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CONSTRUCTING NEW R-SEQUENCES

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R-sequences play an important role in modern commutative algebra. The purpose of this paper is to show how new R-sequences may be constructed from a given one. In the first section we give some general results, which are applied in the second section to obtain an explicit method of construction.

Recall that a sequence of elements $x_1, \cdots, x_n$ in R is an R-sequence if $(x_1, \cdots, x_n)R \neq R$, x_1 is a nonzero divisor on R, and for $2 \leqq i \leqq n$, x_i is a nonzero divisor on $R/(x_1, \cdots, x_{i-1})R$.

Throughout this paper R will be a commutative noetherian ring which contains a field K. Moreover, R will either be local or graded.

I wish to thank Melvin Hochster for showing me Proposition 1.5, which simplified this paper considerably.

1. It is easy to see that if $x_1, \cdots, x_n \varepsilon R$ and $X_1, \cdots, X_n$ are independent indeterminates over K, and if $\varphi: K[X_1, \cdots, X_n] \rightarrow R$ by $\varphi(f(X_1, \cdots, X_n)) = f(x_1, \cdots, x_n)$ is a flat monomorphism, then $x_1, \cdots, x_n$ is an R-sequence. The converse, when R is local, is due to Hartshorne [3].

PROPOSITION 1.1 (Hartshorne). *Suppose R is local. If $x_1, \cdots, x_n \varepsilon R$ form an R-sequence then $\varphi: K[X_1, \cdots, X_n] \rightarrow R$ is a flat monomorphism, where φ is the map determined by $\varphi(X_i) = x_i$ for each i and $\varphi(a) = a$ for all $a \in K$.*

REMARK. Saying that φ is a monomorphism is the same as saying that $x_1, \cdots, x_n$ are algebraically independent over K.

COROLLARY 1.2. *Assume R is local. Suppose $f_1, \cdots, f_n$ is a $K[X_1, \cdots, X_n]$-sequence, and each $f_i \in (X_1, \cdots, X_n)K[X_1, \cdots, X_n]$. Suppose also that $x_1, \cdots, x_n$ is an R-sequence. Then*

$$f_1(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n)$$

is an R-sequence.

Proof. By Proposition 1.1 the map φ is a flat monomorphism. By flatness, since $f_1, \cdots, f_n$ is a $K[X_1, \cdots, X_n]$-sequence, $\varphi(f_1), \cdots, \varphi(f_n)$

is an R-sequence. (The assumption that each $f_i \in (X_1, \cdots, X_n)$ guarantees that the $\varphi(f_i)$ generate a *proper* ideal of R.)

REMARK. It is well-known (e.g., [4, Theorem 119]) that for *any* local noetherian ring R, a permutation of an R-sequence is again an R-sequence. However, if R contains a field, the preceding result yields a very simple proof of this fact. For it is clear that for any permutation σ of $\{1, \cdots, n\}$, $X_{\sigma(1)}, \cdots, X_{\sigma(n)}$ is a $K[X_1, \cdots, X_n]$-sequence. Letting $f_i = X_{\sigma(i)}$, we have $f_i(x_1, \cdots, x_n) = x_{\sigma(i)}$, and so by Corollary 1.2, $x_{\sigma(1)}, \cdots, x_{\sigma(n)}$ is an R-sequence.

We now give a graded analogue of Proposition 1.1. For in order to use Corollary 1.2 we need $K[X_1, \cdots, X_n]$-sequences.

PROPOSITION 1.3. *Assume R is graded, and let $x_1, \cdots, x_n$ be homogeneous elements of R of positive degree. Then $x_1, \cdots, x_n$ is an R-sequence* $\Leftrightarrow$ (i) *$x_1, \cdots, x_n$ are algebraically independent over K, and* (ii) *R is a free $K[x_1, \cdots, x_n]$-module.*

Proof. Let $A = K[x_1, \cdots, x_n]$.

($\Leftarrow$) Assume (i) and (ii). Hence A is a polynomial ring in n variables and thus $x_1, \cdots, x_n$ is an A-sequence. Since R is A-free, any A-sequence is an R-sequence.

($\Rightarrow$) (i) follows from [5, p. 199].

