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NOETHERIAN FIXED RINGS

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One of the basic questions of noncommutative Galois theory is the relation between a ring R and the ring S fixed by a group of automorphisms of R. This paper explores what happens when the group is finite and the fixed ring S is assumed to be Noetherian. Easy examples show that R may not be Noetherian; however, in this paper it is shown that R is Noetherian with some rather natural assuptions. More precisely we prove the Theorem 2: Let S be a semiprime ring. Assume that G is a finite group of automorphisms of S and that S has no $|G|$-torsion. If S^G is left noetherian then S is left noetherian.

Theorem 2 answers a question raised by Fisher and Osterburg [4]. This result rests on calculations which can best be described as belonging to noncommutative Galois theory. The basic theorem here may be of independent interest.

THEOREM 1. *Let R be a semisimple artinian ring. If G is a finite group of automorphisms of R and $|G|$ is invertible in R then R is a finitely generated ring R^G-module.*

The proof of Theorem 1 follows the spirit of Karchenko's work on polynomial identity rings ([6]).

1. A proof of Theorem 1. We will repeatedly need Levitzki's fixed ring theorem ([8]): Suppose R is a semisimple artinian ring. If G is a finite group of automorphisms of R with $|G|$ invertible in R then R^G is semisimple artinian.

LEMMA 1. *If Theorem 1 is true when G is a simple group then it is true for an arbitrary finite G.*

Proof. By induction on the length of a composition series for G. If G is not already simple choose $H \Delta G$ with $1 \neq H \neq G$. By Levitzki's theorem R^H is semisimple artinian. G/H acts on R^H and R^H has no $|G/H|$-torsion; by induction R^H is a finitely generated right R^G-module. Again, induction shows that R is a finitely generated right R^H-module. The lemma follows.

We eventually assume that G is simple. In that case either G consists entirely of outer automorphisms or entirely of inner automorphisms.

LEMMA 2. *Let B be a simple artinian ring and let G be a finite group of outer automorphisms of B. Then B is a finitely generated right B^G-module.*

Proof. By [1], B^G is a simple ring and B is a free module over B^G of rank $|G|$. (Cf. [5] for the case of a division ring.)

LEMMA 3. *Let B be a simple artinian ring and let G be a finite group of inner automorphisms of B. Assume $|G|$ is invertible in B. Then B is a finitely generated right B^G-module.*

Proof. Let F be the center of B.

For each $g \in G$ pick one $x \in B$ such that ${}^g b = xbx^{-1}$ for all $b \in B$. Call the finite set so chosen, $\bar{G}$. Then collection of sums, $F\bar{G}$, is a finite dimensional algebra over F. Since $1/|G| \in F$, Maschke's theorem for twisted group algebras ([9]) states that $F\bar{G}$ is a separable algebra. Thus there is a finite extension field K of F such that K is a splitting field for each simple constituent of $F\bar{G}$.

$K \otimes_F B$ is a simple artinian ring with center K. G acts on $K \otimes_F B$ by

$${}^g(k \otimes b) = k \otimes {}^g b .$$

Obviously this action, too, is induced by inner automorphisms. A straight-forward calculation shows that $(K \otimes B)^G = K \otimes B^G$. Similarly, if $K \otimes B$ is a finitely generated right $(K \otimes B)^G$-module then B is a finitely generated B^G-module.

Thus we replace B with $K \otimes_F B$ and assume each simple constituent of $F\bar{G}$ is a total matrix ring with entires in F. Let $\mathscr{E}$ be the set of centrally primitive idempotents in $F\bar{G}$.

The crux of this lemma is to show that if $e \in \mathscr{E}$ then eBe is a finitely generated right B^G-module. An element of B^G commutes with elements of $F\bar{G}$ so it certainly commutes with e; hence eBe is a right B^G-module. Let ε_{ij} be a set of matrix units for $eF\bar{G}$. If x is in eBe, set

$$\pi_{ij}(x) = \sum_k \varepsilon_{ki} x \varepsilon_{jk}$$

$\pi_{ij}(x)$ commutes with each of the matrix units. Since F is the center of B, it commutes with $eF\bar{G}$. Thus it commutes with $F\bar{G}$. In other words, $\pi_{ij}(x)$ is in B^G. The map $\pi_{ij}: eBe \to B^G$ is a right B^G-module map by the argument at the beginning of this paragraph. We claim that the map

$$\sum_{i,j} \pi_{ij}: eBe \longrightarrow \bigoplus_{i,j} \sum B^G$$

is injective. For if $\sum_k \varepsilon_{ki} x \varepsilon_{jk} = 0$ for all i and j, multiple on the right by ε_{ij}:

$$\varepsilon_{ii} x \varepsilon_{jj} = 0 \quad \text{for all } i \text{ and } j.$$

Hence $exe = 0$. But $x \in eBe$ implies $exe = x$. We finish this paragraph by noticing that Levitzki's theorem says that B^G is right noetherian. Since eBe is isomorphic to a submodule of a finitely generated B^G-module, eBe is finitely generated.

