PACIFIC JOURNAL OF MATHEMATICS
Vol. 69, No. 1, 1977

FINITE DIRECT SUMS OF CYCLIC VALUATED p-GROUPS

ROGER HUNTER, FRED RICHMAN AND ELBERT WALKER

The purpose of this paper is to characterize finite direct sums of cyclic valuated p-groups in terms of numerical invariants.

Our starting point is the subgroup problem for finite abelian p-groups. It follows from the proof of Ulm's theorem that an isomorphism f between two subgroups of a finite abelian p-group A is induced by an automorphism of A exactly when f preserves heights in A. This suggests a device for dealing with such subgroups—we give each element of a subgroup a value, namely, its height in the containing group. Now the containing group is dispensed with and we work only with the subgroup and its valuation, which we call a valuated p-group. Theorem 32 of [2] shows that all of the information pertaining to the original embedding is essentially captured by this process. There is a natural definition of a morphism of valuated p-groups, yielding the category $\mathscr{T}_p$ of finite valuated p-groups. We then observe that $\mathscr{T}_p$ has direct sums and set about characterizing the simplest objects in $\mathscr{T}_p$, the cyclics and their direct sums. The invariants for finite abelian p-groups are provided by functors from the category of abelian groups to the category of vector spaces, each functor picking out the number of cyclic summands of a given order in any decomposition of the group into cyclics. We carry out a parallel program for direct sums of cyclics in $\mathscr{T}_p$, obtaining in Theorem 2 a complete set of invariants. An example is then given to show that the objects of $\mathscr{T}_p$ are not all direct sums of cyclics. Thus arises the question of finding some criterion for a valuated p-group to be a direct sum of cyclics. We provide such a criterion in Theorem 3, making use of the functional invariants used for direct sums of cyclics. The paper concludes with a proof that p^2-bounded valuated groups in $\mathscr{T}_p$ are direct sums of cyclics. This bound is the best possible, as the example referred to above is bounded by p^3.

Valuations. Once and for all, fix a prime p. All groups considered will be finite abelian p-groups. Let A be such a group. Define $p^0A = A$ and $pA = \{pa: a \in A\}$. Clearly pA is a subgroup of A. The subgroups p^nA are defined inductively in the obvious way. The *height* $h(a)$ of a nonzero element a of A is defined by $h(a) = n$ where $a \in p^nA \backslash p^{n+1}A$. Set $h(0) = \infty$. We agree that $\infty < \infty$ and

$n < \infty$ for all $n = 0, 1, \cdots$. Thus h defines a function—the *height function*—from A to the set $\{0, 1, \cdots\} \cup \{\infty\}$. Note that $hpa > ha$, $h(a + b) \geqq \min\{ha, hb\}$, and $hna = ha$ if $(p, n) = 1$. Given a subgroup $A \subseteq B$, we value each element of A with its height in B. Formally, a *valuation* of A is a function $v: A \to \{0, 1, \cdots\} \cup \{\infty\}$ such that

(i) $vpa > va$
(ii) $vna = va$ if $(p, n) = 1$
(iii) $v(a + b) \geqq \min\{va, vb\}$
(iv) $va = \infty$ if and only if $a = 0$.

We call A together with a valuation a *valuated group*. A morphism $f: A \to B$ of valuated groups is a group homomorphism such that $va \leqq vfa$ for each a in A. We write $\mathscr{F}_p$ for the *category of* (finite) *valuated p-groups* so obtained. The direct sum of valuated groups A and B is the group direct sum $A \oplus B$ with valuation $v(a, b) = \min\{va, vb\}$.

Condition (i) ensures $h(a) \leqq v(a)$ for all elements a of A. If $A \in \mathscr{F}_p$ and has valuation v such that $v = h$, we say A is a *group*. A map $f: A \to B$ of $\mathscr{F}_p$ is an *embedding* if f is one-to-one and $va = vfa$ for all a in A. If A is a subgroup of a valuated group B and the inclusion $A \subseteq B$ is a embedding, then the quotient B/A is valuated by $v(b + A) = \max\{v(b + a): a \in A\}$.

The following theorem is a special case of Theorem 12, [1].