(ii) A is a graded subring of R, with grading induced by that of R. That is, if $R = \bigoplus \Sigma R_k$, let $A_k = A \cap R_k$. Then ΣA_k is a direct sum, which we claim equals A. Since each x_i is homogeneous, $x_i \in A_{m_i}$ for some integer $m_i \geqq 1$. Also, $K \subset R$ and R is graded, so $K \subset R_0$, and therefore $K = A_0$. Since every element g of A is a polynomial in the x_i's with coefficients in K, it follows that $g \in \bigoplus \Sigma A_k$. Hence $A = \bigoplus \Sigma A_k$. Thus, with the grading on A induced by that of R, and with the original grading on R, R is a graded A-module. Now by [2, Ch. VIII, Thm. 6.1] since A_0 is a field and R is a graded A-module, if $\mathrm{Tor}_1^A(R, A_0) = 0$ then R is A-free. Thus to prove (ii) it suffices to show that $\mathrm{Tor}_1^A(R, K) = 0$.

We compute $\mathrm{Tor}_1^A(R, K)$ by taking a projective resolution of K over A and tensoring it with R. Since $x_1, \cdots, x_n$ are algebraically independent over K, they form an A-sequence, and so the Koszul complex of the x's over A is exact and therefore yields a free A-resolution of K. Tensoring it with R gives the Koszul complex of the x's over R. But since by hypothesis the x's form an R-sequence, this Koszul complex has zero homology ([1, Cor. 1.2] or

[2, Ch. VIII, 4.3]). In particular, the first homology group, $\mathrm{Tor}_1^A(R, K)$, is 0, and we are done.

We have a graded analogue of Corollary 1.2. Its proof is nearly identical to the latter's and so we omit it.

COROLLARY 1.4. *Suppose R is graded and $x_1, \cdots, x_n$ is an R-sequence, where each x_i is homogeneous of positive degree. Suppose $f_1, \cdots, f_n$ is a $K[X_1, \cdots, X_n]$-sequence with each $f_i \in (X_1, \cdots, X_n)$. Then $f_1(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n)$ is an R-sequence.*

We close this section with a proposition due to M. Hochster.

PROPOSITION 1.5. *Let S be a graded Macaulay ring such that S_0 is local. Let $x_1, \cdots, x_n$ be homogeneous elements of S. If* $\mathrm{rank}\,(x_1, \cdots, x_n) = n$ *then $x_1, \cdots, x_n$ is an S-sequence.*

Proof. Let $M = M_0 + \sum_{i \geqq 1} S_i$, where M_0 is the maximal ideal of S_0. Then M is maximal in S and contains every proper homogeneous ideal of S. Let $I = (x_1, \cdots, x_n)$, and localize at M. Then in the local Macaulay ring S_M, $\mathrm{rank}\,(f_M) = n$, so $x_1, \cdots, x_n$ is an S_M-sequence, by [4, Thms. 129 and 136]. Let $\mathscr{K}$ denote the Koszul complex of the x's over S. Then $\mathscr{K} \otimes_S S_M$ is acyclic since it is the Koszul complex of the x's over S_M. Hence for each $i \geqq 1$, the ith homology module $H_i(\mathscr{K} \otimes S_M) = 0$. Since S_M is S-flat we have $H_i(\mathscr{K}) \otimes S_M = 0$, so $\mathrm{ann}\,(H_i(\mathscr{K})) \not\subset M$. Since the x's are homogeneous, $\mathscr{K}$ is a complex of graded S-modules and hence $H_i(\mathscr{K})$ is also graded. But the annihilator of a graded module is a homogeneous ideal. Thus $\mathrm{ann}\,(H_i(\mathscr{K})) = S$ and so $H_i(\mathscr{K}) = 0$ for all $i \geqq 1$. Therefore $\mathscr{K}$ is acyclic, and so by [1, Prop. 2.8], $x_1, \cdots, x_n$ is an S-sequence.

2. Any permutation σ in the symmetric group $\mathscr{S}_n$ acts as an automorphism on the polynomial ring $K[X_1, \cdots, X_n]$ by

$$(\sigma f)(X_1, \cdots, X_n) = f(X_{\sigma(1)}, \cdots, X_{\sigma(n)}) .$$

The next lemma is the key to our construction.

