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ENUMERATION OF DOUBLY UP-DOWN PERMUTATIONS

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It is well known that $A(n)$, the number of up-down permutations of $\{1,2,\cdots,n\}$ satisfies

$$\sum_{n=0}^{\infty} A(2n)\frac{z^{2n}}{(2n)!} = \sec z,$$

$$\sum_{n=0}^{\infty} A(2n+1)\frac{z^{2n+1}}{(2n+1)!} = \tan z.$$

In the present paper generating functions are obtained for up-down (down-up) permutations in which the peaks themselves are in an up-down configuration.

In a previous paper the writer obtained generating functions for the number of up-down (and down-up) permutations counting the rises among the "peaks".

1. Let $Z_n = \{1,2,\cdots,n\}$ and let $(a_1,a_2,\cdots,a_n)$ be an arbitrary [**4**, pp. 105–112] up-down permutation of Z_n. Then $(b_1,b_2,\cdots,b_n)$, where

$$b_i = n - a_i + 1 \qquad (i = 1,2,\cdots,n)$$

is a down-up permutation and vice versa. Thus, for $n > 1$, there is a one-to-one correspondence between up-down and down-up permutations so that it suffices to consider the former.

In the present paper we are concerned with up-down (and down-up) permutations of Z_n in which it is required that the peaks themselves satisfy the up-down or down-up conditions. Thus let $(a_1,a_2,\cdots,a_n)$ denote an up-down permutation of Z_n so that

$$(1.1) \qquad a_{2k-1} < a_{2k}, \quad a_{2k} > a_{2k+1} \qquad (k = 1,2,\cdots,[n/2]).$$

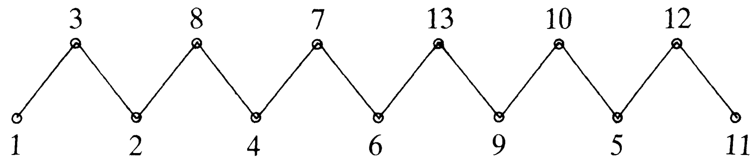
Then the additional requirement is either

$$(1.2) \qquad a_{4k-2} < a_{4k}, \quad a_{4k} > a_{4k+2} \qquad (k = 1,2,\cdots,[n/4])$$

or

$$(1.3) \qquad a_{4k-2} > a_{4k}, \quad a_{4k} < a_{4k+2} \qquad (k = 1,2,\cdots,[n/4]).$$

For example the permutation $(1,3,2,8,4,7,6,13,9,10,5,12,11)$



satisfies both (1.1) and (1.2).

Next let $(a_1, a_2, \cdots, a_n)$ be a down-up permutation of Z_n so that

$$(1.4) \qquad a_{2k-1} > a_{2k}, \quad a_{2k} < a_{2k+1} \qquad (k = 1,2,\cdots,[n/2]).$$

Then the additional requirement is either

$$(1.2)' \qquad a_{4k-3} > a_{4k-1}, \quad a_{4k-1} < a_{4k+1} \qquad (k = 1,2,\cdots,[n/4])$$

or

$$(1.3)' \qquad a_{4k-3} < a_{4k-1}, \quad a_{4k-1} < a_{4k+1} \qquad (k = 1,2,\cdots,[n/4]).$$

Thus there are four possibilities, namely

I. (1.1) and (1.2),
II. (1.1) and (1.3),
III. (1.4) and (1.2)′,
IV. (1.4) and (1.3)′.

There are various relations between these varieties of permutations; however they depend upon the residue of n (mod 4).

In order to derive generating functions it will be convenient to define the following enumerants. Let $A_{RF}(n)$ denote the number of up-down permutations of Z_n such that the peaks

$$(a_2, a_4, \cdots, a_{2[n/2]})$$

begin with a rise, $a_2 < a_4$, and end with a fall. Thus for this case it is necessary that $n \equiv 2$ or 3 (mod 4).

We define $A_{RR}(n)$, $A_{FR}(n)$, $A_{FF}(n)$ in a similar way. Note that for RR, $n \equiv 0$ or 1 (mod 4); for FR, $n \equiv 2$ or 3 (mod 4); for FF, $n \equiv 0$ or 1 (mod 4).

