PACIFIC JOURNAL OF MATHEMATICS
Vol 70, No 1, 1977

ENUMERATING NORMAL BUNDLES OF IMMERSIONS AND EMBEDDINGS OF PROJECTIVE SPACES

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All normal bundles of immersions $P^m \to R^{2m-\epsilon}$ for $m \geqq 7$, $\epsilon \leqq 2$ are classified. Those represented by embeddings are identified, and, for $\epsilon \leqq 1$, those which compress to an immersion $P^m \to R^{2m-\epsilon-1}$ are identified.

1. Introduction. The notation of [10] and [11] is used. Let all manifolds be differentiable, and all vector bundles real. Write P^m for real projective m-space, and let h be the canonical line bundle over P^m. If V is a manifold, let $[V \subseteq R^n]$ and $[V \subset R^n]$, respectively, be the set of regular homotopy classes of immersions $V \to R^n$ and the set of isotopy classes of embeddings $V \to R^n$. If ξ is a stable vector bundle over a complex X, let $A_k(X;\xi)$ be the set of equivalence classes of stabilized k-plane bundles over X representing ξ. (Equivalently, let $A_k(X;\xi)$ be the set of fiber-homotopy classes of liftings of the classification map $X \to BO$ to BO_k.) Thus, if $\dim V = m, [V \subseteq R^n] = A_{n-m}(V;\nu_V)$, where ν_V is the stable normal bundle of V.

Let $V_k(X;\xi)$ be the set of equivalence classes of k-plane bundles over X which represent ξ, i.e., the set of equivalence classes of $A_k(X;\xi)$ under ordinary bundle equivalence. There is a naturally defined action

$$\gamma\colon KO^{-1}(X) \times A_k(X;\xi) \to A_k(X;\xi)$$

such that the orbits of γ are precisely the elements $V_k(X;\xi)$; if $\dim X \leqq 2k-2$, $A_k(X;\xi)$ is an Abelian affine group and γ is an affine action, i.e., for any $\alpha \in KO^{-1}(X)$, $\gamma(\alpha, \ )\colon A_k(X;\xi) \to A_k(X;\xi)$ is an affine isomorphism. In that range, the so-called metastable range, let $A_k^0(X;\xi)$, an Abelian group, be the difference group of $A_k(X;\xi)$, provided the latter is nonempty. Any nonempty Abelian affine group is identified with its difference group by identifying some element with 0. This choice is arbitrary, and different choices may result in different expressions of the action γ and the corresponding equivalence relation on $A_k(X;\xi)$. The statements of theorems 1 and 2 are based on *some* choice.

Recall [11] that $[P^m \subseteq R^{2m-\epsilon}] \cong A_{m-\epsilon}(P^m, \nu)$, where ν is the stable normal bundle of P^m, and is an Abelian affine group (called the immersion group) if $m \geqq 7$ and $\epsilon \leqq 2$ (the only cases considered here). The orbits of the natural action of $KO^{-1}(P^m)$ on that immersion group then are the sets of classes which have equivalent normal bundles. Let $\sim$ be the equivalence relation on $P^mR^{2m-\epsilon}$ determined thereby.

THEOREM 1. (Main result). *Let $m \geqq 7$. Then the immersion groups, $[P^m \subseteq R^{2m-\epsilon}]$ for $\epsilon \leqq 2$, the subsets consisting of those immersions which come from embeddings and those which compress (only if $\epsilon \leqq 1$) to immersions in $R^{2m-1-\epsilon}$, and the relation $\sim$ on the immersion group can be read off table* 1.

Proof. The immersion groups themselves and the embedding and compressing facts are direct from theorem 0.1 of **[11]**. Using explicit descriptions of the action of $KO^{-1}(P^m)$ (as given in tables 2 & 3), the equivalence questions can all be settled.

TABLE 1

Case	immersion group	those which come from embeddings	those which compress	generators of the relation $\sim$	number of distinct normal bundles
$m = 2^r$	$[P^m \subseteq R^{2m}] \cong Z$	odds	just 0	$i \sim -i$, all $i \in Z$	∞
	$[P^m \subseteq R^{2m-1}] \cong Z_2$	none	none	$0 \sim 1$	1
	$[P^m \subseteq R^{2m-2} = \varnothing$	–	–	–	0
$m \equiv 0(1)$ not a power of 2	$[P^m \subseteq R^{2m}] \cong Z$	evens	just 0	$i \sim -i$, all $i \in Z$	∞
	$[P^m \subseteq R^{2m-1}] \cong Z_2$	just 0	just 0	none	2
	$[P^m \subseteq R^{2m-2}] \cong 0$	0	?	–	1
$m \equiv 1(4)$	$[P^m \subseteq R^{2m}] \cong Z_2$	just 0	both	none	2
	$[P^m \subseteq R^{2m-1}] \cong Z_2 \oplus Z_2 \oplus Z_2$	$(0, i, 0)$, all $i \in Z_2$	$(i, j, 0)$, all $i, j \in Z_2$	$(0, i, j) \sim (1, i, j)$ unless $i = j = 0$	5
	$[P^m \subseteq R^{2m-2}] \cong Z_2 \oplus Z_2$	if $m = 2^r + 1$, none. Otherwise, $(0,0)$ and $(0,1)$	?	$(0,1) \sim (1,1)$	3
$m \equiv 2(4)$	$[P^m \subseteq R^{2m}] \cong Z$	evens	just 0	$i \sim -i$, all $i \in Z$	∞
	$[P^m \subseteq R^{2m-1}] \cong Z_4$	just 0	0 and 2	$1 \sim 3$	3
	$[P^m \subseteq R^{2m-2}] \cong Z_2$	just 0	?	none	2