LEMMA 2.1. *Let σ be the cyclic permutation $(1, 2, \cdots, n)$, of order n. Let K be a field, with $a \in K$. Define a homogeneous polynomial $f \in K[X_1, \cdots, X_n]$ by $f(X_1, \cdots, X_n) = X_1^m - ag$, where $g = \prod_{t=1}^k X_{i_t}^{m_t}$, $2 \leqq i_1 < i_2 < \cdots < i_k \leqq n$, each $m_t \geqq 1$, and $\sum_{t=1}^k m_t = m$.*

If $a^n \neq 1$, then the only common zero of $f, \sigma f, \cdots, \sigma^{n-1} f$ in K^n is $(0, \cdots, 0)$.

Proof. We first treat a special case where the basic idea of the proof is not obscured by details. Suppose that $k = n - 1$, i.e., that *each* X_i, $2 \leqq i \leqq n$, divides the monomial g. Let $(z_1, \cdots, z_n) \in K^n$ be a common zero of $f, \sigma f, \cdots, \sigma^{n-1} f$. We have the following system of equations:

$$\begin{aligned} z_1^m &= a z_2^{m_2} \cdots z_{n-1}^{m_{n-1}} z_n^{m_n} \\ z_2^m &= a z_3^{m_2} \cdots z_n^{m_{n-1}} z_1^{m_n} \\ &\vdots \\ z_n^m &= a z_1^{m_2} \cdots z_{n-2}^{m_{n-1}} z_{n-1}^{m_n} . \end{aligned}$$

Equating the product of the left sides with the product of the right sides, and using the fact that $\sum_{i=2}^n m_i = m$, we obtain:

$$\left(\prod_{i=1}^n z_i\right)^m = a^n \left(\prod_{i=1}^n z_i\right)^{m_2} \cdots \left(\prod_{i=1}^n z_i\right)^{m_n} = a^n \left(\prod_{i=1}^n z_i\right)^m .$$

But $a^n \neq 1$, so $\prod_{i=1}^n z_i = 0$ and thus some $z_j = 0$. For all i such that $i \neq j$, z_j appears on the right side of the ith equation of the system above. Hence $z_i = 0$. Thus $(z_1, \cdots, z_n) = (0, \cdots, 0)$.

In the general case we shall break up the system of n equations into a number of subsystems, for each of which the preceding argument can be used.

Let $H = \langle \sigma^{i_1}, \cdots, \sigma^{i_k} \rangle$ be the subgroup of the cyclic group $\langle \sigma \rangle$ generated by $\sigma^{i_1}, \cdots, \sigma^{i_k}$. Thus H is cyclic, of order dividing n. In fact, $H = \langle \sigma^b \rangle$ where b is the greatest common divisor of $n, i_1, \cdots, i_k$.

We claim that if X_r divides $\sigma^s(g)$, then $r \equiv s \pmod b$. For $r = \sigma^s(i_c)$ for some c, $1 \leqq c \leqq k$. Thus $r \equiv s + i_c \pmod n$. Since b is a common divisor of i_c and n, it follows that $r \equiv s \pmod b$.

Now consider $\prod_{s=1}^n \sigma^s(g)$. It is clearly invariant under σ. But if $\sigma(\prod_{i=1}^n X_i^{a_i}) = \prod_{i=1}^n X_i^{a_i}$, then $a_1 = a_2 = \cdots = a_n$. Now since deg $g = m$, $\deg (\prod_{s=1}^n \sigma^s g) = nm$. Thus $\prod_{s=1}^n \sigma^s g = \prod_{i=1}^n X_i^m$. On the other hand, for any r,

$$\prod_{s=1}^n \sigma^s g = \Big(\prod_{s \equiv r \pmod b} \sigma^s g\Big)\Big(\prod_{s \not\equiv r \pmod b} \sigma^s g\Big) ,$$

and if $r \not\equiv s \pmod b$ then X_r does not divide $\sigma^s g$. Therefore

$$\prod_{s \equiv r \pmod b} \sigma^s g = \prod_{s \equiv r \pmod b} X_s^m = \Big(\prod_{s \equiv r \pmod b} X_s\Big)^m .$$

Now suppose $(z_1, \cdots, z_n)$ is a common zero of $f, \sigma f, \cdots, \sigma^{n-1} f$. Then for all $1 \leqq s \leqq n$, $z_s^m = a(\sigma^s g)(z_1, \cdots, z_n)$. Hence

$$\Big(\prod_{s \equiv r \pmod b} z_s\Big)^m = a^{n/b} \prod_{s \equiv r \pmod b} (\sigma^s g)(z_1, \cdots, z_n) = a^{n/b} \Big(\prod_{s \equiv r \pmod b} z_s\Big)^m .$$

Since $a^n \neq 1$, it follows that $a^{n/b} \neq 1$, and so $z_s = 0$ for some $s \equiv r \pmod b$. We shall show that $z_t = 0$ for *every* $t \equiv r \pmod b$.