We also define $C_{RF}(n)$, $C_{RR}(n)$, $C_{FR}(n)$, $C_{FF}(n)$ in an analogous manner for down-up permutations. Then for RF, $n \equiv 1$ or 2 (mod 4), for RR, $n \equiv 0$ or 3 (mod 4), for FR, $n \equiv 1$ or 2 (mod 4), for FF, $n \equiv 0$ or 3 (mod 4).

We shall accordingly consider the following enumerants:

$$(1.5)\qquad \begin{cases} A_{RF}(4n+3), & A_{RF}(4n+2) \\ A_{RR}(4n+1), & A_{RR}(4n) \\ A_{FR}(4n+3), & A_{FR}(4n+2) \\ A_{FF}(4n+1), & A_{FF}(4n) \end{cases}$$

$$(1.6)\qquad \begin{cases} C_{RF}(4n+1), & C_{RF}(4n+2) \\ C_{RR}(4n+3), & C_{RR}(4n) \\ C_{FR}(4n+1), & C_{FR}(4n+2) \\ C_{FF}(4n+3), & C_{FF}(4n). \end{cases}$$

Reading a permutation both from left to right and from right to left, we get the following relations connecting the enumerants.

$$(1.7)\qquad \begin{cases} A_{RR}(4n+1) = A_{FF}(4n+1) \\ A_{RR}(4n) = C_{FF}(4n) \\ A_{FF}(4n) = C_{RR}(4n) \\ A_{RF}(4n+2) = C_{RF}(4n+2) \\ A_{FR}(4n+2) = C_{FR}(4n+2) \\ C_{RR}(4n+3) = C_{FF}(4n+3). \end{cases}$$

Put

$$(1.8)\qquad \begin{cases} y_{RF}(x) = \sum_{n=0}^{\infty} A_{RF}(4n+3)\dfrac{x^{4n+3}}{(4n+3)!} & (A_{RF}(3) = 2) \\ y_{RR}(x) = \sum_{n=0}^{\infty} A_{RR}(4n+1)\dfrac{x^{4n+1}}{(4n+1)!} & (A_{RR}(1) = 1) \\ y_{FR}(x) = \sum_{n=0}^{\infty} A_{FR}(4n+3)\dfrac{x^{4n+3}}{(4n+3)!} & (A_{FR}(3) = 2) \\ y_{FF}(x) = \sum_{n=0}^{\infty} A_{FF}(4n+1)\dfrac{x^{4n+1}}{(4n+1)!} & (A_{FF}(1) = 1) \end{cases}$$

$$(1.9)\qquad \begin{cases} z_{RF}(x) = \sum_{n=0}^{\infty} A_{RF}(4n+2)\dfrac{x^{4n+2}}{(4n+2)!} & (A_{RF}(2) = 1) \\ z_{RR}(x) = \sum_{n=0}^{\infty} A_{RR}(4n)\dfrac{x^{4n}}{(4n)!} & (A_{RR}(0) = 1) \\ z_{FR}(x) = \sum_{n=0}^{\infty} A_{FR}(4n+2)\dfrac{x^{4n+2}}{(4n+2)!} & (A_{FR}(2) = 1) \\ z_{FF}(x) = \sum_{n=0}^{\infty} A_{FF}(4n)\dfrac{x^{4n}}{(4n)!} & (A_{FF}(0) = 1) \end{cases}$$

$$(1.10)\quad\begin{cases} \bar{y}_{RF}(x) = \sum_{n=0}^{\infty} C_{RF}(4n+1)\dfrac{x^{4n+1}}{(4n+1)!} & (C_{RF}(1)=1) \\ \bar{y}_{RR}(x) = \sum_{n=0}^{\infty} C_{RR}(4n+3)\dfrac{x^{4n+3}}{(4n+3)!} & (C_{RR}(3)=2) \\ \bar{y}_{FR}(x) = \sum_{n=0}^{\infty} C_{FR}(4n+1)\dfrac{x^{4n+1}}{(4n+1)!} & (C_{FR}(1)=1) \\ \bar{y}_{FF}(x) = \sum_{n=0}^{\infty} C_{FF}(4n+3)\dfrac{x^{4n+3}}{(4n+3)!} & (C_{FF}(3)=2) \end{cases}$$

