

SIDON SETS ASSOCIATED WITH A CLOSED SUBSET OF A COMPACT ABELIAN GROUP

E. V. DUDLEY

Déchamps-Gondim in [1] announced that a Sidon set E contained in the dual of a connected compact abelian group G is associated with each compact subset K of G having interior, in the sense that there exists a finite subset F of E and some constant such that this constant times the maximum absolute value of any $E \setminus F$ -spectral trigonometric polynomial on K majorizes the sum of the absolute values of the Fourier transform. It is readily shown that if G is not connected not all Sidon sets have this property. In [7], Ross described the class of all Sidon sets which are associated with all compact sets K having interior. In this paper, the Sidon sets associated with a particular set K are analysed and characterized.

1. Introduction.

1.1. Throughout this paper, the symbol G is used to denote an arbitrary infinite, compact, abelian group, the symbol X denotes its character group and λ , Haar measure on G . For E a subset of X , we call an integrable function an E -spectral function if its Fourier transform vanishes off E . For any space $F(G)$ of integrable functions, the space of all E -spectral functions belonging to $F(G)$ is denoted by $F_E(G)$. We denote by $\text{Trig}(G)$, the space of all complex-valued trigonometric polynomials on G and by $A(G)$, the space of all functions with absolutely convergent Fourier series. The usual norm on $A(G)$ is denoted by $\|\cdot\|_A$. All other notation not explained in this paper appears in López and Ross [6].

DEFINITION 1.2 (see López and Ross [6] p. 109). Let K be a nonvoid compact subset of G and E a subset of X . We say that E and K are *strictly associated* if there exists a constant $\kappa > 0$ such that

$$\|f\|_A \leq \kappa \|\hat{\xi}_K f\|_U \quad \text{for all } f \in \text{Trig}_E(G)$$

where $\hat{\xi}_K$ denotes the characteristic function of K . In particular if E and G are strictly associated, we say that E is a *Sidon set*. We say that E and K are *associated* if $E \setminus F$ and K are strictly associated for some finite subset F of E .

DEFINITION (see López and Ross [6] p. 110). Let X_0 denote a subgroup of an abelian group X . A subset E of X is called X_0 -subtransversal provided that each coset of X_0 intersects E in at most one point. We say that E is almost X_0 -subtransversal if $E \setminus F$ is X_0 -subtransversal for some finite subset F of E .

THEOREM 1.3 (see [6] Theorem 8.22, López and Ross). A Sidon set E is strictly associated with every compact subset of G with nonvoid interior if and only if E is X_0 -subtransversal for all finite subgroups X_0 of X .

Ross proved the above theorem after previously showing that the theorem was true if the property "strictly associated" is replaced with "associated" and the property " X_0 -subtransversal" is replaced with "almost X_0 -subtransversal." Both results are generalizations of previous results due to Déchamps-Gondim [1, 2, 3] which show the theorems to work for G connected. In this case, the condition that E be X_0 -subtransversal for each finite subgroup X_0 of X is trivial since X is torsion free and has only one finite subgroup, $\{1\}$.

In this paper, a characterization is given of the Sidon sets which are associated with a given compact subset K of G for which $\lambda K = \lambda(\text{int } K) > 0$. In §2, we define the notion of a K -basic subset of X and establish several elementary properties. In §3, we show that Sidon K -basic sets are those which are associated with K .

2. Definition and properties of K -basic sets.

DEFINITION 2.1. Let K be a nonvoid compact subset of G and let E be a subset of X . We call E a K -basic set if the zero function is the only function, $f \in \text{Trig}_E(G)$ such that $\xi_K f = 0$. We denote by $D(K)$, the set of all K -basic sets and we let $D_1(K) = \{E \in D(K) : 1 \in E\}$.

2.2. In the following lemma, we list without proof certain easy consequences of the definition.

LEMMA. Let K, K_1 be compact subsets of G . Then the following are true.

- (i) Subsets of K -basic sets are K -basic sets.
- (ii) A K -basic set is a $K + a$ -basic set for each $a \in G$.
- (iii) If $K_1 \supseteq K$, then a K -basic set is a K_1 -basic set.
- (iv) Translates of K -basic sets are K -basic sets. In particular, for each $E \in D(K)$ and each $\chi \in E$, we have $\chi^{-1}E \in D_1(K)$.
- (v) Inverses of K -basic sets are K -basic sets.

2.3. We now characterize the property of a set $E \subseteq X$ being X_0 -subtransversal for all finite subgroups X_0 of X in terms of the ideas introduced in 2.1.

