(p) = x$ we have,

$$h(p) = Ad(\theta(p)^{-1})\bar{h}(x) = Ad(\theta(p)^{-1})\Big(\sum_\nu [\bar{g}_\nu(x), \bar{\phi}_\nu(x)]\Big)$$

$$= \sum_\nu [Ad(\theta(p)^{-1})\bar{g}_\nu(x), Ad(\theta(p)^{-1})\bar{\phi}_\nu(x)]$$

$$= \sum_\nu [g_\nu(p), \phi_\nu(p)] = \Big(\sum_\nu [g_\nu, \phi_\nu]\Big)(p).$$

2. A classification theorem. In this section, in addition to the assumptions made in the introduction, we assume K is a simple Lie group with trivial center. We first make some observations.

Given a principal K-bundle $\pi: \mathcal{P} \to X$ we construct the associated fiber bundle $\mathcal{A} \to X$ with fiber $\mathcal{K}$ where K acts on $\mathcal{K}$ via the adjoint representation of K. Each $p \in \mathcal{P}$ with $\pi(p) = x$ gives a linear isomorphism $\phi_p: \mathcal{K} \to \mathcal{A}_x$. Since $Ad: K \to \mathrm{Lis}(\mathcal{K})$ actually takes values in $\mathrm{Aut}(\mathcal{K})$ we see $\mathcal{A}$ is a bundle of Lie algebras. Therefore $\Gamma^r(\mathcal{A})$, the space of C^r-sections of $\mathcal{A}$, is a Lie algebra with pointwise bracket. There is a natural isomorphism $\mathcal{H} \to \Gamma^r(\mathcal{A})$ given by $h \to \tilde{h}$ where $\tilde{h}(x) = \phi_p(h(p))$ for each $x \in X$ where $p \in \pi^{-1}(x)$ [3]. This isomorphism is an isomorphism of $\mathcal{F}(X)$-modules and is a homeomorphism with respect to the C^r-topologies.

Now suppose $\pi_i: \mathcal{P}_i \to X$ are principal K-bundles, $i = 1, 2$, with gauge groups H_i and $\mathcal{H}_i$ the Lie algebra of H_i. For $x_0 \in X$ the ideal $\mathcal{H}_{ix_0}$

is closed. Let $\psi: H_1 \to H_2$ be a C^1-group isomorphism. There is an induced Lie algebra isomorphism $\psi_*: \mathscr{H}_1 \to \mathscr{H}_2$ given by

$$\psi_*(h)(p) = \frac{d}{dt}\bigg|_{t=0} [\psi(\exp(th))](p)$$

ψ_* is a topological isomorphism and so for each $x_0 \in X$ $\psi_*(\mathscr{H}_{1x_0})$ is a closed ideal having property s in $\mathscr{H}_2$. If we write $\mathscr{I} = \psi_*(\mathscr{H}_{1x_0})$ and refer to the discussion of section 1 we have ideals $\mathscr{K}_x \subset \mathscr{K}$ for each $x \in X$. There are apparently two possible cases.

Case 1. $\mathscr{K}_x = \mathscr{K}$ for all $x \in X$.

We argue this cannot occur. Since $\mathscr{I}$ is an ideal with property s $\mathscr{I}$ is an $\mathscr{F}(X)$-submodule. If $\mathscr{K}_x = \mathscr{K}$ for all x in X we shall show $\mathscr{I} = \mathscr{H}_2$ which is impossible since $\mathscr{H}_{1x_0} \neq \mathscr{H}_1$. To show $\mathscr{I} = \mathscr{H}_2$ we regard $\mathscr{I}$ as a closed $\mathscr{F}(X)$-submodule of $\Gamma^r(\mathscr{A}_2)$. Then for $x \in X$, $v \in \mathscr{A}_{2x}$ there is $h \in \mathscr{I}$ for which $h(x) = v$. One now uses the $\mathscr{F}(X)$-module structure to show for any $x \in X$ and for any r-jet $\xi \in j^r_x\mathscr{A}_2$ there is an $h \in \mathscr{I}$ for which $j^r_x h = \xi$. Since $\mathscr{I}$ is a closed submodule we conclude $\mathscr{I} = \Gamma^r(\mathscr{A}_2)$ by applying a "global" version of a well-known theorem of Whitney. We refer to **[5]**, Corollary 1.6, p. 25.