For $1 \leqq j \leqq k$, X_{i_j} divides g: Thus $X_t = \sigma^{t-i_j}(X_{i_j})$ divides $\sigma^{t-i_j}(g)$, say $x_t h = \sigma^{t-i_j}(g)$. Now $\sigma^{t-i_j}(f) = \sigma^{t-i_j}(X_1^m) - a\sigma^{t-i_j}(g) = X_{t-i_j}^m - ax_t h$. If $z_t = 0$, then $z_{t-i_j}^m = 0$ since $(z_1, \cdots, z_n)$ is a zero of $\sigma^{t-i_j}(f)$, and so $z_{t-i_j} = 0$. Thus for all j and for all q with $q \equiv s \pmod{i_j}$, we have $z_q = 0$. This implies $z_t = 0$ for all $t \equiv r \pmod b$. Since r was arbitrary, $(z_1, \cdots, z_n) = (0, \cdots, 0)$.

THEOREM 2.2. *Let K, σ, a, and f be as in the preceding lemma. Then $f, \sigma f, \cdots, \sigma^{n-1} f$ is a $K[X_1, \cdots, X_n]$-sequence.*

Proof. Let $I = (f, \sigma f, \cdots, \sigma^{n-1} f)$ and let $R = K[X_1, \cdots, X_n]$. Let $S = \bar{K}[X_1, \cdots, X_n]$, where $\bar{K}$ is the algebraic closure of K. By Lemma 2.1 the variety of IS in $\bar{K}^n$ contains only the origin. Hence by the Nullstellensatz, the radical of IS is the maximal ideal $(X_1, \cdots, X_n)S$. Therefore $\operatorname{rank}(IS) = n$, and so by Proposition 1.5 $f, \sigma f, \cdots, \sigma^{n-1} f$ is an S-sequence. Now $S = R \otimes_K \bar{K}$, so S is R-free. Hence S is faithfully R-flat, and thus $f, \sigma f, \cdots, \sigma^{n-1} f$ is also an R-sequence.

Combining Theorem 2.2 with Corollaries 1.2 and 1.4, we have:

COROLLARY 2.3. *Suppose R contains a field K, and $x_1, \cdots, x_n$ is an R-sequence. Define $f \in K[X_1, \cdots, X_n]$ as in Lemma 2.1, and assume $a^n \neq 1$. If R is local, or if R is graded and each x_i is homogeneous of positive degree, then*

$$f(x_1, \cdots, x_n),\ (\sigma f)(x_1, \cdots, x_n),\ \cdots,\ (\sigma^{n-1} f)(x_1, \cdots, x_n)$$

is an R-sequence.

REMARK. Since f is a homogeneous polynomial of positive degree, when the original R-sequence consists of homogeneous elements of positive degree, the same is true for the resulting R-sequence. Thus in the graded case as well as in the local case, the procedure may be iterated.

EXAMPLE. Let $R = K[X, Y, Z]$, where X, Y, Z are independent indeterminates. By Theorem 2.2, if $a^2 \neq 1$, then $X^2 - aYZ$, $Y^2 - aXZ$, $Z^2 - aXY$ is an R-sequence, and if $b \in K$ and $b^3 \neq 1$, then $X^3 - bY^3$, $Y^3 - bZ^3$, $Z^3 - bX^3$ is another. Hence by Corollary 2.3, $(X^2 - aYZ)^3 - b(Y^2 - aXZ)^3$, $(Y^2 - aXZ)^3 - b(Z^? - aXY)^3$, $(Z^2 - aXY)^3 - b(X^2 - aYZ)^3$ is again an R-sequence, as is $(X^3 - bY^3)^2 - a(Y^3 - bZ^3)(Z^3 - bX^3)$, $(Y^3 - bZ^3)^2 - a(Z^3 - bX^3)(X^3 - bY^3)$, $(Z^3 -$

$bX^3)^2 - a(X^3 - bY^3)(Y^3 - bZ^3)$.

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