THEOREM. *Let E be a subset of X . Then the following are equivalent.*

(i) *The set E is X_0 -subtransversal for all finite subgroups X_0 of X .*

(ii) *The set E is a K -basic set for all compact subsets K of G having positive Haar measure.*

(iii) *For some compact set K_1 having positive Haar measure, the set E is a K -basic set for all compact sets $K \subseteq K_1$ having positive Haar measure.*

Proof. That (i) implies (ii) is a simple consequence of López and Ross 8.12. Now suppose that (i) does not hold. Then there exist characters χ_1 and χ_2 of finite order such that $\chi\chi_1$ and $\chi\chi_2$ are elements of E for some $\chi \in X$. The set $K = \{x \in G: \chi_1(x) = \chi_2(x) = 1\}$ is a closed neighbourhood of the origin in G on which the E -spectral polynomial $\chi\chi_1 - \chi\chi_2$ is identically zero. We conclude that E is not a K -basic set. It is clear that (ii) implies (iii). Suppose that (iii) holds and let K_2 be any compact set having positive Haar measure. We note that the function $\xi_{-K_2} * \xi_{K_1}: a \rightarrow \lambda((K_2 + a) \cap K_1)$ is a nontrivial continuous function and hence $\lambda((K_2 + a) \cap K_1) > 0$ for some $a \in G$. Hence E is a $(K_2 + a) \cap K_1$ -basic set by (iii). Noting 2.2(ii) and 2.2(iii) and the fact that $K_2 \supseteq (K_2 + a \cap K_1) - a$, we conclude that E is a K_2 -basic set.

COROLLARY 2.4. *If G is a connected compact abelian group, then each subset E of X is a K -basic set for all compact subsets K of G having positive Haar measure.*

Proof. As noted in 1.3, condition (i) of the previous theorem is trivially satisfied for all subsets E of X .

2.5. We now show that each set $E \subseteq X$ possesses at least one maximal K -basic subset and all maximal K -basic sets E possess a characteristic property.

THEOREM. *Let E be a subset of X and K a compact subset of G . Then the following is true.*

(i) *The set E contains a maximal K -basic subset E_0 .*

(ii) *A subset E_0 of E is a maximal K -basic set if and only if for each $f \in \text{Trig}_E(G)$, there exists a unique polynomial $f_0 \in \text{Trig}_{E_0}(G)$ such that $\xi_K(f - f_0)$ is the zero function.*

Proof. To show (i), let $\mathcal{C} = \{E_1 \in D(K): E_1 \subseteq E\}$ and order \mathcal{C} by inclusion. Note that $\emptyset \in D(K)$. It is easily seen that each ascending chain $\{E_\alpha\}$ of sets in \mathcal{C} is bounded above by $\bigcup_\alpha E_\alpha$ and hence by Zorn's lemma, the set \mathcal{C} has a maximal element.

Now let E_0 be a maximal K -basic subset of E . Assume firstly that some nonzero element $f \in \text{Trig}_E(G)$ has the property that $\xi_K(f - f_0)$ is nonzero for each $f_0 \in \text{Trig}_{E_0}(G)$. If we set $f = \sum a_\lambda \chi$ where each $\chi \in E$, then we deduce that for some $\chi \in E$, we have $\xi_K(\chi - f_0)$ is nonzero for all $f_0 \in \text{Trig}_{E_0}(G)$. It follows readily that $E_0 \cup \{\chi\}$ is a K -basic set contradicting the maximality of E_0 . Now assume that for some $f \in \text{Trig}_E(G)$, there exist distinct polynomials f_1 and f_2 belonging to $\text{Trig}_{E_0}(G)$ such that $\xi_K(f - f_1) = \xi_K(f - f_2) = 0$. It follows that $\xi_K(f_1 - f_2) = 0$ contradicting the fact that E_0 is a K -basic set.

On the other hand, if E_0 is any subset of E having the property that for each $f \in \text{Trig}_E(G)$, there exists a unique element $f_0 \in \text{Trig}_{E_0}(G)$ such that $\xi_K(f - f_0) = 0$, then E_0 is a K -basic set because a polynomial $f_0 \in \text{Trig}_{E_0}(G)$ such that $\xi_K f_0 = 0$ is necessarily the zero polynomial since $\xi_K(f_0 - f) = 0$ for $f = f_0$ and $f = 0$. Further, E_0 is maximal since if $\chi \notin E_0$ then $\xi_K(\chi - f_0) = 0$ for some $f_0 \in \text{Trig}_{E_0}(G)$ and hence $E_0 \cup \{\chi\}$ is not a K -basic set.

3. Characterization of sets associated with K .

3.1. In this section, we show by a modification of Déchamps-Gondim's arguments that E is a K -basic Sidon set if and only if E and K are associated where K is a compact set such that $\lambda K = \lambda(\text{int } K) > 0$. We commence with the following theorem.