TABLE 1

Case	immersion group	those which come from embeddings	those which compress	generators of the relation $\sim$	number of distinct normal bundles
$m=7$	$[P^7 \subseteq R^{14}] \cong Z_2$	just 0	both	$0 \sim 1$	1
	$[P^7 \subseteq R^{13}] \cong Z_2 \oplus Z_4$	$(i,0)$, all $i \in Z_2$	all	all are equivalent	1
	$[P^7 \subseteq R^{12}] \cong Z_4 \oplus Z_8$	$(i,0)$, all $i \in Z_4$	?	$(0,0) \sim (0,i) \sim (1,i)$ $(2,0) \sim (2,i) \sim (3,i)$ all $i \in Z_8$	2
$m = 2^r - 1$, other than 7	$[P^m \subseteq R^{2m}] \cong Z_2$	just 0	both	$0 \sim 1$	1
	$[P^m \subseteq R^{2m-1}] \cong Z_2 \oplus Z_4$	$(i,0)$, all $i \in Z_2$	all	$(0,i) \sim (1,i+1)$, all $i \in Z_4$	4
	$[P^m \subseteq R^{2m-2}] \cong Z_4 \oplus Z_8$	$(i,0)$, all $i \in Z_4$	?	$(0,i) \sim (1,i+5)$ $(2,i) \sim (3,i+1)$ all $i \in Z_8$	16
$m \equiv 3(4)$, $m+1$ not a power of 2	$[P^m \subseteq R^{2m}] \cong Z_2$	just 0	both	none	2
	$[P^m \subseteq R^{2m-1}] \cong Z_2 \oplus Z_4$	$(i,0)$, all $i \in Z_2$	all	$(1,i) \sim (1,i+2)$, all $i \in Z_4$	6
	$[P^m \subseteq R^{2m-2}] \cong Z_4 \oplus Z_8$	$(i,0)$, all $i \in Z_4$	?	$(1,i) \sim (3,i+2)$ $(2,i) \sim (2,i+4)$ all $i \in Z_8$	20

The method. In the proof of Theorem 2, two different approaches for enumeration of immersions are used. We need the methods of **[11]** to determine which regular homotopy classes of immersions contain embeddings; while in the context of **[10]**, one may determine the actions which relate different immersions with the same normal bundle. The appendix of this paper, §4, relates these two approaches, when an Adams resolution is used. Note that in the range of dimensions considered in **[11]**, the Atiyah–Hirzebruch spectral sequence agrees with the Adams spectral sequence for $\mathcal{K}(Z)$.

2. Explicit description of the action. Recall that (in the notation of **[10]**) for $m \geqq 1$:

$$KO^{-1}(P^m) \cong \begin{cases} Z_2 \oplus Z_2 \text{ if } m \not\equiv 3(4), \text{ generated by } \rho \text{ and } \psi \\ Z_2 \oplus Z_2 \oplus Z \text{ if } m \equiv 3(4), \text{ generated by } \rho, \psi, \text{ and } \tau \end{cases}$$

Briefly, ρ is classified by a map $P^m \to O$ whose image is any point not in SO, ψ is classified by the Whitehead map $P^m \to SO$, and τ by the composition $P^m \to P^m/P^{m-1} \cong S^m \xrightarrow{\epsilon^m} SO$ where ϵ^m represents a generator of $\pi_m(SO) \cong Z$. Thus $h_0\rho = 1$, and $h_i\rho = 0$ for all $i > 0$; $h_0\psi = 0$ and $h_i\psi = u^i$ for all $i > 0$; while $h_m\tau = u^m$ for $m = 3$ or 7 and $h_i\tau = 0$ in all other cases.

We shall write αx for $\gamma(\alpha, x)$, for any $\alpha \in KO^{-1}(P^m)$ and any $x \in [P^m \subseteq R^n]$. The action of ρ has an easily visualizable geometric meaning, namely, if f: $P^m \to R^n$ is any immersion, $\rho[f] = [T \circ f]$, where T: $R^n \to R^n$ is reflection about any hyperplane. We shall not actually need to make use of this geometric interpretation.

THEOREM 2. *Let $m \geqq 7$. Then, for $\epsilon \leqq 2$, the affine group $[P^m \subseteq R^{2m-\epsilon}]$ can be identified with a direct sum of cyclic groups such that the action of $KO^{-1}(P^m)$ and the morphism (now a group homomorphism) $\mathscr{I}_{2m-\epsilon}$: $[P^m \subseteq R^{2m-\epsilon}] \to [P^m \subseteq R^{2m-\epsilon+1}]$ (recall $[P^m \subseteq R^{2m+1}] = 0$) are described in the following two tables.* (Table 2 for $m \equiv 0$, 1, or 2 mod 4; Table 3 for $m \equiv 3(4)$.)

