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A GENERALIZATION OF CARISTI'S THEOREM WITH APPLICATIONS TO NONLINEAR MAPPING THEORY

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Suppose X and Y are complete metric spaces, $g: X \to X$ an arbitrary mapping, and $f: X \to Y$ a closed mapping (thus, for $\{x_n\} \subset X$ the conditions $x_n \to x$ and $f(x_n) \to y$ imply $f(x) = y$). It is shown that if there exists a lower semicontinuous function φ mapping $f(X)$ into the nonnegative real numbers and a constant $c > 0$ such that for all x in X, $\max\{d(x, g(x)), cd(f(x), f(g(x)))\} \leqq \varphi(f(x)) - \varphi(f(g(x)))$, then g has a fixed point in X. This theorem is then used to prove surjectivity theorems for nonlinear closed mappings $f: X \to Y$, where X and Y are Banach spaces.

1. Introduction. The following fact is well-known in the theory of linear operators;

(1.1) *Let X and Y be Banach spaces with D a dense subspace of X, and let $T: D \to Y$ be a closed linear mapping with dual T'. Suppose the following two conditions hold:*

(i) $N(T') = \{0\}$.

(ii) *For fixed* $c > 0$, $\text{dist}(x, N(T)) \leqq c\,||Tx||$, $x \in D$.

Then $T(D) = Y$.

Proof. Because T is a closed mapping it routinely follows from (ii) that $T(D)$ is closed in Y (e.g., [15, p. 72]), whence it follows from the Hahn-Banach theorem (cf. [17, p. 205]) that $(N(T'))^{\perp} = T(D)$ where $(N(T'))^{\perp}$ denotes the annihilator in Y of the nullspace of T'. By (i), $(N(T'))^{\perp} = Y$.

It is our objective in this paper to give a nonlinear generalization of the above along with more technical related results. The key to our approach is an application of a new generalized version of Caristi's fixed point theorem. While our method parallels that of Kirk and Caristi [12], these new results differ from those of [12] and the earlier 'normal solvability' results of others, e.g., Altman [1], Browder [3-6], Pohozhayev [13, 14], and Zabreiko-Krasnoselskii [18], in that by using the improved fixed point theorem we are able to replace the usual closed range assumption with the assumption that the mapping be closed (in conjunction with a condition which in the linear case reduces to (ii)). Before doing this, however, we state and prove our fixed point theorem.

2. The fixed point theorem. The following theorem reduces to the theorem of Caristi [7, 8] in the case that $X = Y$, f is the identity mapping, and $c = 1$. (We should remark that Caristi's theorem is essentially equivalent to a theorem stated earlier by Ekeland [9]. A simple proof along the general lines below is implicit in Brøndsted [2]. A similar proof is given by Kasahara in [10], and in [16] Wong gives a simplified version of Caristi's original transfinite induction argument.)

THEOREM 2.1. *Let X and Y be complete metric spaces and $g: X \to X$ an arbitrary mapping. Suppose there exists a closed mapping $f: X \to Y$, a lower semicontinuous mapping $\varphi: f(X) \to R^+$, and a constant $c > 0$ such that for each $x \in X$,*

$$(*) \qquad \begin{cases} d(x, g(x)) \leqq \varphi(f(x)) - \varphi(f(g(x))) \, , & \textit{and} \\ cd(f(x), f(g(x))) \leqq \varphi(f(x)) - \varphi(f(g(x))) \, . \end{cases}$$

Then there exists $\bar{x} \in X$ such that $g(\bar{x}) = \bar{x}$.

Proof. We introduce a partial order $\leqq$ in X as follows. For $x, y \in X$ define $x \leqq y$ provided

$$\begin{cases} d(x, y) \leqq \varphi(f(x)) - \varphi(f(y)) \, , & \text{and} \\ cd(f(x), f(y)) \leqq \varphi(f(x)) - \varphi(f(y)) \, . \end{cases}$$

Let $\{x_\alpha\}_{\alpha \in I}$ be any chain in X, i.e., suppose $(I, \leqq)$ is a totally ordered set with $x_\alpha \leqq x_\beta$ iff $\alpha \leqq \beta$. Then $\{\varphi(f(x_\alpha))\}_{\alpha \in I}$ is a decreasing net in R^+ so there exists $r \geqq 0$ such that $\varphi(f(x_\alpha)) \downarrow r$. Let $\varepsilon > 0$. Then there exists $\alpha_0 \in I$ such that $\alpha \geqq \alpha_0$ implies

$$r \leqq \varphi(f(x_\alpha)) \leqq r + \varepsilon$$

and so for $\beta \geqq \alpha$,

$$\begin{aligned} d(x_\alpha, x_\beta) &\leqq \varphi(f(x_\alpha)) - \varphi(f(x_\beta)) \leqq \varepsilon \, , \quad \textit{and} \\ cd(f(x_\alpha), f(x_\beta)) &\leqq \varphi(f(x_\alpha)) - \varphi(f(x_\beta)) \leqq \varepsilon \, . \end{aligned}$$

