PACIFIC JOURNAL OF MATHEMATICS
Vol. 69, No. 2, 1977

INNER-OUTER FACTORIZATION OF FUNCTIONS WHOSE FOURIER SERIES VANISH OFF A SEMIGROUP

HOWARD LEWIS PENN

Let G be a compact, connected, Abelian group. Its dual, Γ, is discrete and can be ordered. Let Γ_1 be a semigroup which is a subset of the positive elements for some ordering, but which contains the origin of Γ. Let $H^p(\Gamma_1)$ be the subspace of $L^p(G)$ consisting of functions which have vanishing off Γ_1. The question that this paper is concerned with is what conditions on a function in $H^p(\Gamma_1)$ assure an inner-outer factorization.

An inner function is a function $f \in H^\infty(\Gamma_1)$ such that $|f|=1$ a.e. (dx) on G. A function $f \in H^p(\Gamma_1)$ is said to be outer if

$$\int_G \log |f(x)| = \log \left| \int_G f(x)dx \right| > -\infty .$$

A function $f \in H^1(\Gamma_1)$ is said to be in the class $LRP(\Gamma_1)$ if $\log |f| \in \Gamma_1(G)$ and $\log |f|$ has Fourier coefficients equal to zero off $\Gamma_1 \cup -\Gamma_1$. The main result of the paper is that if Γ_1 is the intersection of half planes and $f \in H^1(\Gamma_1)$ with $\int_G \log |f(x)| dx > -\infty$ then f has an inner-outer factorization if and only if $\log |f|$ is in $LRP(\Gamma_1)$.

A semigroup, P, in Γ_1 is called a half plane if $P \cup -P = \Gamma$ and $P \cap -P = \{0\}$. Helson and Lowdenslager [2] proved that if Γ_1 is a half plane then every function $f \in H^p(\Gamma_1)$ with $\int \log |f| dx > -\infty$ has a factorization as a product of an outer function, $h \in H^p(\Gamma_1)$ and an inner function, g, and this factorization is unique up to multiplication by constants of magnitude 1. From now on we shall assume $\int \log |f| dx > -\infty$.

Helson and Lowdenslager also showed [3] that if u is a real function such that u and e^u are summable, and v is the conjugate function of u with respect to the half plane, Γ_1, then e^{u+iv} is an outer function in $H^1(\Gamma_1)$. Conversely, if a summable outer function has the represention e^{u+iv} with u and v real then u is summable and v is equal to its conjugate modulo 2π except for an additive constant.

Let P be a half plane which contains Γ_1. Then, for $u \in L^1_R(G)$ there exists a conjugate function, v, which is unique if we assume $v(0) = 0$, such that $u + iv$ has its Fourier series supported on P. The function, v, is in L^p, $p < 1$. If u has its Fourier coefficients supported only on $\Gamma_1 \cup -\Gamma_1$ then $u + iv$ has its Fourier coefficients supported only on Γ_1 [4, Chap. 8, §7]. Therefore, $f \in LRP(\Gamma_1)$ if and only if $\log|f| \in L^1$ and $\log|f|$ is the real part of a function whose Fourier coefficients vanish off Γ_1.

THEOREM. *Assuming that* $f \in H^1(\Gamma_1)$ *and* $\int \log|f|\,dx > -\infty$, *and that* $\Gamma_i = \bigcap_{i \in I} P_i$, *then* f *has an inner-outer factorization if and only if* $\log|f|$ *has its Fourier coefficients vanish off* $\Gamma_1 \cup -\Gamma_1$.

Proof. Assume $f \in LRP(\Gamma_1)$. Let $u = \log|f| \in L^1(G)$. Take any $i \in I$ and consider P_i. $\Gamma_1 \subset P_i$ and f has an inner-outer factorization with respect to P_i. The outer factor is given by e^{u+iv_i} where v_i is the conjugate function to u with respect to P_i. Since u has its Fourier coefficients supported on $\Gamma_1 \cup -\Gamma_1$, it follow that v_i also has its Fourier coefficients supported there. Therefore, v_i is the same as the conjugate function of u with respect to any of the other half planes P_j, $j \in I$. Therefore, the outer factor of f in $H^1(P_i)$ is given by e^{u+iv_i}. Also, if P_j is any of the other half planes whose intersection gives Γ_1, then the outer factor f in $H^1(P_j)$ is e^{u+iv_j}. Therefore, $e^{u+iv_i} \in \bigcap_{i \in I} H^1(P_i)$, which is just equal to $H^1(\Gamma_1)$. For each half plane P_i, $i \in I$, we have that the inner factor is given by $fe^{-(u+iv_i)}$. Therefore, the inner factor is also in $H^1(\Gamma_1)$.

Conversely, assume that f has an inner-outer factorization, gh, in $H^1(\Gamma_1)$. Choose P_j, $j \in I$, then the outer factor, h, of f in $H^1(\Gamma_1)$, and hence in $H^1(P_j)$, is given by e^{u+v_j}, where v_j is the conjugate function of $u = \log|f|$ with respect to P_j. Since this is true for all P_j, $j \in I$, it follows that e^{iv_j} is the same regardless of which half plane, P_j is used. Now assume P_k is another of the half planes whose intersection is Γ_1. Then $e^{iu_k} = e^{iu_j}$ where v_k is the conjugate function of u with respect to P_j. It follows that $v_k(x) = v_j(x) + 2n\pi$ where n might change from point to point. We will now show that $n = 0$. Consider the function $h^{1/2}$ which is outer in $H^2(\Gamma_1) \subset H^1(\Gamma_1)$. It follows that $\log|h^{1/2}| = u/2$. The conjugate function of $u/2$ with respect to p_j is $v_j/2$ and its conjugate function with respect to P_k is $v_k/2$. By the Helson and Lowdenslager theorem $h^{1/2} = e^{(u+iv_j)2}$ and also $h^{1/2} = e^{(u+iv_k)/2}$. Therefore,