$$(1.11)\quad\begin{cases} \bar{z}_{RF}(x) = \sum_{n=0}^{\infty} C_{RF}(4n+2)\dfrac{x^{4n+2}}{(4n+2)!} & (C_{RF}(2)=1) \\ \bar{z}_{RR}(x) = \sum_{n=0}^{\infty} C_{RR}(4n)\dfrac{x^{4n}}{(4n)!} & (C_{RR}(0)=1) \\ \bar{z}_{FR}(x) = \sum_{n=0}^{\infty} C_{FR}(4n+2)\dfrac{x^{4n+2}}{(4n+2)!} & (C_{FR}(2)=1) \\ \bar{z}_{FF}(x) = \sum_{n=0}^{\infty} C_{FF}(4n)\dfrac{x^{4n}}{(4n)!} & (C_{FF}(0)=1). \end{cases}$$

In view of (1.7), we get

$$(1.12)\quad\begin{cases} y_{RR}(x) = y_{FF}(x), \quad z_{RR}(x) = \bar{z}_{FF}(x), \quad z_{FF}(x) = \bar{z}_{RR}(x) \\ z_{RF}(x) = \bar{z}_{RF}(x), \quad z_{FR}(x) = \bar{z}_{FR}(x), \quad \bar{y}_{RR}(x) = \bar{y}_{FF}(x). \end{cases}$$

Note that, for example, in taking

$$A_{RF}(3) = A_{FR}(3) = 2,$$

we are listing the up-down permutation $(1,3,2)$ and $(2,3,1)$ both under RF and FR. This is done so that the recurrences given below will be satisfied. A like remark applies in a number of other instances, as is evident from an examination of (1.8), ..., (1.11).

In the remainder of the paper we evaluate the sixteen enumerants defined in (1.8), ..., (1.11). For a summary of results see §6 below.

2. Evaluation of $y_{RF}(x)$, $y_{RR}(x)$, $y_{FF}(x)$, $y_{FR}(x)$. We consider first $y_{RF}(x)$. The method employed is to take a typical permutation of Z_n and consider the effect of removing the largest element. This is indeed the method used in [1]. Clearly the element removed must be a peak. The given permutation breaks into two pieces one of which may be vacuous. Thus for $A_{RF}(4n+3)$ we get the recurrence

$$A_{RF}(4n+3)=\sum_{k=0}^{n-1}\binom{4n+2}{4k+3}A_{RF}(4k+3)A_{RF}(4n-4k-1)$$

$$(n \geqq 1, \quad A_{RF}(3)=2). \tag{2.1}$$

It follows that the generating function $y_{RF}(x)$ satisfies the differential equation

$$y'_{RF}(x)=y^2_{RF}(x)+x^2. \tag{2.2}$$

Now put

$$y_{RF}(x)=-\frac{U'(x)}{U(x)}, \tag{2.3}$$

where

$$U(x)=\sum_{n=0}^{\infty} a_n \frac{x^{4n}}{(4n)!} \qquad (a_0=1). \tag{2.4}$$

Substituting (2.3) in (2.2) we get

$$U''(x)+x^2U(x)=0. \tag{2.5}$$

This implies the recurrence

$$a_{n+1}+(4n+1)(4n+2)a_n=0 \qquad (n=0,1,2,\cdots)$$

and therefore

$$a_n=(-1)^n 1.5.9\ldots(4n-3).2.6.10\ldots(4n-2). \tag{2.6}$$

Thus

$$\frac{a_n}{(4n)!}=\frac{(-1)^n}{3.7.11\ldots(4n-1).4.8.12\ldots4n}=\frac{(-1)^n}{4^{2n}}\frac{1}{n!\,(3/4)_n},$$

where

$$(a)_n=a(a+1)\ldots(a+n-1).$$

Hence (2.4) becomes

$$U(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{2n}}\frac{x^{4n}}{n!\,(3/4)_n}={}_0F_1\left(-;\frac{3}{4};\frac{x^4}{16}\right) \tag{2.7}$$

in the notation of generalized hypergeometric functions [5, Ch. 5].