THEOREM. *Let K be a compact subset of G with nonvoid interior and let $m \in \mathbb{N}$. Then there exists a positive constant $\delta(m, K)$ such that for all K -basic sets E having at most m elements and for all $f \in \text{Trig}_E(G)$, we have*

$$\int_G |\xi_K f|^2 d\lambda \geq \delta(m, K) \sum_{\chi \in E} |f(\chi)|^2.$$

The proof imitates the classical proof by Zygmund [8, p. 60] which was observed to work for connected groups by Déchamps-Gondim [3, Lemme 6.3]. Connectedness is used specifically to show that for a certain nonzero polynomial f , we have $\int_{K_1} f d\lambda \neq 0$ for some $K_1 \subseteq K$. For the above theorem, this will continue to hold since the polynomial f will be an E -spectral polynomial for some K -basic set E .

3.2. A critical argument which is needed later requires the existence of a unique solution to a system of linear equations. The following corollary to 3.1 which is a modified version of Lemme 6.4 of Déchamps-Gondim [3] provides this.

COROLLARY. *Let K be a compact subset of G with nonvoid interior and let m be a positive integer greater than one. Let g be a nonnegative integrable function such that $\lambda(K \setminus \text{supp } g) < m^{-2}\delta(m, K)$ where $\delta(m, K)$ is the constant appearing in 3.1 and let $\{\chi_j: j = 1, 2, \dots, r\}$ be a K -basic set where $2 \leq r \leq m$. We set*

$$K_n = \{x \in K: g(x) > n^{-1}\}$$

and let H be the $r \times r$ -matrix

$$H_{jk} = \hat{g}(\chi_j \chi_k^{-1}).$$

For each n , such that $\lambda(K \setminus K_n) < m^{-2}\delta(m, K)$, we have

$$\det H \geq \delta_n \text{ where } \delta_n = \min\{1, n^{-m}(\delta(m, K) - m^2\lambda(K \setminus K_n))^m\}.$$

Proof. Since H is an hermitian matrix, there exists a diagonal matrix $D = \{\delta_{jk} d_j\}_{jk}^r$ and a unitary matrix $U = \{u_{jk}\}_{jk}^r$ such that $D = UHU^*$. We calculate:

$$\begin{aligned} \det H &= \prod_{j=1}^r d_j \\ &= \prod_{j=1}^r \sum_{k,s}^r u_{jk} H_{ks} \overline{u_{js}} \\ &= \prod_{j=1}^r \sum_{k,s}^r u_{jk} \overline{u_{js}} \hat{g}(\chi_k \chi_s^{-1}) \\ &= \prod_{j=1}^r \int_G g \left(\sum_{k,s}^r u_{jk} \overline{u_{js}} \chi_k \chi_s^{-1} \right) d\lambda \\ &= \prod_{j=1}^r \int_G g \left| \sum_s^r \overline{u_{js}} \chi_s \right|^2 d\lambda \\ &\geq \prod_{j=1}^r \int_G \xi_{K} g \left| \sum_s^r \overline{u_{js}} \chi_s \right|^2 d\lambda. \end{aligned}$$

We have the estimate for $j = 1, 2, \dots, r$:

$$\begin{aligned} &\int_G \xi_{K} g \left| \sum_s^r \overline{u_{js}} \chi_s \right|^2 d\lambda \\ &\geq \int_G \xi_{K_n} g \left| \sum_s^r \overline{u_{js}} \chi_s \right|^2 d\lambda \\ &\geq n^{-1} \int_G \xi_{K} g \left| \sum_s^r \overline{u_{js}} \chi_s \right|^2 d\lambda \end{aligned}$$

$$\begin{aligned}
& - n^{-1} \int_G \xi_{K \setminus K_n} \left| \sum_s \overline{u_{j_s}} \chi_s \right|^2 d\lambda \\
& \geq n^{-1} \delta(m, K) - m^2 n^{-1} \lambda(K \setminus K_n) \\
& \text{by the previous theorem} \\
& = n^{-1} (\delta(m, K) - m^2 \lambda(K \setminus K_n)).
\end{aligned}$$

The corollary follows.

It is clear that a modification of Déchamps-Gondim's arguments is needed in the above theorem since the set $\{\chi_j: j = 1, 2, \dots, r\}$ is not necessarily a K_n -basic set and hence we do not necessarily have the inequality

$$\int_G \xi_{K_n} \left| \sum_s \overline{u_{j_s}} \chi_s \right|^2 d\lambda \geq \delta(m, K).$$

3.3. We need to be able to choose a suitable nonnegative integrable function which, in particular, satisfies the requirement of 3.2. Hence we have

LEMMA. *Let K be a compact subset of G such that $\lambda K = \lambda \text{int } K > 0$ and let $\varepsilon > 0$. There exists a nonnegative function $g \in A(G)$ such that $\text{supp } g \subseteq K$ and $\lambda\{x \in K: g(x) \neq 1\} < \varepsilon$.*

Proof. Let $V = \text{int } K$ and choose a compact set $F \subseteq V$ such that $\lambda(V \setminus F) < \varepsilon$. Since $A(G)$ is a normal family of functions on G (see Hewitt and Ross [5](34.21)), it contains a function g such that $g(x) = 1$ for $x \in F$ and $g(x) = 0$ for $x \in G \setminus V$.