$$h = h^{1/2}h^{1/2} = e^{u+i(v_j+v_k)/2} .$$

Hence

$$v_k(x) = (v_j(x) + v_k(x))/2 + 2n\pi .$$

So,

$$v_k(x) = v_j(x) + 4n\pi .$$

Now consider $h^{1/4} = e^{(u+iv_j)/4} = e^{(u+iv_k)/4}$. Therefore,

$$h = h^{3/4}h^{1/4} = e^{3(u+iv_k)/4}e^{(u+iv_j)/4} = e^{u+i(3v_k+v_j)/4} .$$

Hence,

$$v_k(x) = v_j(x) + 8n\pi .$$

By considering the 2^mth roots of h we can show that the difference between v_k and v_j must be $2^{m+1}n\pi$. This must hold for all values of m. The only integer for which this is true is 0. Therefore u has the same conjugate function with respect to each of the half planes.

We will show that u has its Fourier coefficients supported of $\Gamma_1 \cup -\Gamma_1$. Suppose that $\hat{u}(\gamma) \neq 0$, where $\gamma \notin \Gamma_1 \cup -\Gamma_1$. Then there exists P_j, $j \in I$ such that $\gamma \notin P_j$. There also exists P_k, $k \in I$, such that $\gamma \notin -P_k$. Let v_j be the conjugate functions of u with respect to the half plane, P_j and let v_k be the conjugate function of u with respect to P_k. Since $\gamma \notin P_j$, we have

$$\hat{v}_j(\gamma) = i\hat{u}(\gamma)$$

[4, Chap. 8, §7]. Likewise, since $\gamma \notin -P_k$ it follows that $\gamma \in P_k \backslash 0$ and that

$$\hat{v}_k(\gamma) = -i\hat{u}(\gamma) .$$

But since $\hat{u}(\gamma) \neq 0$, we have $\hat{v}_j(\gamma) \neq \hat{v}_k(\gamma)$, and hence v_j and v_k are different functions. But we have just shown that u has the same conjugate function with respect to each half plane. Therefore $\hat{u}(\gamma) = 0$ and u has its Fourier series supported on $\Gamma_1 \cup -\Gamma_1$. Therefore $f \in LRP(\Gamma_1)$.

COROLLARY. *If $f \in H^1(\Gamma_1)$ where Γ_1 is the intersection of half planes and $f \in LRP(\Gamma_1)$, then $f = p_1 p_2$ where $p_1, p_2 \in H^2(\Gamma_1)$ and* $|p_1|^2 \equiv |p_2|^2 \equiv |f|$

EXAMPLE. In [1] Ebenstein discusses the H^p functions on a semigroup which is the intersection of a countable collection of half planes. This semigroup fulfills the hypothesis of the theorem. Let T^ω be the compact group which is the Cartesian product of countably many circles. The dual $\sum_{i=I}^{\infty} Z$, is the direct sum of countably many copies of the integers. Define $A \subset \sum_{i=1}^{\infty} Z$ by

$$A = \{x : x_i \geqq 0 \quad \text{for all } i\} .$$

We may define $H^p(T^\omega)$, $p \geqq 1$ as the subset of $L^p(T^\omega)$ consisting of these functions whose Fourier coefficients vanish off A. The semigroup, A, is the intersection of half planes P_i defined as follows:

$$P_i = \{x: x_i \geqq 0\ ,\quad \text{if } x_i = 0 \text{ then } x_1 \geqq 0\ ,$$
$$\text{if } x_1, x_2, \cdots, x_j = 0 \text{ then } x_{j+1} = 0\}\ .$$

Therefore the theorem applies to $H^p(T^\omega)$.

REMARK. One might hope that certain theorems which hold for the H^p spaces of the disk would remain true, at least for the class $LRP(\Gamma_1)$. One such theorem is Szego's theorem which states if $w \in L^1(dx)$ and $w \geqq 0$, then

$$\inf_{g \in A_0} \int |1 - g|^2 w dx = \exp \int \log(w) dx$$

where A_0 consists of those polynomials supported on Γ_1, with zero-th coefficient equal to zero. This theorem is true if Γ_1 is a half plane [4, Chap. 8, § 3]. Rudin has an example [5, Theorem 4.4.8] of a function, f, which is outer, but does not span. This same function can be used to show that Szego's theorem fails even for the class LRP.

REFERENCES

1. Samuel E. Ebenstein, *Some H^p spaces which are uncomlimented in L^p*, Pacific J. Math., **43** (1972), 327–339.

2. H. Helson and D. Lowdenslager, *Prediction theory and Fourier analysis in several variables* I, Acta. Math., **99** (1958), 105–202.

3. ———, *Prediction theory and Fourier analysis in several variables* II, Acta. Math., **106** (1961), 175–213.

4. Walter Rudin, *Fourier Analysis on Groups*, Interscience, New York, (1962).

5. ———, *Function Theory in Polydiscs*, Benjamin, New York, (1969).

Received November 19, 1974 and in revised form November 8, 1976. The work in this paper was completed as part of the author's doctorial thesis at the University of Michigan.

UNITED STATES NAVAL ACADEMY
ANNAPOLIS, MD 21402