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MODULES WHOSE QUOTIENTS HAVE FINITE GOLDIE DIMENSION

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If M is a module and M is a submodule of M, then N is irreducible in M if N cannot be written as a proper intersection of two submodules of M. The purpose of this note is to study modules whose submodules can be written as a finite intersection of irreducible submodules. Such modules are characterized by the fact that their quotients all have finite Goldie dimension, so they are called q.f.d. modules.

The main result is: A module M is q.f.d. if and only if every submodule N has a finitely generated submodule T such that N/T has no maximal submodules. Because T is finitely generated this generalizes a theorem of Shock (using his ideas), who showed a q.f.d. module M having the property that every subquotient of M has a maximal submodule must be noetherian (and conversely, of course).

The q.f.d. condition also arises in the study of Krull dimension because a module with Krull dimension must be q.f.d. [1].

First, a remark, which we isolate as a lemma.

LEMMA. *A module M is* q.f.d. *if and only if each quotient of M has finite dimensional socle* (possible zero).

Proof. Every nonzero module has a quotient with nonzero socle. (Take $A \subset M$ to be maximal with respect to not containing m, $0 \neq m \in M$.) So an infinite direct sum of modules has a quotient with infinite dimensional socle.

THEOREM. *A module M is* q.f.d. *if and only if every submodule N contains a finitely generated submodule T, such that N/T has no maximal submodules.*

Proof. Suppose that X is a module such that every finitely generated submodule of X is contained in a maximal submodule of X. Having chosen maximal submodules $M_1, \cdots, M_n$ and elements $x_1, \cdots, x_n$ such that $x_i \notin M_i$, but $x_i \in M_j$ for $j > i$, choose a maximal submodule M_{n+1} containing $x_1R + \cdots + x_nR$ and an element x_{n+1} not in M_{n+1}.

Let $\bar{X} = X/\bigcap_{i=1}^{\infty} M_i$. Then, $\bar{X} = \bigcap_{i=1}^{n} M_i \oplus \bigcap_{i=n+1}^{\infty} M_i$, because, if we denote the right hand summand by M, we have a strict descending chain $M \supset M \cap M_n \supset M \cap M_n \cap M_{n-1} \cdots$ so that M has the same composition length as $X/\bigcap_{i=1}^{n} M_i$.

Thus, $\bar{X}$ has direct sums of arbitrary size, and so is infinite dimensional.

Conversely, we wish to show that if for every $N \subset M$, we can always find such a T, then M has no quotient which contains an infinite direct sum. If M/K does, then by the lemma we can find a K' with M/K' having an infinite direct snm of simple submodules. Let $K' \subset S \subset M$ be such that S/K' is this infinite direct sum. Choose T finitely generated such that S/T has no maximal submodules. Then $S/T + K'$ has no maximal submodules. But $S/T + K'$ is a homomorphic image of the semisimple module S/K' so $S/T + K'$ is semisimple, and always has maximal submodules if it is not zero. So, it must be zero and $S = T + K'$. Since T is finitely generated S/K' must be, a clear impossibility.

REMARK. The above naturally raises the question, when are finitely generated modules finite dimensional? We observe the following:

PROPOSITION. *If cyclic modules are finite dimensional then finitely generated modules are.*

Proof. Let $E(M)$ denote the injective hull of the module M. Let M be generated by $\{m_1, \cdots, m_r\}$. Let $E(M) = E(m_1R) \oplus K_1$, and write $m_2 = a_1 + k_1$. Then $K_1 = E(k_1R) \oplus K_2$. So $E(M) = E(m_1R) \oplus E(k_1R) \oplus K_2$, and $m_1R + m_2R \subset E(m_1R) \oplus E(k_1R)$. Continue in this fashion to get $E(M)$ as a finite direct sum of injective hulls of cyclic modules. These are finite dimensional, so M is.

Since there are non-noetherian valuation rings (ideals are linearly ordered), there are non-noetherian rings whose finitely generated modules are finite dimensional. The result cited in the first paragraph is 4.10 of [4].

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