Next we show that if e and f are different elements of $\mathscr{E}$ then fBe is a finitely generated right B^G-module. (Of course it is a B^G-module as above.) Since B is simple, $BeB = B$. Thus we can choose $v_i \in fBe$ and $u_i \in eBf$ so that

$$f = \sum_i v_i u_i .$$

Define $\varphi: fBe \rightarrow \bigoplus \sum_i eBe$ by $\varphi(y) = (u_i y)$, a right B^G-module map. $\varphi(y) = 0 \Rightarrow u_i y = 0$ for each $i \Rightarrow (\sum v_i u_i)y = 0 \Rightarrow fy = 0$. But $fy = y$. Hence φ is injective. Finish the argument as before.

Because $B = \sum_{e,f \in \varepsilon} fBe$, B is a finitely generated right B^G-module.

Proof of Theorem 1. Induct on the order of G. Assume G is simple.

Let e be a centrally primitive idempotent in R. eR is a simple artinian ring. Moreover the stabilizer $H = \mathrm{Stab}_G(e)$ acts on eR and $1/|H|e \in eR$. By Lemmas 2 and 3, eR is a finitely generated right $(eR)^H$-module.

Claim. $(eR)^H = e(R^G)$.

Certainly $e(R^G) \subseteqq (eR)^H$. Let $G = \bigcup_{\gamma \in \Gamma} \gamma H$ be a coset decomposition of G with $1 \in \Gamma$. G permutes the centrally primitive idempotents of R and for $\alpha \neq \beta$ in Γ, ${}^{\alpha}e \neq {}^{\beta}e$. Equivalently, if $\gamma \neq 1$ is in Γ, $e({}^{\gamma}e) = 0$. If $x \in (eR)^H$ define $t_\Gamma(x) = \sum_{\gamma \in \Gamma} ({}^{\gamma}x)$. If $g \in G$, $\{g\gamma \mid \gamma \in \Gamma\}$ are also coset representatives for H. Thus ${}^{g}t_\Gamma(x) = t_\Gamma(x)$. That is, $t_\Gamma(x) \in R^G$. But $et_\Gamma(x) = x$ by the remarks above about multiplying idempotents. Thus $(eR)^H \subseteqq (eR^G)$.

We now know that eR is a finitely generated right $e(R^G)$-module. That means eR is a finitely generated R^G-module. Since $R = \sum_e eR$, we are done.

2. Theorem 2 and its relatives.

LEMMA 4. *Let A be a semiprime ring. Assume G is a finite group of automorphisms of A and A has no $|G|$-torsion. Then tr_G does not vanish on any nonzero right ideal of A.*

$$(\text{Here } tr_G(a) = \sum_{g \in G} ({}^g a).)$$

Proof. Suppose I is a right ideal of A with $tr_G(I) = 0$. If $J = \sum_{g \in G} {}^g I$ then J is a G-invariant right ideal of A with $tr_G(J) = 0$. By [2], J is nilpotent. But the only nilpotent right ideal in a semiprime ring is 0.

Proof of Theorem 2. S^G is left Goldie, so according to [6], S is (semiprime) left Goldie. Let R be the left quotient ring for S; R is semisimple artinian. By Theorem 1 we can find a finite set of generators $x_1, \cdots, x_n$ for R as a right R^G-module. Choose a regular t and s_i both in S such that $x_i = t^{-1}s_i$.

$R = \sum_{i=1}^n t^{-1}s_i R^G \Rightarrow tR = \sum_i s_i R^G$. But $tR = R$ since t is invertible. Thus we assume $x_i \in S$.

Define $T: S \to \bigoplus \sum_{i=1}^n S^G$ by $T(a) = [tr_G(ax_i)]_{i=1}^n$. T is clearly a left S^G-module map. We will be done once we prove that T is injective.

$T(a) = 0$ implies $tr_G(ax_i) = 0$ for all i. But tr_G is a *right* R^G-module map. Thus $tr_G(aR) = 0$. By the previous lemma, $a = 0$.

We have actually proved that S is a finitely generated S^G-module!

One might well ask whether the requirement that S have no $|G|$-torsion can be dropped. Consider the following counterexample. Let F be a field of characteristic $p > 2$ and let Φ be the free group on x and y. If S denotes the ring of two-by-two matrices over the group algebra $F[\Phi]$ then S is semiprime but not noetherian. Let G be the multiplicative subgroup of S generated by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}.$$

G is isomorphic to the semidirect product of $Z/p \oplus Z/p \oplus Z/p$ with $Z/2$. Since char $F \neq 2$, $S^{\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]}$ is the collection of diagonal matrices. The only diagonal matrices fixed by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are the scalar matrices. Now a simple calculation shows that S^G consists of those scalars in the center of $F[\Phi]$. But it is well known that the center is F, a patently noetherian ring.