Thus $\{f(x_\alpha)\}$ is a Cauchy net in Y while $\{x_\alpha\}$ is a Cauchy net in X. By completeness there exist $\bar{y} \in Y$ and $\bar{x} \in X$ such that $f(x_\alpha) \to \bar{y}$ and $x_\alpha \to \bar{x}$. Since f is a closed mapping, $f(\bar{x}) = \bar{y}$ and lower-semicontinuity of φ yields $\varphi(f(\bar{x})) \leqq r$. Moreover, if $\alpha, \beta \in I$ with $\alpha \leqq \beta$, then

$$\begin{aligned} d(x_\alpha, x_\beta) &\leqq \varphi(f(x_\alpha)) - \varphi(f(x_\beta)) \leqq \varphi(f(x_\alpha)) - r \, ; \\ cd(f(x_\alpha), f(x_\beta)) &\leqq \varphi(f(x_\alpha)) - r \, . \end{aligned}$$

Taking limits with respect to β yields

$$d(x_\alpha, \bar{x}) \leqq \varphi(f(x_\alpha)) - r \leqq \varphi(f(x_\alpha)) - \varphi(f(\bar{x}))\ ;$$
$$cd(f(x_\alpha), f(\bar{x})) \leqq \varphi(f(x_\alpha)) - \varphi(f(\bar{x}))\ .$$

This proves that $x_\alpha \leqq \bar{x}$, $\alpha \in I$.

Having thus shown that every totally ordered set in $(X, \leqq)$ has an upper bound we apply Zorn's lemma to obtain maximal element $x \in X$. By (*), $x \leqq g(x)$; hence $x = g(x)$.

3. Applications. If X and Y are topological vector spaces and $f\colon X \to Y$, then f is said to be *Gâteaux differentiable* at $x \in X$ if there exists a (possibly unbounded) *linear* operator $L\colon X \to Y$ such that for each $w \in X$,

$$t^{-1}(f(x + tw) - f(x)) \longrightarrow Lw \quad \text{as} \quad t \longrightarrow 0^+\ .$$

The operator $L = df_x$ is called the Gâteaux derivative of f at x and we use df'_x to denote the dual of df_x in the usual sense (e.g., [17, p. 194]).

We now state a theorem which is an immediate generalization of the theorem of the introduction. Notationally, we let $B_\delta(\cdot)$ denote the closed ball centered at $(\cdot)$ with radius δ. Also, $N(df'_x)$ denotes the nullspace of df'_x in Y^*, the space of all continuous linear functionals on Y, and $(N(df'_x))^\perp$ denotes its annihilator in Y.

THEOREM 3.1. *Let X and Y be Banach spaces and $f\colon X \to Y$ a* (nonlinear) *closed mapping which is Gâteaux differentiable at each $x \in X$ with derivative df_x. Let df'_x denote the dual of df_x, and suppose for each $x \in X$ and fixed $c > 0$:*

(i)′ $N(df'_x) = \{0\}$.

(ii)′ *There exists $\delta = \delta(x) > 0$ such that if $y \in B_\delta(f(x)) \cap f(X)$, then for some $v \in f^{-1}(y)$,*

$$||x - v|| \leqq c\,||f(x) - y||\ .$$

Then $f(X) = Y$.

It is obvious that (i)′ reduces to (i) in the linear case and it is a routine matter to show that (ii)′ similarly reduces to (ii). In contrast with the linear case, however, we do not show directly that (ii)′ implies closedness of the range of f. Instead we derive Theorem 3.1 from the following more general result which follows quite easily from Theorem 2.1.

THEOREM 3.2. *Suppose X is a complete metric space, Y a Banach space, and $f\colon X \to Y$ a closed mapping. Suppose for $y_0 \in Y$ there exist constants $c > 0$, $p < 1$ such that:*

(a) *Corresponding to each* $x \in X$ *there exists* $\delta = \delta(x) > 0$ *such that if* $y \in B_\delta(f(x)) \cap f(X)$, *then*

$$d(x, v) \leqq c \, \|f(x) - y\|$$

for some $v \in f^{-1}(y)$.

(b) *For each* $y \in f(X)$ *there exists a sequence* $\{y_j\}$ *in* $f(X)$ *with* $y_j \neq y$ *for each* j *such that* $y_j \to y$ *and a sequence* $\{\xi_j\}$ *of nonnegative real numbers such that for each* j

$$\|\xi_j(y_j - y) - (y_0 - y)\| \leqq p \, \|y_0 - y\| \, .$$

Then $y_0 \in f(X)$.

The following geometric lemma, implicit in [12], will facilitate the proof of Theorem 3.2.