Case 2. $\mathscr{K}_x = \mathscr{K}$ for some x.

In this case there is some x_1 for which $\mathscr{K}_{x_1} = (0)$ since K is simple. We claim there cannot be an $x_2 \neq x_1$, for which $\mathscr{K}_{x_2} = 0$. For if there were then we would have $\mathscr{I} \subset \mathscr{H}_{2x_1} \cap \mathscr{H}_{2x_2}$. But the codimension of $\mathscr{I}$ in $\mathscr{H}_2$ equals the codimension of $\mathscr{H}_{1x_0}$ in $\mathscr{H}_1$ which equals the codimension of $\mathscr{H}_{2x_1}$ in $\mathscr{H}_2$ so $\mathscr{I} \subset \mathscr{H}_{2x_1} \cap \mathscr{H}_{2x_2}$ is not possible. Therefore in the present case we see there is a unique $x_1 \in X$ for which $\mathscr{I} = \mathscr{H}_{2x_1}$.

Thus we see that a C^1 isomorphism $\psi: H_1 \to H_2$ gives rise to a bijection $\bar{\psi}: X \to X$ defined by

$$\psi_*(\mathscr{H}_{1x}) = \mathscr{H}_{2\bar{\psi}(x)}.$$

Now let $h \in \mathscr{H}_1$, $f \in \mathscr{F}(X)$. We have $\bar{\psi}: X \to X$ and we write $\bar{\psi}_*(f) = f \circ \bar{\psi}^{-1}$.

LEMMA 2.1. $\psi_*(fh) = \bar{\psi}_*(f)\psi_*(h)$.

Proof. Let $p_2 \in \mathscr{P}_{2x}$ let $\lambda = \bar{\psi}_*(f)(x)$. Then

$$\begin{aligned}\psi_*(fh)(p_2) &= \psi_*(fh - \lambda h)(p_2) + \psi_*(\lambda h)(p_2)\\ &= \psi_*((f-\lambda)h)(p_2) + \lambda\psi_*(h)(p_2).\end{aligned}$$

Let $x' = \bar{\psi}^{-1}(x)$ and let $p_1 \in \mathscr{P}_{1x'}$. Then $(f-\lambda)h(p_1) = (f(x')-\lambda)h(p_1) = 0$ by choice of λ. Thus $(f-\lambda)h \in \mathscr{H}_{1x'}$ and so $\psi_*((f-\lambda)h) \in \mathscr{H}_{2x}$ so $\psi_*((f-\lambda)h)(p_2) = 0$. Thus

$$\psi_*(fh)(p_2) = \lambda\psi_*(h)(p_2) = (\bar{\psi}_*(f)\cdot\psi_*(h))(p_2)$$

as desired.

LEMMA 2.2. *The map $\bar{\psi}$: $X \to X$ is a C^r-diffeomorphism.*

Proof. We need only show $\bar{\psi}^{-1}$ is C^r. It is enough to show that if $f \in \mathscr{F}(X)$ then $f \circ \bar{\psi}^{-1}$ is C^r. Choose $x_0 \in X$, U a neighborhood of x_0 $\mathscr{P}_2$ trivial over U. Then let V be a neighborhood of x_0 with $\bar{V} \subset U$. Let k be a section of $\mathscr{A}_2$ over U which in the local trivialization has constant principal part. We can then cut k down to get a new section, again called k, defined on all of X and agreeing with the original k on V. Then choose $h \in \Gamma^r(\mathscr{A}_1)$ such that $\psi_*(h) = k$. (We are identifying $\mathscr{H}_i$ and $\Gamma(\mathscr{A}_i)$). Now by Lemma we have $\psi_*(fh) = (f\circ\bar{\psi}^{-1})\psi_*(h) = (f\circ\bar{\psi}^{-1})k$. When we view the C^r-section $(f\circ\bar{\psi}^{-1})k$ in our local trivialization we conclude $f\circ\bar{\psi}^{-1}$ is C^r on V. So we conclude $f\circ\bar{\psi}^{-1}$ is C^r and hence $\bar{\psi}^{-1}$ is C^r.