TABLE 2

Case	immersion group	$\mathscr{I}_{2m-\epsilon}$	Action of ρ	Action of ψ
$m = 2^r$	$[P^m \subseteq R^{2m}] \cong Z$	0	$\rho i = -i$, all $i \in Z$	trivial
	$[P^m \cong R^{2m-1}] \cong Z_2$	0	$\rho i = i+1$, all $i \in Z_2$	$\psi i = i+1$, all $i \in Z_2$
	$[P^m \subseteq R^{2m-2}] = \varnothing$	–	–	–
$m \equiv 0(4)$, not a power of 2	$[P^m \subseteq R^{2m}] \cong Z$	0	$\rho i = -i$, all $i \in Z$	trivial
	$[P^m \subseteq R^{2m-1}] \cong Z_2$	0	trivial	trivial
	$[P^m \subseteq R^{2m-2}] \cong 0$	0	trivial	trivial

TABLE 2

Case	immersion group	$\mathcal{J}_{2m-\epsilon}$	Action of ρ	Action of ψ
$m \equiv 1(4)$	$[P^m \subseteq R^{2m}] \cong Z_2$	0	trivial	trivial
	$[P^m \subseteq R^{2m-1}] \cong$ $Z_2 \oplus Z_2 \oplus Z_2$	$\mathcal{J}_{2m-1}(i,j,k)$ $= k$	$\rho(i,j,k) =$ $(i+k,j,k)$	$\psi(i,j,k) =$ $(i+j,j,k)$
	$[P^m \subseteq R^{2m-2}] \cong$ $Z_2 \oplus Z_2$	$\mathcal{J}_{2m-2}(i,j)$ $= (i,j,0)$	trivial	$\psi(i,j) =$ $(i+j,j)$
$m \equiv 2(4)$	$[P^m \subseteq R^{2m}] \cong Z$	0	$\rho i = -i$	trivial
	$[P^m \subseteq R^{2m-1}] \cong Z_4$	0	$\rho i = -i$	trivial
	$[P^m \subseteq R^{2m-2}] \cong Z_2$	$\mathcal{J}_{2m-2}1 = 2$	trivial	trivial

TABLE 3

Case	immersion group	$\mathcal{J}_{2m-\epsilon}$	Action of ρ	Action of ψ	Action of τ
$m = 7$	$[P^7 \subseteq R^{14}] \cong Z_2$	0	trivial	$i = i+1$	$i = i+1$
	$[P^7 \subseteq R^{13}] \cong$ $Z_2 \oplus Z_4$	$\mathcal{J}_{13}(i,j)$ $= \bar{j} \in Z_2$, the mod 2 reduction of $j \in Z_4$	trivial	$\psi(i,j) =$ $(i+1, j+2i+1)$ where $2\colon Z_2 \to Z_4$ is the monomorphism	$\tau(i,j) =$ $(i, j+1)$
	$[P^7 \subseteq R^{12}] \cong$ $Z_4 \oplus Z_8$	$\mathcal{J}_{12}(i,j)$ $= (\bar{i}, \bar{j})$ $Z_2 \oplus Z_4$	trivial	$\psi(i,j) =$ $(-i+1, j-2i+5)$	$\tau(i,j) =$ $(i, j+1)$
$m = 2^r - 1$ other than 7	$[P^m \subseteq R^{2m}] \cong Z_2$	0	trivial	$\psi i = i+1$	trivial
	$[P^m \subseteq R^{2m-1}] \cong$ $Z_2 \oplus Z_4$	$\mathcal{J}_{2m-1}(i,j)$ $= \bar{j}$	trivial	$\psi(i,j) =$ $(i+1, j+2i+1)$	trivial
	$[P^m \subseteq R^{2m-2}] \cong$ $Z_4 \oplus Z_8$	$\mathcal{J}_{2m-2}(i,j)$ $= (\bar{i}, \bar{j})$	trivial	$\psi(i,j) =$ $(-i+1, j-2i+5)$	trivial
$m \equiv 3(4)$, $m+1$ not a power of 2	$[P^m \subseteq R^{2m}] \cong Z_2$	0	trivial	trivial	trivial
	$[P^m \subseteq R^{2m-1}] \cong$ $Z_2 \oplus Z_4$	$\mathcal{J}_{2m-1}(i,j)$ $= \bar{j}$	trivial	$\psi(i,j) =$ $(i, j+2i)$	trivial
	$[P^m \subseteq R^{2m-2}] \cong$ $Z_4 \oplus Z_8$	$\mathcal{J}_{2m-2}(i,j)$ $= (\bar{i}, \bar{j})$	trivial	$\psi(i,j) =$ $(-i, j+2i)$	trivial

Proof. In all cases, we utilize the results of **[10]** and **[11]**. We also make use of the natural affine isomorphism $[P^m \subseteq R^{2m-\epsilon}] \cong V_{m-\epsilon}(P^m, -(m+1)h)$; the actions are stated in terms of the former affine group, but are actually computed in terms of the latter. (See §4 for a complete discussion of the relation between the Adams resolutions of the difference groups of those two Abelian affine groups.) For even m, the stated results are the only ones which agree with [**11**, Thm. 0.1] and [**10**, Thm. 3.8]. For $m \equiv 1(4)$, we need use only those results together with lemmas of this paper. We give a complete proof of Case IV only, assuming that the reader is familiar with the notation of **[10]**.

Let $m \equiv 3(4)$. The action of ρ is trivial on $[P^m \subseteq R^{2m-2}]$, since, by [**10**, 3.7], ρ is the null element of ν, the normal bundle of P^m in dimension $m-2$. Since $\mathcal{I}_{2m-2}$ and $\mathcal{I}_{2m-1}$ are onto, ρ must, by naturality, act trivially on the other two affine groups.