Alternatively we may write [**5**, p. 108]

$$U(x) = (\tfrac{1}{2}x)^{\frac{1}{2}}\Gamma(\tfrac{3}{4})J_{-1/4}(\tfrac{1}{2}x^2), \tag{2.8}$$

where $J_{-1/4}(z)$ denotes the Bessel function of order $-1/4$.

In the next place, we have for $A_{RR}(4n+1)$ the recurrence

$$A_{RR}(4n+1) = \sum_{k=0}^{n-1} \binom{4n}{4k+3} A_{RF}(4k+3)A_{RR}(4n-4k+1) \qquad (n>0, \quad A_{RR}(1)=1). \tag{2.9}$$

This gives

$$y'_{RR}(x) = y_{RF}(x)y_{RR}(x) + 1. \tag{2.10}$$

Hence, by (2.3), (2.10) becomes

$$U(x)y'_{RR}(x) + U'(x)y_{RR}(x) = U(x)$$

and therefore

$$y_{RR}(x) = \frac{1}{U(x)}\int U(x), \tag{2.11}$$

where generally

$$\int f(x) = \int_0^x f(t)\,dt. \tag{2.12}$$

By (1.12) this implies

$$y_{FF}(x) = \frac{1}{U(x)}\int U(x). \tag{2.13}$$

Note that, by (2.7) and (2.12),

$$\int U = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{2n}} \frac{x^{4n+1}}{n!\,(3/4)_n(4n+1)}. \tag{2.14}$$

The enumerant $A_{FR}(4n+1)$ satisfies the recurrence

$$A_{FR}(4n+3) = \sum_{k=0}^{n} \binom{4n+2}{4k+1} A_{FF}(4k+1)A_{RR}(4n-4k+1) \qquad (n \geqq 0, \quad A_{FR}(3)=2). \tag{2.15}$$

Thus

$$y'_{FR}(x) = y_{FF}(x)y_{RR}(x).$$

Hence, by (2.11) and (2.13),

$$y'_{FR}(x) = \left\{\frac{1}{U(x)}\int U(x)\right\}^2. \tag{2.16}$$

It can be verified that

$$U^2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{2n}} \frac{(n+1)_n x^{4n}}{n!\,(3/4)_n(3/4)_n}. \tag{2.17}$$

3. Evaluation of $z_{RF}(x)$, $z_{RR}(x)$, $z_{FF}(x)$, $z_{FR}(x)$. As above we have first

$$A_{RF}(4n+2) = \sum_{k=0}^{n-1} \binom{4n+1}{4k+3} A_{RF}(4k+3)A_{RF}(4n-4k-2) \qquad (n>0, \quad A_{RF}(2)=1). \tag{3.1}$$

This yields

$$z'_{RF}(x) = y_{RF}(x)z_{RF}(x) + x. \tag{3.2}$$

Hence, by (2.3),

$$U(x)z'_{RF}(x) + U'(x)z_{RF}(x) = xU(x),$$

so that

$$z_{RF}(x) = \frac{1}{U(x)}\int xU. \tag{3.3}$$

Note that

$$\int xU = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{2n}} \frac{x^{4n+2}}{n!\,(3/4)_n(4n+2)}. \tag{3.4}$$

Next

$$A_{RR}(4n) = \sum \binom{4n-1}{4k+3} A_{RF}(4k+3)A_{RR}(4n-4k-4) \qquad (n \geqq 1, \quad A_{RR}(0)=1).$$

Thus

$$z'_{RR}(x) = y_{RF}(x) z_{RR}(x). \tag{3.5}$$

It follows that

$$z_{RR}(x) = \frac{1}{U(x)}. \tag{3.6}$$

For $A_{FF}(4n)$ we have

$$\begin{aligned} A_{FF}(4n) = \sum_{k=0}^{n-1} \binom{4n-1}{4k+1} A_{FF}(4k+1) A_{RF}(4n-4k-2) \\ (n>0, \quad A_{FF}(0)=1). \end{aligned} \tag{3.7}$$