We now define a bundle isomorphism $\tilde{\psi}$ such that the following commutes:

$$\begin{array}{ccc} \mathscr{A}_1 & \xrightarrow{\tilde{\psi}} & \mathscr{A}_2 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\bar{\psi}} & X \end{array}$$

Let $\alpha_x \in \mathscr{A}_{1x}$. Choose a section $h \in \Gamma^r(\mathscr{A}_1)$ such that $h(x) = \alpha_x$. Define $\tilde{\psi}(\alpha_x)$ by $\tilde{\psi}(\alpha_x) = \psi_*(h)(\bar{\psi}(x))$. This is independent of the choice of h for if h_1 were another section with $h_1(x) = \alpha_x$ then $h - h_1$ vanishes at x. Hence $\psi_*(h-h_1)$ vanishes at $\bar{\psi}(x)$ so $\psi_*(h)(\bar{\psi}(x)) = \psi_*(h_1)(\bar{\psi}(x))$. It is clear that the diagram commutes and that $\tilde{\psi}$ mapping $\mathscr{A}_{1x}$ to $\mathscr{A}_{2\bar{\psi}(x)}$ is a Lie algebra isomorphism.

LEMMA 2.3. *$\tilde{\psi}$ is C^r.*

Proof. We work locally trivializing $\mathscr{A}_1$. Let U be open in X, $V \subset U$ also open, γ: $U \times R^m \to \mathscr{A}_1 \mid U$ be a trivialization of $\mathscr{A}_1$ over U. Using this we see there are C^r-sections $h_1, \cdots, h_m \in \Gamma^r(\mathscr{A}_1)$ such that for each x in the subset V, $h_1(x), \cdots h_m(x)$ give a basis for the fiber over x which corresponds to the standard basis of R^m under γ. We claim

$\tilde{\psi} \circ \gamma : V \times R^m \to \mathscr{A}_2$ is given by

$$\tilde{\psi} \circ \gamma (x, \xi^1, \cdots, \xi^m) = \sum_{i=1}^{m} \xi^i \psi_*(h_i)(\bar{\psi}(x)).$$

If so then $\tilde{\psi}$ is C^r. But given $\xi^1, \cdots, \xi^m$ choose $f^i \in \mathscr{F}(X)$, $f^i(x) = \xi^i$. Then by Lemma 2.1 we see

$$\begin{aligned}\tilde{\psi}(\gamma(x, \xi^1, \cdots, \xi^m)) &= \tilde{\psi}\Big(\sum_{i=1}^{m} \xi^i h_i(x)\Big) = \tilde{\psi}\Big(\Big(\sum_{i=1}^{m} f^i h_i\Big)(x)\Big) \\ &= \psi_*\Big(\sum_{i=1}^{m} f^i h_i\Big)(\bar{\psi}(x)) \\ &= \sum_{i=1}^{m} \bar{\psi}_*(f^i)(\bar{\psi}(x))\psi_*(h_i)(\bar{\psi}(x)) \\ &= \sum_{i=1}^{m} \xi^i \psi_*(h_i)(\bar{\psi}(x)).\end{aligned}$$

Let $p \in \mathscr{P}_{1x}$. Then $\phi_p^1: \mathscr{K} \to \mathscr{A}_{1x}$ is a Lie algebra isomorphism. If $q \in \mathscr{A}_{2\bar{\psi}(x)}$ then we have a Lie algebra isomorphism $\phi_q^2: \mathscr{K} \to \mathscr{A}_{2\bar{\psi}(x)}$. (Note the superscripts tell which bundle is being used).