The actions of ψ and τ on $A_{m-\epsilon}(P^m; \nu)$ are explicitly given in [**10**, Thm. 3.8], for $\epsilon \leqq 1$. Using 4.2 below, those actions can be seen to be the ones described above for those cases. We thus need only to examine the action of ψ and τ on $[P^m \subseteq R^{2m-2}] \cong A_{m-2}(P^m, \nu)$.

We deal with τ first. By 3.4, τ acts trivially on the difference group, since τ "lives" on the top cell of P^m. The action of τ on $A_{m-2}(P^m; \nu)$ is thus a pure translation, and is determined solely by $s_a\tau \in A^0_{m-2}(P^m; \nu)$, which then does not depend on the choice of a. Pinching any $S^m \subset P^m \times S^1$ which is the boundary of a smooth disc, we obtain an onto map $q: P^m \times S^1 \to (P^m \times S^1) \vee S^{m+1}$; and $F_{\tau,\nu}$ can be chosen to be the composition

$$P^m \times S^1 \xrightarrow{q} (P^m \times S^1) \vee S^{m+1} \xrightarrow{p_1 \vee 1} P^m \vee S^{m+1} \xrightarrow{f \vee \epsilon^{m+1}} BO$$

where $f: P^m \to BO$ classifies ν, ϵ^{m+1} represents the generator of $\pi_{m+1}(BO)$ in that dimension, and p_1: $P^m \times S^1 \to P^m$ is the projection. By definition of the action γ, τ has a fixed point if and only if $F_{\tau,\gamma}$ lifts to BO_{m-2}, and if τ has a fixed point, $s_a\tau = 0$ for $a =$ fixed point, and hence for all a. Choose $g: P^m \to BO_{m-2}$ which represents some immersion of P^m in R^{2m-2}. If $m \geqq 15$, $\pi_{m+1}(BO_{m-2}) \to \pi_{m+1}(BO)$ is onto, by Barrat and Mahowald **[1]**. Thus a map $\zeta: S^{m+1} \to BO_{m-2}$ may be chosen which projects to ϵ^{m+1}, and $(g \circ p_1 \vee \zeta) \circ q$ is the desired lifting. M. Mahowald has personally assured us the existence of ζ in the case $m = 11$ also; thus τ acts trivially for all $m \geqq 11$. (It is not actually necessary to make use of this unpublished fact; a modification of the argument using quaternionic bundles in the proof of [**10**, 3.8] suffices for all $m \equiv 3(8)$.) The only remaining case is $m = 7$; by naturality, $s_a\tau$ must then be an element of order 8. Since there is a certain freedom in the choice of generators, it is possible to insist that $s_a\tau = (0, 1)$.

We now look at the action of ψ. The $\mathcal{K}(Z)$-Adams resolution of BO_{m-2} over BO yields the tower (where all spaces are over BO, and $k_r = BO \times K(Z_2, r)$, the Eilenberg–Maclane space in the category of BO-sectioned spaces):

$$\begin{array}{ccccc}
k_m & \xrightarrow{\lambda} & E_3 & & \\
 & & \downarrow{\scriptstyle \pi_3} & & \\
k_{m-2} \times k_m & \xrightarrow{\lambda} & E_2 & \xrightarrow{\beta} & k_{m+1} \\
 & & \downarrow{\scriptstyle \pi_2} & & \\
k_{m-2} \times k_m & \xrightarrow{\lambda} & E_1 & \xrightarrow{\alpha} & k_m \times k_{m+1} \\
 & & \downarrow{\scriptstyle \pi_1} & & \\
P^m & \xrightarrow{f} & BO & \xrightarrow{w_{m-1}, \hat{w}_{m+1}} & k_{m-1} \times k_{m+1}
\end{array}$$

where E_3 is an m-approximation of BO_{m-2}, and where

$$\hat{w}_{m+1} = w_{m+1} + w_1 w_m + (w_2 + w_1^2) w_{m-1}$$

$$\lambda^* \alpha^* (\iota_m \otimes 1) = (Sq^2 \iota_{m-2} + w_2 \iota_{m-2}) \otimes 1$$

$$\lambda^* \alpha^* (1 \otimes \iota_{m+1}) = (Sq^2 Sq^1 \iota_{m-2} + (w_3 + w_2 w_1) \iota_{m-2}) \otimes 1 + 1 \otimes Sq^1 \iota_m$$

$$\lambda^* \beta^* \iota_{m+1} = (Sq^2 \iota_{m-1} + w_2 \iota_{m-1}) \otimes 1 + 1 \otimes Sq^1 \iota_m$$

Let g be a lifting of f to BO_{m-2}, and let g_1 be the projection of g down to E_1. Then we have affine morphisms:

$$[P^m ; E_3]_{g_1} \xrightarrow{(\pi_1)^*} [P^m ; E_3]_f = A_{m-2}(P^m ; \nu) \xrightarrow{(\pi_3)_*} [P^m ; E_2]_f$$

By using the results of **[12]**, we can calculate that $[P^m ; E_2]_f^0 \cong Z_4 \oplus Z_4$ and $[P^m ; E_3]_{g_1}^0 \cong Z_2 \oplus Z_4$; and then, by the results of **[10]**, the action of ψ on both of those groups can be established. The stated results are the only ones (up to choice of generators) which agree with these calculations. We omit the details.