LEMMA. *Let* Y *be a normed linear space with* $a, b, c \in Y$. *Suppose for* $\xi \geqq 1$ *and* $p < 1$,

$$(*) \qquad \|\xi(a - b) - (c - b)\| \leqq p \, \|c - b\| \, .$$

Then

$$\|a - b\| \leqq (1 + p)(1 - p)^{-1}[\|b - c\| - \|a - c\|] \, .$$

Proof.

$$\begin{aligned} \|\xi(a - c)\| - \|(1 - \xi)(b - c)\| & \\ \leqq \|\xi(a - c) + (1 - \xi)(b - c)\| & \\ = \|\xi(a - b) - (c - b)\| & \\ \leqq p \, \|b - c\| \, . & \end{aligned}$$

Thus $\|\xi(a - c)\| \leqq (\xi - 1 + p) \, \|b - c\|$, i.e.,

$$\|a - c\| \leqq [1 - \xi^{-1}(1 - p)] \, \|b - c\|$$

from which (using (*) and the triangle inequality)

$$\begin{aligned} \|b - c\| - \|a - c\| &\geqq \{1 - [1 - \xi^{-1}(1 - p)]\} \, \|b - c\| \\ &= \xi^{-1}(1 - p) \, \|b - c\| \\ &\geqq \xi^{-1}(1 - p)\xi(1 + p)^{-1} \, \|a - b\| \\ &= (1 - p)(1 + p)^{-1} \, \|a - b\| \, . \end{aligned}$$

Proof of Theorem 3.2. Suppose $y_0 \notin f(X)$. Let $x \in X$ and $y = f(x)$, and let $\{y_j\}$ be the sequence defined by (b). Since $y_j \to y$, j may be chosen so large that $\|y_j - y\| \leqq \delta(x)$. We also assume $\xi_j \geqq 1$. (Note that since $y_0 \neq y$, (b) implies $\xi_j \to +\infty$.) With j thus fixed we apply the lemma to the inequality in (b) and obtain

$$(1) \qquad 0 < \|y - y_j\| \leqq (1+p)(1-p)^{-1}[\|y - y_0\| - \|y_j - y_0\|] .$$

By (a) there exists $v \in f^{-1}(y_j)$ such that

$$(2) \qquad d(x, v) \leqq c\,\|y - y_j\| .$$

Define $g: X \to X$ by taking $g(x) = v$ with v obtained as above, and define $\varphi: f(X) \to R^+$ by

$$\varphi(f(x)) = c(1+p)(1-p)^{-1}\,\|f(x) - y_0\| .$$

Then clearly φ is continuous on $f(X)$ and together (1) and (2) yield

$$\begin{cases} d(x, g(x)) \leqq \varphi(f(x)) - \varphi(f(g(x))) , & \textit{and} \\ c\,\|f(x) - f(g(x))\| \leqq \varphi(f(x)) - \varphi(f(g(x))) . \end{cases}$$

By Theorem 2.1 there exists $\bar{x} \in X$ such that $g(\bar{x}) = \bar{x}$, contradicting (1).

In order to derive Theorem 3.1 from Theorem 3.2 we need an elementary fact from linear algebra. Let X and Y be locally convex topological vector spaces and suppose $L: X \to Y$ is a linear operator. The *dual* L' of L (cf. [17, p. 194]) is defined on a subset D of Y^* by the relation

$$\langle x, L'y'\rangle = \langle Lx, y'\rangle , \quad y' \in D , \quad x \in X$$

where X^* and Y^* denote respectively the spaces of continuous linear functionals on X and Y and where by assumption $\langle \cdot, L'y'\rangle \in X^*$. If $(N(L'))^{\perp}$ denotes the annihilator of $N(L')$ in Y it routinely follows from the Hahn-Banach theorem that $(N(L'))^{\perp} \subset \overline{L(X)}$. (For, suppose there exists $y_0 \in (N(L'))^{\perp}$ with $y_0 \notin \overline{L(X)}$. Then there exists $y' \in Y^*$ such that $\langle y_0, y'\rangle \neq 0$ while $\langle z, y'\rangle = 0$ for all $z \in \overline{L(X)}$; hence $\langle Lu, y'\rangle = \langle u, L'y'\rangle = 0$ for all $u \in X$ yielding $L'y' = 0$, i.e., $y' \in N(L')$. Since $y_0 \in (N(L'))^{\perp}$ implies $\langle y_0, y'\rangle = 0$, we have a contradiction.)

We now follow an approach of Browder [4, 6]. With X as above and Y a Banach space the asymptotic direction set of the mapping $f: X \to Y$ in the direction $x \in X$ is the set

$$D_x(f) = \bigcap_{\varepsilon > 0} \mathrm{cl}(\{y \in Y \mid y = \xi(f(u) - f(x)) , \ \xi \geqq 0, \ u \in X, \ \|f(u) - f(v)\| < \varepsilon\}).$$

The following is a minor variant of Proposition 1 of [4, 6]. We include the proof to show that continuity of df_x is not essential.