Now $(\phi_q^2)^{-1} \circ \tilde{\psi} \circ \phi_p^1: \mathscr{K} \to \mathscr{K}$ lies in $\mathrm{Aut}(\mathscr{K})$. Let $\mathscr{E} = \{(p, q) \mid p \in \mathscr{P}_{1x}$ and $q \in \mathscr{P}_{2\bar{\psi}(x)}$ for some $x \in X\}$. $\mathscr{E}$ is the total space of the fiber product of $\mathscr{P}_1$ and $\bar{\psi}^*\mathscr{P}_2$. We have a map $\rho : \mathscr{E} \to \mathrm{Aut}(\mathscr{K})$, $\rho(p, q) = (\phi_q^2)^{-1} \circ \tilde{\psi} \circ \phi_p^1$. ρ is continuous and $\mathscr{E}$ is connected so ρ takes values in one of the connected components of $\mathrm{Aut}(\mathscr{K})$. Since K is a simple group the identity component of $\mathrm{Aut}(\mathscr{K})$ is $\mathrm{Aut}^\circ(\mathscr{K}) = Ad(K)$. Suppose $\sigma \in \mathrm{Aut}(\mathscr{K})$ and that $\rho(E) \subset \mathrm{Aut}^\circ(\mathscr{K})\sigma = Ad(K)\sigma$. Let $q \in \mathscr{P}_2$, $k \in K$. Then $\phi_{qk}^2 = \phi_q^2 \circ Ad(k)$. So $\rho(p, qk) = Ad(k^{-1}) \circ \rho(p, q)$. We conclude that for each $p \in \mathscr{P}_{1x}$ there is a unique $\mu(p)$ in $\mathscr{P}_{2\bar{\psi}(x)}$ for which $\rho(p, \mu(p)) = \sigma$. We then have a map $\mu : \mathscr{P}_1 \to \mathscr{P}_2$ covering $\bar{\psi}$. K acts freely on the right of both $\mathscr{P}_1$ and $\mathscr{P}_2$. We now show there is an automorphism $\bar{\sigma}$ of K, induced by σ, such that if a new action of K on $\mathscr{P}_2$ is defined by $q * k = q\bar{\sigma}(k)$, (the right side being the original action) then μ becomes K-equivariant. We have $\sigma \in \mathrm{Aut}(\mathscr{K})$. $\tau \to \sigma\tau\sigma^{-1}$ is an automorphism of $\mathrm{Aut}(\mathscr{K})$ and hence restricts to an automorphism of $\mathrm{Aut}^\circ(\mathscr{K}) = Ad(K)$. Using the isomorphism $Ad: K \to Ad(K)$ we see a unique automorphism $\bar{\sigma}$ is induced. $\bar{\sigma}$ satisfies the equation $Ad(\bar{\sigma}(k)) = \sigma Ad(k)\sigma^{-1}$. Now we show $\mu(pk) = \mu(p) * k$ for $p \in \mathscr{P}_1$, $k \in K$. We need only show $\rho(pk, \mu(p) * k) = \sigma$. But

$$\begin{aligned}\rho(pk, \mu(p) * k) &= \rho(pk, \mu(p)\bar{\sigma}(k)) = Ad(\bar{\sigma}(k))^{-1} \circ \rho(p, \mu(p)) \circ Ad(k) \\ &= Ad(\bar{\sigma}(k))^{-1} \circ \sigma \circ Ad(k) = \sigma Ad(k)^{-1}\sigma^{-1}\sigma Ad(k) = \sigma\end{aligned}$$

so we are done.

DEFINITION 2.4. Let $\pi: \mathcal{P} \to X$ be a principal K-bundle, τ an automorphism of K. The principal K-bundle $\pi^\tau: \mathcal{P}^\tau \to X$ is defined by introducing the new action $*: \mathcal{P} \times K \to P$, $p * k = p\tau(k)$. We say π^τ is conjugate to π by τ.

Considering the previous discussion we have now proved

THEOREM 2.5. *Under the assumptions made above if* $\psi: H_1 \to H_2$ *is a* C^1 *isomorphism then there is a* C^r*-diffeomorphism* $\bar{\psi}: X \to X$ *and an automorphism* $\bar{\sigma}$ *of* K *such that* $\pi_1 \cong \bar{\psi}^*(\pi_2^{\bar{\sigma}})$.

REMARK. Of course if $\bar{\sigma}$ is an inner automorphism we get $\pi_2^{\bar{\sigma}} \cong \pi_2$ and $\bar{\sigma}$ can be dropped.