Comparison with previous results. We remark that the choices of generators in Theorems 1 and 2 of this paper, and the same choices in Theorem 0.1 of **[11]**, can be made identical.

3. For any complex X, $\rho \in KO^{-1}(X)$ may be defined to be the element classified by a constant map to a point in 0 which is not in

SO. For any stable bundle ξ over X, let $s\colon A_k(X;\xi)\to A_{k+1}(X;\xi)$ be suspension, i.e., Whitney sum with a trivial line bundle.

LEMMA 3.1. *For any stable bundle ξ over a complex X, and for any $x\in A_k(X;\xi)$, $\rho sx = s\rho x = sx$.*

Proof. Since $i\colon BO_k\to BO_{k+1}$ is a map of spaces over BO, $\rho s = s\rho$. Let $f\colon X\to BO$ classify ξ, and let $g\colon X\to BO_k$ classify x. Then the composition $X\times S^1\xrightarrow{g\times\epsilon^1} BO_k\times BO_1\to BO_{k+1}$ (where ϵ^1 represents the generator of $\pi_1(BO_1)$) is a lifting of $F_{\rho,\xi}$ which agrees with $i\circ g$ on X. Thus ρ acts trivially on sx.

COROLLARY 3.2. *If $f\colon V\to R^n$ is any immersion, where V is any manifold, then $\rho\mathcal{I}_n[f]=\mathcal{I}_n[f]\in[V\subseteq R^{n+1}]$.*

We say that a space F has *homotopy width k* if for some n, F is n-connected and has no homotopy above dimension $n+k$.

LEMMA 3.3. *Let $\pi\colon E\to B$ be a fibration with fiber a stable space F (all spaces having the homotopy type of CW complexes), and let $f\colon X\to B$ be a map, where X is a CW complex. Let $\gamma\colon \pi_1(B^X,f)\times[X;E]_f\to[X;E]_f$ be the left action as given in* **[9]**; *where $[X;E]_f$ is the Abelian affine group of vertical homotopy classes of liftings of f to E. Suppose that $\alpha\in\pi_1(B^X,f)$ is trivial on the $(k-1)$-skeleton of X, i.e., is represented by a homotopy $f_t\colon X\times I\to B$ such that $f_0=f_1=f$ and $f_t\mid X^{k-1}=f\mid X^{k-1}$ for all t: then $\gamma(\alpha,\ )\colon [X;E]_f\to[X;E]_f$ is a pure translation; i.e., there exists $s\alpha\in[X;E]_f^0$ such that $\gamma(\alpha,x)=x+s\alpha$ for all $x\in[X;E]_f$.*

Proof. By **[14]**, there exists a sectioned fibration $BF\in\mathcal{T}Y(\text{fib})$, where Y is of the homotopy type of a CW complex, such that $P_YBF\to BF$ is the universal example of fibrations with fiber the homotopy type of F and with base the homotopy type of a CW complex. Furthermore, BF can be delooped in the category $\mathcal{T}Y(\text{fib})$; and by **[4]**, $\Omega Y\sim\mathcal{F}$, the H-space of all base-point preserving self-homotopy equivalences of F. A classifying map θ can then be chosen, and we have a diagram (essentially diagram (1–1) of **[9]**):

$$
\begin{tikzcd}
F \arrow[r] & E \arrow[r] \arrow[d, "\pi"] & P_YBF \arrow[d] \\
X \arrow[r, "f"] & B \arrow[r, "\theta"] \arrow[dr, "\beta"'] & BF \arrow[d, shift left] \\
 & & Y \arrow[u, shift left]
\end{tikzcd}
$$

By a straightforward obstruction theory argument, $\pi_i(\mathcal{F}) = 0$ for $i \geqq k$, thus $\pi_i(Y) = 0$ for all $i > k$. Another obstruction theory argument then shows that $\beta_{\#}\colon \pi_1(B^X, f) \to \pi_1(B^Y, \beta \circ f)$ sends α to the identity. Thus, by [**9**, 3.1], $\gamma(\alpha, \ )$ is a pure translation.

COROLLARY 3.4. *If ξ is a stable bundle over a complex X of dimension $m \leqq 2k - 2$, and if $\alpha \in KO^{-1}(X)$ is trivial on the $(m-k)$-skeleton of X, i.e., lies in the kernel of $KO^{-1}(X) \to KO^{-1}(X^{m-k})$, then $\alpha\colon A_k(X;\xi) \to A_k(X;\xi)$ is a pure translation.*

Proof. $A_k(X;\xi) = [X; E_m]_f$, where $f\colon X \to BO$ classifies ξ, and $\pi\colon E_m \to BO$ is obtained by killing all homotopy groups of the fiber of $BO_k \to BO$ above m. The fiber of π is then a stable space of homotopy width $m - k + 1$. Apply 3.3, and we are done.

4. Appendix. Comparing three classification theorems for immersions. In [**6**], Hirsch proved the following classification theorem for immersions. Let τ be the tangent bundle of the m-dimensional manifold V, and let $L^n(\tau)$ be the bundle over V whose fiber over $x \in V$ is the space of linear injective maps $\tau_x \to R^n$. The differential of an immersion $f\colon V \to R^n$ determines a cross-section $L(f)$ of $L^n(\tau)$. Hirsch proved that the map $[V \subseteq R^n] \to [V; L^n(\tau)]_1$ (where 1 is the identity map on V) which maps the regular homotopy class $[f]$ to $[L(f)]$ is a bijection for $n > m$.