THEOREM 1. *Each A in $\mathscr{F}_p$ has an embedding in a group $\hat{A}$ of $\mathscr{F}_p$.*

Thus, each valuation on A arises by restricting the height function on some containing group.

Direct Sums of Cyclics. Each element a of a valuated group A determines the sequence

$$\bar{v}(a) = (va, vpa, vp^2a, \cdots)$$

called the *value sequence of* a. A valuated group is *cyclic* if it is cyclic as an abelian group. Clearly two cyclic valuated groups are isomorphic in $\mathscr{F}_p$ if and only if their generators have the same value sequence. We can express this fact without reference to generators in terms of functorial invariants which are defined for all valuated groups. A *value sequence* is an increasing sequence $(\alpha_1, \alpha_2, \cdots)$, $\alpha_i \in \{0, 1, \cdots\} \cup \{\infty\}$. Let $\mu = (\alpha_1, \alpha_2, \cdots)$ and $\nu = (\beta_1, \beta_2, \cdots)$ be value sequences. Define $\mu \geqq \nu$ if $\alpha_i \geqq \beta_i$ for $i = 1, 2, \cdots$, and $\mu > \nu$ if $\mu \geqq \nu$ and $\alpha_i \neq \beta_i$ for some i. Define

$$A(\mu) = \{a \in A: \bar{v}(a) \geqq \mu\}$$

and $A(\mu)^*$ to be the subgroup generated by the set $\{a \in A: \bar{v}(a) > \mu\}$. Notice that $A(\mu)$ and $A(\mu)^*$ are subgroups of A as an abelian group. Define $f(\mu, A)$ to be the rank of the vector space

$$(A(\mu) + pA)/(A(\mu)^* + pA) .$$

Obviously $f(\mu, A)$ is an invariant of A. If $A \oplus B$ is a direct sum in $\mathscr{T}_p$, then it is easy to show that

$$f(\mu, A \oplus B) = f(\mu, A) + f(\mu, B) .$$

As defined, value sequences are infinite. However, from this point on we adopt the convention of writing only those terms which are not ∞. Thus $(1, 2, 4, \infty, \infty, \cdots)$ is written $(1, 2, 4)$.

LEMMA 1. *Let A be a cyclic valuated group. Then*

$$f(\mu, A) = \begin{cases} 1 & \text{if } \mu \text{ is the value sequence of a generator of } A, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let ν be the value sequence of a generator z of A. If $\mu > \nu$ then $z \notin A(\mu)$ so $A(\mu) \subseteqq pA$. When $\mu \not\geqq \nu$ there is some entry of μ which is strictly smaller than the corresponding entry of ν. Since $\bar{v}(x) \geqq \nu$ for all x in A, $x \in A(\mu)$ implies $\bar{v}(x) > \mu$ so $A(\mu) = A(\mu)^*$. Thus, for $\mu \neq \nu$ we have $f(\mu, A) = 0$. On the other hand, if $\mu = \nu$ then $A(\mu) = A$ and $A(\mu)^* \subseteqq pA$ so $f(\mu, A) = \text{rank}\,(A/pA) = 1$.

Since the invariants $f(\mu, A)$ are additive, they form a complete set of invariants for direct sums of cyclics in $\mathscr{T}_p$.

THEOREM 2. *Two direct sums of cyclics A and B are isomorphic in $\mathscr{T}_p$ if and only if $f(\mu, A) = f(\mu, B)$ for all value sequences μ.*

By the *rank* $r(A)$ of a valuated group A we mean the rank of A as an abelian group. If A is a direct sum of cyclics in $\mathscr{T}_p$, it is an immediate consequence of Lemma 1 that

$$\sum_{\mu} f(\mu, A) = r(A) .$$

The converse is also true.

THEOREM 3. *Let $A \in \mathscr{T}_p$. Then A is a direct sum of cyclics if and only if $\sum_\mu f(\mu, A) = r(A)$.*

The proof of this theorem must wait until some theory of the

invariants $f(\mu, A)$ has been developed. The first step is a useful lemma whose proof is trivial.