However, the $|G|$-torsion restriction is not needed when S is (semiprime) commutative or, more generally, when S has no nilpotent elements. There are several difficulties in proving the last statement along the lines of Theorem 2. First, there are division rings on which tr_G vanishes. Even if this objection is met, our induction and restriction techniques all ignore the question of fidelity of action. Reconsider, for instance, Lemma 4. The Bergman-Isaacs theorem states that if H is a group of *automorphisms* of J and $tr_H(J) = 0$

then $J = 0$. Thus implicit in our argument is the proposition that $tr_G(J) = 0 \Rightarrow tr_{G/K}(J) = 0$ where K is the kernel of the action of G on J. The implication is true because J has no $|K|$-torsion.

We avoid these complications (and, of course, replace them with other complications) by refining the notion of trace. Let G be a finite group acting on a ring R. If $\wedge$ is a subset of G define $t_\wedge$: $R \to R$ by

$$t_\wedge(r) = \sum_{\lambda \in \wedge} ({}^\lambda r) .$$

$t_\wedge$ is an R^G-bimodule map. Notice that $tr_G \equiv t_G$.

LEMMA 5. *Let G be a finite group acting on the division ring D. Then there is a subset $\wedge \subseteq G$ such that $t_\wedge$ is a mapping from D onto D^G.*

Proof. Suppose we can find $\wedge$ such that $t_\wedge$ is a nonzero function from D into D^G. Say $d \in D$ such that $t_\wedge(d) = w \neq 0$. If $x \in D^G$, $t_\wedge(dw^{-1}x) = t_\wedge(d)w^{-1}x = x$. Thus $t_\wedge$ is surjective.

We argue by induction on the length of a composition series for G. If G is simple and does not act faithfully then G acts trivially; choose $\wedge = \{1\}$. If G is simple group of automorphisms, a result of Faith ([3]) shows that t_G is not identically zero.

When G is not simple choose $H \Delta G$ with $H \neq 1$ and $H \neq G$. By induction there is a subset $A \subseteq H$ such that t_A: $D \to D^H$ is surjective. G/H acts on D^H, so we can find $C \subseteq G/H$ such that t_C: $D^H \to D^G$ is surjective. If B consists of representatives in G for elements of C then $t_C = t_B$. Now $t_{B \cdot A} = t_B \cdot t_A$ is the desired map.

Let S be a ring without nilpotent elements. Suppose G is a finite group of automorphisms of S such that S^G is left noetherian. By [7] S is a semiprime left Goldie ring. By the Faith-Utumi theorem the quotient ring, R, of S has no nilpotent elements. Let e be a centrally primitive idempotent of R.

LEMMA 6. *$S \cap eR$ is a finitely generated left S^G-module.*

Proof. We first observe that the left quotient ring of $S \cap eR$ in eR is the entire division ring eR. Choose z and s in S with z regular such that $e = z^{-1}s$. Then $s = ze \in S \cap eR$. If $x \in eR$ choose q and w in S with q regular such that $qx = w$. Then $(sq)x = sw$. But sq and sw are in $S \cap eR$ with sq regular when considered as an element in eR.

$H = \mathrm{Stab}_G(e)$ is a group which acts on $S \cap eR$. Pick a transversal, $G = \Gamma \cdot H$. As in Theorem 1, if $a \in S^H \cap eR$ then

$$t_\Gamma(a) \in S^G \quad \text{and} \quad e \cdot t_\Gamma(a) = a .$$

Thus t_Γ is an injective left S^G-module map from $S^H \cap eR$ into S^G.

The Galois theory for division rings ([5]) as applied to eR implies that eR is a finite dimensional right $(eR)^H$-vector space. As in the proof of Theorem 2 we can choose a basis $x_1, \cdots, x_n$ in $S \cap eR$. Use Lemma 5 to find $\wedge \subseteq H$ so that $t_\wedge$ is nondegenerate on eR. Define $T: S \cap eR \to \bigoplus \sum_{i=1}^n S^G$ by

$$T(a) = [t_{\Gamma \cdot \wedge}(ax_i)]_{i=1}^n .$$

It is easy to check that T is a well defined left S^G-module map. The lemma is completed by showing that T is injective. Suppose $a \neq 0$ and $T(a) = 0$. Then $t_\Gamma \cdot t_\wedge(ax_i) = 0$ for each i. Since t_Γ is injective, $t_\wedge(ax_i) = 0$ for each i. That is, $t_\wedge(a \cdot eR) = 0$. But eR is a division ring: $a \cdot eR = eR$. We have contradicted the nonvanishing of $t_\wedge$.

THEOREM 3. *Let S be a ring without nilpotent elements. If G is a finite group of automorphisms of S and S^G is left noetherian then S is left noetherian* (in fact, is finitely generated as an S^G-module).

Proof. So far we have proved that $\sum_e (S \cap eR)$ is a finitely generated left S^G-module, where the sum is taken over the centrally primitive idempotents of R.

As observed in the first paragraph of Lemma 6, $S \cap eR$ contains an element invertible in eR. Consequently there is an element $d \in \Sigma(S \cap eR)$ which is invertible in R. Define $f: S \to \Sigma(S \cap eR)$ by $f(s) = sd$. Since f is an injective left S^G-module map, S is a finitely generated left S^G-module.

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