In [**5**], Haefliger and Hirsch proved a second classification theorem for immersions in the metastable range. Give τ a metric and let τ_0 and $P(\tau)$ denote the sphere bundle and projective bundle respectively, of τ. A cross section g of $L^n(\tau)$ determines a Z_2-equivariant map $\bar{g}\colon \tau_0 \to S^{n-1}$ where Z_2 acts via the antipodal map, and $\bar{g}$ determines a cross-section of the S^{n-1} bundle $(\tau)^{n-1}$ associated to the double cover $\tau_0 \to P(\tau)$. In [**5**], it is proved that the induced map $[V; L^n(\tau)]_1 \to [P(\tau); (\tau)^{n-1}]_1$ is a bijection for $2n > 3m + 1$.

Finally, in [**8**], James and Thomas proved that $[M \subseteq R^n]$ is in one-to-one correspondence with $A_{n-m}(V; \nu_V)$ if $n - m > 1$. The bijection can be obtained as follows. Let $\bar{\nu}$ be a vector bundle in the stable class of ν_V and let $\bar{\nu}_l$ denote the bundle of orthonormal l-frames of $\bar{\nu}$. Then, if $l = \dim \bar{\nu} - n + m$, we can identify $A_{n-m}(V; \nu_V)$ with $[V; \bar{\nu}_l]_1$. There are natural maps $L^n(\tau) \to L^{n+l}(\tau \oplus \epsilon^l)$ and $\bar{\nu}_l \to L^{n+l}(\tau \oplus \epsilon^l)$ where the first map is the obvious one and the second map takes an l-frame $R^l \to \bar{\nu}_x$ to the composite $\tau_x \times R^l \to \tau_x \times \bar{\nu}_x \to R^{n+l}$. (The second map comes from a trivialization of $\tau \oplus \bar{\nu}$.) The induced maps

$$[V; L^n(\tau)]_1 \to [V; L^{n+l}(\tau \oplus \epsilon^l)]_1$$
$$[V; \bar{\nu}_l]_1 \to [V; L^{n+l}(\tau \oplus \epsilon^l)]_1$$

are bijections for $n - m > 1$.

Assume from now on that $2n > 3m + 1$. Then, if M immerses in R^n, each of the above classification theorems induces an affine group structure on $[M \subseteq R^n]$. The difference groups (in the notation of **[13]**) are, respectively:

$$\{V; S'_V(L^n(\tau))\}_1^{-1}$$
$$\{P(\tau); S'_{P(\tau)}((\tau)^{n-1})\}_1^{-1}$$
$$\{V; S'_V(\bar{\nu}_l)\}_1^{-1}$$

where S'_V and $S'_{P(\tau)}$ denote fiberwise unreduced suspension. By **[13]**, for each spectrum $\mathscr{E}$ satisfying the conditions of [**13**, §2], there are spectral sequences

$$\{E_r(V, S'_V(L^n(\tau)); \mathscr{E})_1\}$$
$$\{E_r(P(\tau), S'_{P(\tau)}((\tau)^{n-1}); \mathscr{E})_1\}$$
$$\{E_r(V, S'_V(\bar{\nu}_l); \mathscr{E})_1\}$$

converging to quotients of

$$\{V; S'_V(L^n(\tau))\}_1^*$$
$$\{P(\tau); S'_{P(\tau)}((\tau)^{n-1})\}_1^*$$
$$\{V; S'_V(\bar{\nu}_l)\}_1^*$$

respectively.

The purpose of this appendix is to show that these difference groups and the portions of the above Adams spectral sequences converging to them are isomorphic (by isomorphisms consistent with the bijections discussed in the first three paragraphs of this section). These isomorphisms will be made explicit for $\mathscr{E} = \mathscr{K}(Z_2)$, the Eilenberg–MacLane spectrum for Z_2.

By naturality of the spectral sequences and the existence of the maps $L^n(\tau) \to L^{n+l}(\tau \oplus \epsilon^l)$ and $\bar{\nu}_l \to L^{n+l}(\tau \oplus \epsilon^l)$, it suffices to show that $\{E_r(V, S'_V(L^n(\tau)); \mathscr{E})_1\}$ and $\{E_r(P(\tau), S'_{P(\tau)}((\tau)^{n-1}); \mathscr{E})_1\}$ are isomorphic.

Let h be the canonical line bundle over $P(\tau)$. Then in the notation of **[11]**, $\{P(\tau); S'_{P(\tau)}((\tau)^{n-1})\}_1^{-1} = \pi_{nh}^{n-1}(P(\tau))$. By the Thom isomorphism theorem $\pi_{nh}^{n-1}(P(\tau)) \cong \pi_{(n+l)h}^{n+l-1}(D(lh), (lh)_0)$ where $D(lh)$ and $(lh)_0$ are the disc and sphere bundles of lh, respectively **[3]**. Since the stable vector bundle determined by a line bundle over a finite-dimensional CW complex has order a power of 2, we can choose l so that $(n+l)h$ is trivial. By choosing a trivialization of $(n+l)h$, we can identify $\pi_{(n+l)h}^{n+l-1}(D(lh),(lh)_0)$ with $\{T(lh); S^{n+l}\}^{-1}$ where $T(lh)$ is the Thom complex of lh. Let $T_V(lh)$ be obtained from $D(lh)$ by identifying the portion of $(lh)_0$ lying over $x \in V$ to a point for each $x \in V$, so that $T_V(lh)$ becomes a V-sectioned space with $T_V(lh)/V = T(lh)$. Then $\{T(lh); S^{n+l}\}^{-1} \cong \{T_V(lh); V \times S^{n+l}\}_V^{-1}$ where $\{\ ;\ \}_V$ denotes stable homotopy in the category of V-sectioned spaces (see **[2]**).