LEMMA 2. *If $a \in A\backslash pA$ and $\bar{v}(a) = \mu$ is maximal in $\{\bar{v}(x): x \in A\backslash pA\}$ then $a \notin A(\mu)^* + pA$. In particular, $f(\mu, A) \neq 0$.*

Theorem 3 would be redundant if every object of $\mathscr{T}_p$ were a direct sum of cyclics. An example to the contrary is therefore in order.

EXAMPLE. A valuated group which is not a direct sum of cyclics. Let $B = \langle a\rangle \oplus \langle b\rangle \oplus \langle c\rangle$ where $0(a) = p^5$, $0(b) = p^3$, $0(c) = p$. Set $x = p^2a + c$, $y = p^2a + pb$ and let A be the subgroup of B generated by x and y with the induced valuation. It is easy to see that $\bar{v}(x) = (0, 3, 4)$, $\bar{v}(y) = (1, 2, 4)$, and $\bar{v}(x - y) = (0, 2)$ are all maximal among $\{\bar{v}(a): a \in A\backslash pA\}$. By Lemma 2, $\sum_\mu f(\mu, A) \geqq 3$. However, $r(A) = 2$ so A cannot be a direct sum of cyclics in $\mathscr{T}_p$.

In view of the above example, we define

$$d(A) = \sum_\mu f(\mu, A) .$$

By the modular law,

$$(A(\mu)^* + pA) \cap A(\mu) = A(\mu)^* + pA \cap A(\mu)$$

so that

$$\frac{A(\mu) + pA}{A(\mu)^* + pA} \cong \frac{A(\mu)}{A(\mu) \cap (A(\mu)^* + pA)} = \frac{A(\mu)}{A(\mu)^* + pA \cap A(\mu)} .$$

This isomorphism is natural, so it follows that each nonzero coset of $(A(\mu) + pA)/(A(\mu)^* + pA)$ has a representative with value sequence μ. For each value sequence μ, choose one representative in $A(\mu)$ for each element of a basis for $(A(\mu) + pA)/(A(\mu)^* + pA)$. We call the union of such sets, one for each μ, a *v-basis* for A. If X is a v-basis for A, each element a of $A(\mu)$ can be written

$$a = \sum u_i x_i + a^* + pb$$

where $x_i \in X$, $\bar{v}(x_i) = \mu$, $(p, u_i) = 1$ and $a^* \in A(\mu)^*$. We call this sum an *X-representation* of a.

LEMMA 3. $d(A) \geqq r(A)$.

Proof. Let H be the subgroup generated by a v-basis for A. As $r(A/pA) = r(A)$ it suffices to prove that $H + pA = A$. Since

$A(\mu) \subseteq H + pA + A(\mu)^*$ we have $A(\mu) \subseteq H + pA$ by induction on μ. Hence $A \subseteq H + pA$.

Let α be the maximum value of the nonzero elements of a (nontrivial) valuated group A. We write $A(\alpha)$ for the subgroup generated by the elements of value α. Then $\alpha \neq \infty$ and $0 \neq A(\alpha) \subseteq A[p]$. It turns out that we can find a set of generators for $A(\alpha)$ by taking multiples of elements in a v-basis X for A. The least natural number n such that $p^n a = 0$ is called the *exponent* of a and is written $e(a)$. Let

$$X_\alpha = \{p^{e(x)-1}x : x \in X \quad \text{and} \quad \alpha \in \bar{v}(x)\} .$$

Thus X_α is obtained by taking those representatives x whose value sequence ends in α, and multiplying by powers of p so that the resulting elements have order p. As yet, there is no guarantee that X_α is not empty. We remedy this in:

LEMMA 4. *$A(\alpha)$ is generated by X_α.*

Proof. Let K be the subgroup generated by X_α. Certainly $K \subseteq A(\alpha)$. Suppose that $K \neq A(\alpha)$, and choose, among those elements a of A which satisfy $p^{e(a)-1}a \in A(\alpha) - K$, one of maximum order, and then of maximal value sequence. Let $\vec{v}(a) = \mu$ and

$$a = \sum u_i x_i + \sum a_j^* + pc ,$$

$v(a_j^*) > \mu$, be an X-representation for a. Set $k = e(a) - 1$. Now $p^k a \in A(\alpha)$ so $p^k A(\mu) \subseteq A(\alpha)$ whence $p^k x_i \in A(\alpha)$ and $p^k a_j^* \in A(\alpha)$. Therefore $p^{k+1}c \in A(\alpha)$, and to avoid contradicting the choice of a we must have $p^{k+1}c \in K$. For the same reason, each a_j^* has order less than p^k. But then $p^k a \in K$, a contradiction.