This gives

$$z'_{FF}(x) = y_{FF}(x) z_{RF}(x). \tag{3.8}$$

Therefore, by (2.13) and (3.3),

$$z'_{FF}(x) = \frac{1}{U^2(x)} \int U(x) \cdot \int xU(x). \tag{3.9}$$

As for $A_{FR}(4n+2)$, we have

$$\begin{aligned} A_{FR}(4n+2) = \sum_{k=0}^{n} \binom{4n+1}{4k+1} A_{FF}(4k+1) A_{RR}(4n-2k) \\ (n \geqq 0, \quad A_{FR}(2)=1). \end{aligned} \tag{3.10}$$

Thus

$$z'_{FR}(x) = y_{FF}(x) z_{RR}(x). \tag{3.11}$$

Then, by (2.13) and (3.6),

$$z'_{FR}(x) = \frac{1}{U^2(x)} \int U(x). \tag{3.12}$$

4. Evaluation of $\bar{y}_{RF}(x)$, $\bar{y}_{RR}(x)$, $\bar{y}_{FF}(x)$, $\bar{y}_{FR}(x)$. To begin with

$$C_{RF}(4n+1)=\sum_{k=0}^{n-1}\binom{4n}{4k+2}C_{RF}(4k+2)A_{RF}(4n-4k-2) \tag{4.1}$$
$$(n>0,\quad C_{RF}(1)=1).$$

This gives

$$\bar{y}'_{RF}(x)=\bar{z}_{RF}(x)z_{RF}(x). \tag{4.2}$$

By (1.12), $\bar{z}_{RF}(x)=z_{RF}(x)$, so that (4.2) reduces to

$$\bar{y}'_{RF}(x)=\{z_{RF}(x)\}^2. \tag{4.3}$$

This formula together with (3.3) determines $\bar{y}_{RF}(x)$.

Next

$$C_{RR}(4n+3)=\sum_{k=0}^{n}\binom{4n+2}{4k+2}C_{RF}(4k+2)A_{RR}(4n-4k) \tag{4.4}$$
$$(n\geqq 0,\quad C_{RR}(3)=1),$$

so that

$$\bar{y}'_{RR}(x)=\bar{z}_{RF}(x)z_{RR}(x)=z_{RF}(x)z_{RR}(x). \tag{4.5}$$

Hence

$$\bar{y}'_{RR}(x)=\frac{1}{U^2(x)}\int xU(x). \tag{4.6}$$

Since by (1.12), $\bar{y}_{FF}(x)=\bar{y}_{RR}(x)$, we have also

$$\bar{y}'_{FF}(x)=\frac{1}{U^2(x)}\int xU(x). \tag{4.7}$$

In the next place

$$C_{FR}(4n+1)=\sum_{k=0}^{n}\binom{4n}{4k}C_{FF}(4k)A_{RR}(4n-4k) \tag{4.8}$$
$$(n\geqq 0,\quad C_{FR}(1)=1).$$

This gives

$$\bar{y}'_{FR}(x)=\bar{z}_{FF}(x)z_{RR}(x)=\{z_{RR}(x)\}^2, \tag{4.9}$$

since $\bar{z}_{FF}(x) = z_{RR}(x)$. Therefore, by (3.6),

$$\bar{y}'_{FR}(x) = \frac{1}{U^2(x)}. \tag{4.10}$$

5. Second solution of (2.5). One solution of the differential equation

$$w'' + x^2 w = 0 \tag{5.1}$$

is given by $w_1 = U(x)$. For a second solution we take

$$w_2 = U(x)V(x), \quad V(x) = \sum_{n=0}^{\infty} b_n \frac{x^{4n+1}}{(4n+1)!} \quad (b_0 = 1). \tag{5.2}$$