The fiber of $T_V(lh)$ over any $x \in V$ can be viewed as $P_{l+m,m} = P^{l+m-1}/P^{l-1}$ (see [7, p. 205] and, in fact, we can identify $T_V(lh)$ with $P_{l+m,m}(\tau)$ if $l \geqq m$, where $P_{l+m,m}(\tau)$ is obtained from $P(\tau \oplus \epsilon^l)$ by identifying each fiber of $P(\tau \oplus \epsilon^{l-m}) \subset P(\tau \oplus \epsilon^l)$ to a point. Assume that V immerses in R^n and take $\tilde{\nu}$ to be an $(n-m)$-plane bundle in the stable class ν. By **[2]**, if $l+n$ is a large power of 2, there is a duality map

$$T_V(lh) \wedge_V S_V(P_{n,m}(\tilde{\nu})) \rightarrow V \times S^{l+n}$$

in the category of V-sectioned spaces which induces isomorphisms

$$\{T_V(lh);\ V \times S^0\}_V^i \cong \{V \times S^0;\ P_{n,m}(\tilde{\nu})\}_V^{i-l-n+1} = \{V;\ P_{n,m}(\tilde{\nu})\}_l^{i-l-n+1}$$

Now it is not difficult to show that the Thom isomorphism

$$\pi_{nh}^i(P(\tau)) \cong \pi_{(n+l)h}^{i+l}(D(lh),(lh)_0) = \{T_V(lh);\ V \times S^0\}_V^{i+l}$$

preserves the Adams spectral sequence; a routine argument using the following lemma then establishes that the above duality map induces an isomorphism of the $\mathscr{E}$-Adams spectral sequence for $\{T_V(lh);\ V \times S^0\}_V^*$ with $\{E_r(V, P_{n,m}(\tilde{\nu}); \mathscr{E})_l\}$ when each space of $\mathscr{E}$ is compact.

LEMMA 4.1. *Let $\bar{\mathscr{E}}$ be a spectrum of B-sectioned bundles with compact CW fibers, where B is a compact CW complex and let U: $W \wedge_B W^* \rightarrow B \times S^k$ be a k-duality in the category of B-sectioned bundles. (See* **[2]** *for definitions.) Then U induces an isomorphism*

$$\{B \times S^0;\ W \wedge_B \bar{\mathscr{E}}\}_B^i \rightarrow \{W^*;\ S^k \wedge_B \bar{\mathscr{E}}\}_B^{-i}$$

Proof. The lemma is easily proved using the methods of [**2**, §3 and §4].

Let Z be the V-sectioned bundle whose fiber over $x \in V$ is the space of basepoint preserving maps from the fiber $T(lh_x)$ of $T_V(lh)$ to

S^{l+n}. The duality map

$$T_V(lh) \wedge_V S_V(P_{n,m}(\tilde{\nu})) \to V \times S^{l+n}$$

induces a map of V-sectioned spaces $\alpha\colon S_V(P_{n,m}(\tilde{\nu})) \to Z$ which is a $(2n-2m+1)$-equivalence on fibers.

We now define a map of V-sectioned spaces $\beta\colon S'_V(L^n(\tau)) \to Z$ which is a $(2n-2m)$-equivalence on fibers. Given an injective linear map $g_x\colon \tau_x \to R^n$, the Z_2-equivariant map $\bar{g}_x\colon (\tau_0)_x \to S^{n-1}$ determined by g_x defines a cross-section of $(\tau)_x^{m-1}$ which we denote by $\bar{g}_x$ also. In turn, $\bar{g}_x$ determines a specific cross-section $\tilde{g}_x$ of the restriction of the sphere bundle $((n+l)h)_0$ to the fiber $D(lh_x)$ of $D(lh)$ such that $\tilde{g}_x \mid (lh_x)_0$ is the diagonal map $(lh_x)_0 \to (lh_x)_0 \times_V (lh_x)_0$. Composition of $\tilde{g}_x$ with a trivialization of $((n+l)h)_0$ yields a map $g'_x\colon D(lh_x) \to S^{l+n-1}$. We may assume that the trivialization has been chosen so that $g'_x((lh_x)_0)$ is the South pole of S^{l+n-1}. Define β by $\beta([g_x, t])(y) = [g'_x(y), t]$ for $t \in [0,1]$, $y \in D(lh_x)$; where S^{l+n} is identified with the reduced suspension of S^{l+n-1}.

α and β induce isomorphisms of the relevant portions of the spectral sequences $\{E_r(V, S'_V(P_{n,m}(\tilde{\nu})); \mathscr{E})_1\}$ and $\{E_r(V, S'_V(L^n(\tau)); \mathscr{E})_1\}$.