Throughout, α will denote the maximum value of elements in $A\backslash\{0\}$. Denote $A/A(\alpha)$ by $\bar{A}$ and the image of an element a of A under the natural homomorphism $A \to \bar{A}$ by $\bar{a}$. It is readily seen that $\vec{v}(\bar{a}) = (\alpha_1, \cdots, \alpha_n)$ where each $\alpha_i < \alpha$ and that $\vec{v}(a) = (\alpha_1, \cdots, \alpha_n)$ or $\vec{v}(a) = (\alpha_1, \cdots, \alpha_n, \alpha)$. With this in mind, given $\mu = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_i < \alpha$, define $\mu\alpha = (\alpha_1, \cdots, \alpha_n, \alpha)$.

LEMMA 5. *Let X be a v-basis for A. Then a v-basis for $\bar{A}$ can be chosen from the set $\bar{X} = \{\bar{x} : x \in X\}$.*

Proof. Let $\mu = (\alpha_1, \cdots, \alpha_n)$ where each $\alpha_i < \alpha$. Suppose $v(\bar{y}) = \mu$. It is enough to show that

$$\bar{y} = \sum u_i \bar{x}_i + \bar{a}$$

where $\bar{x}_i \in \bar{X}$, $\bar{v}(\bar{x}_i) = \mu$ and $\bar{a} \in \bar{A}(\mu)^* + p\bar{A}$. Now $y \in A(\mu\alpha)$ and we can write

$$y = \sum u_i x_i + \sum a_j^* + pa$$

where $\bar{v}(x_i) = \mu\alpha$ and $\bar{v}(a_j^*) > \mu\alpha$. If $\bar{v}(a_j^*) \neq \mu$ then $\bar{a}_j^* \in \bar{A}(\mu)^*$ as desired. Replace each a_j^* such that $v(a_j^*) = \mu$ with its X-representation. The result is a sum

$$y = \sum u_i x_i + \sum v_k x_k + w$$

with $\bar{v}(x_k) = \mu$ and $\bar{w} \in \bar{A}(\mu)^* + p\bar{A}$. Observing that $\bar{v}(\bar{x}_k) = \bar{v}(\bar{x}_i) = \mu$ completes the proof.

COROLLARY 1. *The set of finite values vA occurring in A is $V(A) = \{\beta \in \mu : f(\mu, A) \neq 0\}$.*

Proof. We use induction on $\alpha = \max vA$. Certainly $V(A) \subseteqq vA$. By Lemma 4, $\alpha \in V(A)$ and if $\alpha \neq \beta \in vA$ then $\beta \in v\bar{A} = V(\bar{A})$ by induction. But, by Lemma 5, $V(\bar{A}) \subseteqq V(A)$.

LEMMA 6. *Suppose $r(A) = d(A)$. Then each v-basis for A is linearly independent in A.*

Proof. Let $X = \{x_i\}$ be a v-basis for A. If H is the subgroup generated by X then $H + pA = A$ and $r(A) = r(A/pA) = \text{card } X$ so X is independent modulo pA. Suppose to the contrary that X is linearly dependent in A and let

$$(*) \qquad \sum p^{r_i} a_i x_i = 0 \ , \quad (p, a_i) = 1$$

be a linear combination with no term zero. Suppose $r = r_1 = \min r_i$ and set

$$y = \sum_I p^{r_i - r} a_i x_i \ .$$

Now $e(y) \leqq r$ and $e(p^{r_i - r} a_i x_i) > r$ for each i. Suppose now that our y is chosen from among all relations like (*) so that y has maximal value sequence μ. Let

$$y = \sum_J u_j x_j + \sum a_k^* + pa$$

be an X-representation for y. Now $\bar{v}(x_j) = \mu = \bar{v}(y)$ implies $e(x_j) \leqq r$ so $I \cap J$ is empty. Thus independence of X modulo pA implies there is a k with

$$a_k^* = \sum s_l x_l$$

where $(p, s_1) = 1$. Now $\bar{v}(a_k^*) > \bar{v}(y)$ implies $e(a_k^*) \leqq e(y) \leqq r$ so $p^r a_k^* = \sum p^r s_l x_l = 0$. But $p^r s_1 x_1 \neq 0$, contradicting the choice of y.