PROPOSITION 3.1. *Let X be a locally convex topological vector space, Y a Banach space, and suppose f is a mapping of X into Y which is Gâteaux differentiable at $x \in X$ with derivative df_x. If*

$N(df_x')$ denotes the nullspace in Y' of the dual of df_x and if $(N(df_x'))^\perp$ denotes its annihilator in Y, then

$$(N(df_x'))^\perp = \overline{df_x(X)} \subset D_x(f) .$$

Proof. The equality is immediate from observations above. To see that $\overline{df_x(X)} \subset D_x(f)$ we follow [6]: Let $\varepsilon > 0$ and $y \in df_x(X)$. Then $y = df_x(w)$ for some $w \in X$ and by differentiability

$$(\#) \qquad t^{-1}(f(x+tw) - f(x)) \longrightarrow y \quad \text{as} \quad t \longrightarrow 0^+ .$$

Letting $x_t = x + tw$ we have for $t > 0$ sufficiently small, $\|f(x_t) - f(x)\| < \varepsilon$. It follows from this and (#) that

$$y \in cl\{\xi(f(u) - f(x)) \mid \xi \geqq 0,\ u \in X,\ \|f(u) - f(x)\| < \varepsilon\} ;$$

i.e., $y \in D_x(f)$. Since $D_x(f)$ is closed, $\overline{df_x(X)} \subset D_x(f)$.

Proof of Theorem 3.1. Let $y_0 \in Y$, $p \in (0, 1)$. It suffices to establish (b) of Theorem 3.2. Suppose $y = f(x) \in f(X)$, $y \neq y_0$. Since $N(df_x') = \{0\}$, $(N(df_x'))^\perp = Y$ and by Proposition 3.1

$$y_0 - f(x) \in D_x(f) .$$

Choose $\varepsilon_j > 0$ with $\varepsilon_j \to 0$. For each j there exists $z_j \in X$ and $\xi_j \geqq 0$ such that

$$(3) \qquad \|\xi_j(f(z_j) - f(x)) - (y_0 - f(x))\| \leqq p\,\|y_0 - f(x)\|$$

and

$$(4) \qquad \|f(z_j) - f(x)\| < \varepsilon_j .$$

Letting $y_j = f(z_j)$, since $p < 1$ (3) implies $y_j \neq y$ for all j. By (4) $y_j \to y$ as $j \to \infty$ and rewriting (3) we have

$$\|\xi_j(y_j - y) - (y_0 - y)\| \leqq p\,\|y_0 - y\| .$$

This completes the proof.

Finally we note that if $\operatorname{int}\overline{f(X)} \neq \emptyset$, it is not necessary in Theorem 3.1 to assume f is differentiable at each $x \in X$.

THEOREM 3.3. *Let X and Y be Banach spaces and $f\colon X \to Y$ a closed mapping. Let $N = \{x \in X \mid f(x) \in \operatorname{int}\overline{f(X)}\}$ and suppose for $x \in X \backslash N$, f is Gâteaux differentiable with derivative df_x where $N(df_x') = \{0\}$. Suppose also that there exists $c > 0$ such that condition* (ii)′ *of Theorem* 3.1 *holds for all $x \in X$. Then $f(X) = Y$.*

Proof. Let $y_0 \in Y$ and suppose $y_0 \notin f(X)$. Fix $p \in (0, 1)$ and let $x \in X$. If $x \in X \backslash N$, then $(N(df'_x))^\perp = Y$ and by Proposition 3.1, $y_0 - f(x) \in D_x(f)$. But also if $x \in N$, i.e., if $f(x) \in \text{int}\,\overline{f(X)}$, then for $\varepsilon > 0$ chosen so that $B_\varepsilon(f(x)) \subset \overline{f(X)}$ it is possible to select $w \in \text{seg}\,[f(x), y_0]$ so that $w \neq y_0$ and $0 < \|f(x) - w\| < \varepsilon$, and because $w \in \overline{f(X)}$ there exists $\{w_j\} \subset f(X)$ such that $w_j \to w$. Since $y_0 - f(x) = \xi(w - f(x))$ for $\xi > 0$ it thus follows that $\xi(w_j - f(x)) \to y_0 - f(x)$ with $\|w_j - f(x)\| < \varepsilon$ for j sufficiently large proving $y_0 - f(x) \in D_x(f)$. Since $y_0 - f(x) \in D_x(f)$, the proof now follows the proof of Theorem 3.1.

We remark that as a consequence of the above theorem, if $f: X \to Y$ is a closed mapping with range dense in Y, then (ii)' of Theorem 3.1 implies $f(X) = Y$.

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