DEFINITION 3.4 (see López and Ross [6] 7.3). Let E be a nonvoid symmetric subset of X and K a nonvoid compact subset of G . Then we say that E is an $FZ(K)$ -set if there exists a positive constant κ and a finite symmetric subset F of E such that for all $f \in \text{Trig}_{E \setminus F}^r(G)$ we have

$$(1) \quad \|f\|_A \leq \kappa \|\xi_K f^+\|_u.$$

3.5 We have the following analogue of 8.16 of López and Ross [6].

THEOREM. *Let K be a compact subset of G such that $\lambda(K) = \lambda(\text{int } K) > 0$. A symmetric subset E of X is an $FZ(K)$ -set if and only if*

- (i) E is a Sidon set
- and
- (ii) E is the union of a symmetric finite set and a K -basic set.

Proof. Suppose that E is a $FZ(K)$ -set. As observed in the proof of 8.3 of López and Ross [6], the statement (1) implies that $E \setminus F$ and K are strictly associated and hence, in particular, the set $E \setminus F$ is a Sidon set. Since the set F is finite, we conclude that E is a Sidon set. Further the set $E \setminus F$ is a K -basic set since the existence of a nonzero $E \setminus F$ -spectral trigonometric polynomial f such that $\xi_K f$ is the zero function would violate the condition that $E \setminus F$ and K are strictly associated.

To prove converse, it is sufficient to show that each symmetric K -basic Sidon set E is an $FZ(K)$ -set. Noting 3.3, we choose a non-negative function $g \in A(G)$ such that $\text{supp } g \subseteq -K$ and

$$\lambda\{x \in -K: g(x) \neq 1\} < m^{-2\delta}(m, K)$$

where m is the positive integer appearing in 8.9 of López and Ross [6]. A slightly modified version of the argument used in 8.16 of López and Ross [6] can now be used to show that E is an $FZ(K)$ -set. In place of (8.15) of López and Ross [6] we have (3.2) and we let $\delta = \delta_2$. The set F_0 which López and Ross choose at the top of page 129 can be taken to be the empty set in our proof. The result follows.

3.6. Similar to 8.18 of López and Ross [6], we have the following.

COROLLARY. *Let K be as in 3.5 and let $E \subseteq X$. The following are equivalent.*

- (i) *The set E is associated with K .*
- (ii) *The set E is the union of a K -basic Sidon set and a finite set.*

Proof. The argument used in 3.5 suffices to show that (i) implies (ii). In order to show the reverse implication, we use the device used in 8.18 of López and Ross [6] which was first used by Drury. Let $G_0 = G \times T$ and let $Y = X \times Z$ be its character group. Let E be a K -basic Sidon subset of X and let $E_0 = E \times \{1\}$. We show that $E_0 \cup E_0^{-1}$ is a $K \times T$ -basic Sidon set. That it is a Sidon set is observed in 8.18 of López and Ross [6]. Suppose that $f = \sum_j a_j(\chi_j, 1) + \sum_j b_j(\phi_j^{-1}, -1)$ is an $E_0 \cup E_0^{-1}$ -spectral trigonometric polynomial such that $\xi_{K \times T} f = 0$. We have $z \xi_K \sum a_j \chi_j + \bar{z} \xi_K \sum b_j \phi_j^{-1} = 0$ for all $z \in T$. This implies that $\xi_K \sum a_j \chi_j = \xi_K \sum b_j \phi_j^{-1} = 0$. Since E and E^{-1} are K -basic sets (note 2.2(v)), we conclude that $a_j = b_k = 0$ for each j, k . Thus we see that $E_0 \cup E_0^{-1}$ is a $K \times T$ -basic set. The statement (i) follows from 3.5 using the argument appearing in 8.18 of López and Ross [6].

REMARKS 3.7. 1. We are unable to show that condition (i) in 3.6 can be replaced with the statement that E and K are strictly associated. In particular, 8.21 of López and Ross [6] admits no easy generalization.

2. In view of 8.24 of López and Ross [6], the assertion that, if K is a compact subset of G with $\text{int } K \neq \emptyset$, then a K -basic Sidon set is associated with K , is probably false.

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UNIVERSITY OF MELBOURNE
PARKVILLE, VICTORIA 3052
AUSTRALIA