Proof of Theorem 3. In fact, we prove the stronger assertion that if X is a v-basis for A, then $A = \bigoplus_{x \in X} \langle x \rangle$. Recall that $\sum_\mu f(\mu, A) = d(A)$. We use induction on card A. Assume the assertion is true for all groups smaller than A. Since each subgroup B of $A(\alpha)$ satisfying $B \cap pA = 0$ is a summand of A, we may assume $A(\alpha) \subseteqq pA$. Thus $r(\bar{A}) = r(A)$. As $d(A) = r(A)$, Lemma 3 and Corollary 1 imply that $d(\bar{A}) = r(\bar{A})$. If X is a v-basis for A, it follows that $\bar{X}$ is a v-basis for $\bar{A}$ and our induction assumption yields

$$\bar{A} = \bigoplus_{x \in X} \langle \bar{x} \rangle .$$

By Lemma 6, $A = \bigoplus_{x \in X} \langle x \rangle$ as an abelian group. Let $a = \sum n_x x$ be a linear combination of elements of X. It is enough to show that $va = \min \{vn_x x\}$. If $\min vn_x x < \alpha$, then

$$v\bar{a} = \min \{vn_x x\} = \min \{vn_x x\} < \alpha$$

implies $va = v\bar{a} = min \{vn_x x\}$. The case $\min vn_x x = \alpha$ is trivial.

We conclude with a theorem which shows there are no non-cyclic indecomposables bounded by p^2 in $\mathscr{T}_p$.

THEOREM 4. *Let $A \in \mathscr{T}_p$. If $p^2 A = 0$ then A is a direct sum of cyclics.*

Proof. The proof is by induction on $|A|$ so we assume all p^2-bounded valuated groups smaller than A are direct sums of cyclics. As in the proof of Theorem 3, $A(\alpha) \subseteqq pA$ may be assumed. Choose y of maximum value so that $0 \neq py \in A(\alpha)$ and set $x = py$. We first show that each element of order p in $A/\langle y \rangle$ lifts to an element of A having the same order and value. Denote the coset $a + \langle y \rangle$ by $\bar{a}$. Suppose $p\bar{a} = 0$. Then $pa = ux$ where u is a unit. Observe that $va \leqq vy$. Let $b = a - uy$. Then $\bar{b} = \bar{a}$, $pb = 0$ and $vb = v\bar{a}$ as required.

Our induction assumption implies $\bar{A}$ is a direct sum of cyclics $\langle \bar{a}_1 \rangle \oplus \cdots \oplus \langle \bar{a}_n \rangle$—we also arrange that $va_i = v\bar{a}_i$ for each i. If $p\bar{a}_i \neq 0$ then the preceding argument yields a z in $A[p]$ with $\bar{z} = p\bar{a}_i$. Thus $pa_i = z + rx$ so that $vpa_i \geqq vz$ and we conclude that $vpa = vp\bar{a}_i$. The map induced by sending $\bar{a}_i \longrightarrow a_i$ is therefore a map $\bar{A} \longrightarrow A$ of $\mathscr{T}_p$ and we have $A \cong \bar{A} \oplus \langle y \rangle$, a direct sum of cyclics.

REFERENCES

1. F. Richman, *The constructive theory of countable abelian p-groups*, Pacific J. Math., **45** (1973), 621-637.
2. F. Richman and E. Walker, *Valuated groups*, Preprint.

Received June 4, 1976. The first author was partially supported by an Australian National University Postdoctoral Travelling Scholarship. The second and third authors were supported by NSF-MPS71-02778-A04.

NEW MEXICO STATE UNIVERSITY