Clearly w_1 and w_2 are linearly independent. Substituting from (5.2) in (5.1) we get

$$2U'(x)V'(x) + U(x)V''(x) = 0. \tag{5.3}$$

This gives

$$V'(x) = \frac{1}{U^2(x)}. \tag{5.4}$$

Comparing (5.4) with (4.10), it is clear that

$$V(x) = \bar{y}_{FR}(x), \tag{5.5}$$

so that the second solution of (5.1) becomes

$$w_2 = U(x)\bar{y}_{FR}(x). \tag{5.6}$$

6. Summary of results.

$$y_{RF}(x) = -\frac{U'(x)}{U(x)} \tag{6.1}$$

$$y_{RR}(x) = y_{FF}(x) = \frac{1}{U(x)} \int U(x) \tag{6.2}$$

$$y'_{FR}(x) = \left\{ \frac{1}{U(x)} \int U(x) \right\}^2 \tag{6.3}$$

$$z_{RF}(x) = \bar{z}_{RF}(x) = \frac{1}{U(x)} \int xU(x) \tag{6.4}$$

$$z_{RR}(x) = \bar{z}_{FF}(x) = \frac{1}{U(x)} \tag{6.5}$$

$$z'_{FF}(x) = \bar{z}'_{RR}(x) = \frac{1}{U^2(x)} \int U(x) \cdot \int xU(x) \tag{6.6}$$

$$z'_{FR}(x) = z'_{FR}(x) = \frac{1}{U^2(x)} \int U(x) \tag{6.7}$$

$$\bar{y}'_{RF}(x) = \{z_{RF}(x)\}^2 \tag{6.8}$$

$$\bar{y}_{RR}(x) = \bar{y}_{FF}(x) = \frac{1}{U^2(x)} \int xU(x) \tag{6.9}$$

$$\bar{y}'_{FR}(x) = \frac{1}{U^2(x)} \tag{6.10}$$

$$U(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{2n}} \frac{x^{4n}}{n!\,(3/4)_n} \tag{6.11}$$

$$U^2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{2n}} \frac{(n+\frac{1}{2})_n x^{4n}}{n!\,(3/4)_n (3/4)_n}. \tag{6.12}$$

Linearly independent solutions of

$$w'' + x^2 w = 0$$

are furnished by

$$w_1 = U(x), \qquad w_2 = U(x)\bar{y}_{FR}(x). \tag{6.13}$$

7. Generalizations. We may define *doubly* up-down permutations as permutations $(a_1, a_2, \cdots, a_n)$ of Z_n such that

$$a_{2k-1} < a_{2k}, \quad a_{2k} > a_{2k+1} \qquad (k = 1, 2, \cdots, [n/2]) \tag{7.1}$$

and

$$a_{4k-2} < a_{4k}, \quad a_{4k} > a_{4k+2} \qquad (k = 1, 2, \cdots, [n/4]). \tag{7.2}$$

Similarly we may define *triply* up-down as permutations that satisfy (7.1) and (7.2) and in addition

$$a_{8k-4} < a_{8k}, \quad a_{8k} > a_{8k+4} \qquad (k = 1, 2, \cdots, [n/8]). \tag{7.3}$$

It is clear how to extend this definition to r-ply up-down permutations. Thus this suggests the enumeration of permutations of these types.

An extension in a different direction is the following. Let $A_3(n)$ denote the number of permutations of Z_n that satisfy

$$a_{3k-2} < a_{3k-1} < a_{3k}, \quad a_{3k} > a_{3k+1} \qquad (k = 1, 2, \cdots, [n/3]). \tag{7.4}$$

Then, as a special case of a result proved in [2], [3], we have

$$\sum_{n=0}^{\infty} A_3(3n) \frac{x^{3n}}{(3n)!} = \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n)!} \right\}^{-1}. \tag{7.5}$$

This suggests the consideration of permutations that satisfy (7.4) and in addition

$$a_{9k-6} < a_{9k-3} < a_{9k}, \quad a_{9k} > a_{9k+3} \qquad (k = 1, 2, \cdots, [n/9]). \tag{7.6}$$

Moreover further restrictions analogous to (7.3) can also be introduced.

However we shall not treat these extensions in the present paper.

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