3. Classical groups. We apply the results of §2 to the groups $SO(2n+1)$ $n \geqq 1$, $U(n)$ $n \geqq 2$, and $SO(2n)$ $n \geqq 3$. Since the center of $SO(2n+1)$ is trivial and the automorphism group of its Lie algebra is connected [**6**, pages 285–6] we get

THEOREM 3.1. *Let* $\pi_i: \mathcal{P}_i \to X$ *be principal* $SO(2n+1)$ *bundles with gauge groups* H_i, $i = 1, 2$. *Suppose* $\psi: H_1 \to H_2$ *is a* C^1 (local) *isomorphism. Then there is a* C^r*-diffeomorphism* $\bar{\psi}: X \to X$ *so that* $\pi_1 \cong \bar{\psi}^*(\pi_2)$.

Now let K be $SO(2n)$ $n \geqq 3$ or $U(n)$ $n \geqq 2$, $\pi_i: \mathcal{P}_i \to X$ be principal K bundles with gauge groups H_i and $\psi: H_1 \to H_2$ a C^r local isomorphism. Let Z denote the center of K. Now $\hat{\mathcal{P}}_i = \mathcal{P}_i/Z$ is a principal K/Z bundle over X. Let $\hat{H}_i$ be the gauge group of $\hat{\mathcal{P}}_i$. In both cases ($SO(2n)$ and $U(n)$) one can show that the Lie algebra isomorphism $\psi_*: \mathcal{H}_1 \to \mathcal{H}_2$ gives Lie algebra isomorphism $\hat{\psi}_*: \hat{\mathcal{H}}_1 \to \hat{\mathcal{H}}_2$ and also that the center of K/Z is trivial. Thus the results of §2 give a C^r diffeomorphism $\phi: X \to X$ and an automorphism σ of K/Z so that $\hat{\pi}_1 \cong \phi^*(\hat{\pi}_2^\sigma)$. Note that if σ is an inner automorphism $\hat{\pi}_2^\sigma \cong \hat{\pi}_2$ so that σ can be dropped. The form of σ not inner is given in [**6**, page 287]. It can be seen that σ lifts to $\sigma: K \to K$ and that $(\mathcal{P}_i/Z)^\sigma = \mathcal{P}_i^\sigma/Z$. We thus get

THEOREM 3.2. *Let* K *be* $SO(2n)$ $n \geqq 3$ *or* $U(n)$ $n \geqq 2$, $\pi_i: \mathcal{P}_i \to X$ *be principal* K *bundles with gauge groups* H_i, $i = 1, 2$. *Suppose* $\psi: H_1 \to H_2$ *is a* (local) C^r *isomorphism. Then there is a* C^r *diffeomorphism* $\bar{\psi}: X \to X$ *and automorphism* $\sigma: K \to K$, *so that* $\mathcal{P}_1/Z \cong \bar{\psi}^*(\mathcal{P}_2 | Z)^\sigma \cong \bar{\psi}^*(\mathcal{P}_2^\sigma)/Z$ *where* Z *is the center of* K.

One can show that $\mathcal{P}_1$ is a "tensor product" of $\bar{\psi}^*(\mathcal{P}_2^\sigma)$ with a principal Z-bundle over X. One way to see this is to use the classification for bundles as given in **[4]**. We state the result in terms of associated vector bundles.

THEOREM 3.3. *Let* $\pi_i: \mathcal{P}_i \to X$ *be principal* SO$(2n)$ $n \geqq 3$ $(U(n)$ $n \geqq 2)$ *bundles with gauge groups* H_i, $i = 1, 2$. *Let* ξ_i *be the real* (complex) *vector bundle associated with* $\mathcal{P}_i$ *using the usual representation of* SO$(2n)(U(n))$. *Suppose* $\psi: H_1 \to H_2$ *is a* (local) C^1-*isomorphism then there is a* C^r *diffeomorphism* $\bar{\psi}: X \to X$, σ *an automorphism of* SO$(2n)(U(n))$, *and* η *a real* (complex) *line bundle so that* ξ_1 *is* SO$(2n)(U(n))$ *isomorphic to* $\bar{\psi}^*(\xi_2^\sigma) \otimes \eta$.

Final remark. We need not have assumed that $\mathcal{P}_1$ and $\mathcal{P}_2$ were bundles over the same manifold X. We could have considered $\pi_1: \mathcal{P}_1 \to X$ and $\pi_2: \mathcal{P}_2 \to Y$. If the gauge groups H_1 and H_2 are (locally) C^1 isomorphic we get a C^r-diffeomorphism $\bar{\psi}: X \to Y$.

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