Let $\mathscr{K} = \mathscr{K}(Z_2)$. We now examine the case $\mathscr{E} = \mathscr{K}$ in greater detail. The Thom isomorphism gives an isomorphism

$$\begin{gathered}\{P(\tau);\, S'_{P(\tau)}((\tau)^{n-1}) \wedge_{P(\tau)} \mathscr{K}\}_1^{i-n} = H^i(P(\tau); Z_2)\\ \downarrow a\\ \{T_V(lh);\, V \times \mathscr{K}\}_V^{i+l} = H^{i+l}(T_V(lh), V; Z_2)\end{gathered}$$

and duality gives an isomorphism

$$\begin{gathered}\{T_V(lh);\, V \times \mathscr{K}\}_V^{i+l}\\ \downarrow b\\ \{V;\, S'_V(P_{n,m}(\tilde{\nu})) \wedge_V \mathscr{K}\}_1^{i-n}\end{gathered}$$

In the notation of **[13]**,

$$\{V;\, S'_V(P_{n,m}(\tilde{\nu})) \wedge_V \mathscr{K}\}_1^{i-n} = [V;\, \Omega_V^k(S_V^j(\mathscr{K}_V(S'_V(P_{n,m}(\tilde{\nu})))))]_1$$

where $j - k = i - n$.

THEOREM 4.2. *There is a fiber-homotopy equivalence of V-sectioned spaces $\mathscr{K}_V(S'_V(P_{n,m}(\tilde{\nu}))) \to V \times \prod_{j=n-m+1}^{n} K(Z_2, j)$ such that $b \circ a\colon H^i(P(\tau); Z_2) \to \bigoplus_{j=0}^{m-1} H^{i-j}(V; Z_2)$ maps $\sum_{j=0}^{m-1} u^j x_j$ to $\bigoplus_{j=0}^{m-1} x_j$ for $x_j \in H^{i-j}(V; Z_2)$; where $u = w_1(h) \in H^1(P(\tau); Z_2)$.*

Proof. $(b \circ a)(u^{\prime})$ is represented by a map

$$f_{\iota}\colon V \times S^{n-j+s} \to S^{\prime}_{V}(P_{n,m}(\tilde{\nu})) \wedge_{V} K(Z_2, s)$$

of V-sectioned spaces for large s. Let $\bar{f}_{\iota}$ be the composite

$$\begin{aligned} V \times (S^{n-j+s} \wedge K(Z_2, r)) &\to S^{\prime}_{V}(P_{n,m}(\tilde{\nu})) \wedge_{V} (K(Z_2, s) \wedge K(Z_2, r)) \\ &\to S^{\prime}_{V}(P_{n,m}(\tilde{\nu})) \wedge_{V} K(Z_2, r+s) \end{aligned}$$

where the first map is obtained by smashing with $K(Z_2, r)$ and the second map is $1 \wedge_{V} \mu$. (where μ is the product map.) Taking the adjoint map, applying Ω^{r}_{V}, and passing to the limit yields $\tilde{f}_{\iota}\colon V \times K(Z_2, n-j) \to \mathscr{K}_{V}(S^{\prime}_{V}(P_{n,m}(\tilde{\nu})))$. Since $\mathscr{K}_{V}(S^{\prime}_{V}(P_{n,m}(\tilde{\nu})))$ is a fiberwise infinite loop space, we can add the f_i to obtain

$$f\colon V \times \prod_{j=n-m+1}^{n} K(Z_2, j) \to \mathscr{K}_{V}(S^{\prime}_{V}(P_{n,m}(\tilde{\nu})))$$

It is readily checked that f is a fiber homotopy equivalence of V-sectioned spaces and that the inverse of f is the desired map.

In the notation of **[13]**, let $\mathscr{E}_s$ be the spectrum with ith space $\mathscr{E}((S^{i})_s)$. Then

$$E_1^{s,t}(P(\tau), S^{\prime}_{P(\tau)}((\tau)^{n-1}); \mathscr{E})_1 = \{P(\tau); S^{\prime}_{P(\tau)}(\tau^{n-1}) \wedge_{P(\tau)} \mathscr{E}_s\}_1^{s-t}$$

$$E_1^{s,t}(V, S^{\prime}_{V}(P_{n,m}(\tilde{\nu})); \mathscr{E})_1 = \{V; S^{\prime}_{V}(P_{n,m}(\tilde{\nu})) \wedge_{V} \mathscr{E}_s\}_1^{s-t}$$

When $\mathscr{E} = \mathscr{K}(Z_2)$ or $\mathscr{K}(Z)$, $\mathscr{E}_s$ for $s > 0$ is equivalent to a product of $\mathscr{K}(Z_2)$'s; hence Theorem 4.2 determines the isomorphism explicitly in either of these two cases. Moreover, a description of the relations associated with the differentials in $\{E_r(P(\tau), S^{\prime}_{P(\tau)}((\tau)^{n-1}); \mathscr{E})\}_1$ determines a description of the relations associated with the differentials in $\{E_r(P(\tau), S^{\prime}_{P(\tau)}((\tau)^{n-1}; \mathscr{E})\}_1$ determines a description of the relations associated with the differentials in $\{E_r(V, S^{\prime}_{V}(P_{n,m}(\tilde{\nu})); \mathscr{E})\}_1$. The reader should be aware that the relations thus obtained are not, in general, the relations which appear in other treatments of similar resolutions of $\tilde{\nu}_l$ or $P_{n,m}(\tilde{\nu})$ (e.g., in **[10]**).

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Received August 11, 1975.

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