PACIFIC JOURNAL OF MATHEMATICS
Vol. 71, No. 1, 1977

MINIMAL AND MAXIMAL SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS

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This paper is concerned with the construction of the minimal and the maximal solutions of the nonlinear boundary value problem

$$u'' = f(x, u, u'), \quad 0 < x < 1$$
$$B^i u = \alpha_i u(i) + \beta_i u'(i) = b_i, \quad i = 0, 1$$

under rather mild assumptions on f. In particular, no assumption of monotonicity is made on $f(x, u, u')$ either in u or u'.

1. **Introduction.** This paper is concerned with the construction of the minimal and the maximal solutions of the nonlinear boundary value problem (BVP);

$$u'' = f(x, u, u'), \quad 0 < x < 1 \tag{1}$$

$$B^i u \equiv \alpha_i u(i) + \beta_i u'(i) = b_i, \quad i = 0, 1. \tag{2}$$

Obviously, when such boundary value problems are not necessarily uniquely solvable, the existence of the minimal and the maximal solutions plays a useful role in both the quantitative and qualitative theory for these classes of problems. Although considerable literature exists (see, for instance, [9]) about the min-max solutions of initial value problems, very little is known for boundary value problems even in the case of scalar equations (1)-(2). The results in the latter direction usually impose some kind of monotonicity assumption on f in its second and third arguments. In this paper, we establish the minimal and the maximal solutions of BVP (1)-(2) under rather mild assumptions on f. In particular, no assumption of monotonicity is made on $f(x, u, u')$ either in u or u'. The approach taken is essentially an extension of the ideas in [4] where a monotone method was developed for the quasilinear case when f depends on u' linearly. In this paper, we extend the results of [4] in two ways. First, we relax the restriction of linearity of f in u'. Secondly, while in [4] a linear iteration scheme was employed to generate a monotone sequence, here we require a nonlinear iteration scheme. This necessitates our proving existence and uniqueness of solutions of the nonlinear iteration scheme, whereas in the linear case one immediately has existence and uniqueness of the iterative procedure.

The main result can be stated as follows: Suppose there exists a lower and an upper solution for BVP (1)-(2) such that the upper solution dominates the lower solution on the interval of interest. Further, suppose f is continuous and continuously differentiable in its second and third argument, and satisfies a Nagumo condition with respect to these lower and upper solutions. Then there exists maximal and minimal solutions for BVP (1)-(2). Moreover, these are obtained as limits of monotone sequences. Since these sequences converge monotonically, they also provide upper and lower bounds which can be improved by iteration. Thus, if BVP (1)-(2) possesses a unique solution, then this method provides an approximation scheme in which the difference between the upper and lower iterates serves as a good error estimate.

One of the basic motivations in [4] was an extension of the methods in [1], [7] and [11] to a one dimensional quasilinear model of a fluid mechanical problem. The main result of this paper, however, may be considered as an important step in developing a comparison principle for boundary value problems since, for example, the minimal and maximal solutions of a scalar (BVP) may naturally serve as upper and lower bounds for the norm of solutions of higher order systems of differential equations satisfying appropriate boundary conditions. This will be explored elsewhere.

2. Notation and hypotheses. Let $R = (-\infty, \infty)$, $I = [0, 1]$, and $\|u\| = \sup_I |u(x)|$. For any pair of functions $u(x)$ and $v(x)$ with $u(x) \leqq v(x)$, $x \in I$, we define the conical segment

$$\langle u, v \rangle = \{w(x) \mid u(x) \leqq w(x) \leqq v(x), x \in I\} .$$

Let prime denote derivative with respect to x and let subscripts denote derivatives with respect to variables other than x, for example, $f_u = \partial f / \partial u(x, u, u')$. We make the following hypotheses:

(H_1) The real constants α_i, β_i in (2) satisfy, $\alpha_0, \alpha_1, \beta_1 \geqq 0$, $\beta_0 \leqq 0$ and $\alpha_i^2 + \beta_i^2 > 0$ for $i = 0, 1$.

(H_2) There exist continuously differentiable functions u_0, v_0 which satisfy

$$u_0(x) \leqq v_0(x) , \quad x \in I; \tag{3}$$

furthermore, u_0 satisfies the inequalities

$$\begin{aligned} u_0'' &\geqq f(x, u_0, u_0') \\ B^i u_0 &\leqq b_i , \quad i = 0, 1 . \end{aligned} \tag{4}$$

and v_0 satisfies (4) with inequalities reversed. Recall that this condition says that u_0 and v_0 are lower and upper solutions of (1)-(2) respectively.

(H_3) f is continuous on $I \times R \times R \to R$ and satisfies a Nagumo condition with respect to u_0, v_0; that is, for $x \in I$, $u \in \langle u_0, v_0 \rangle$, $u' \in R$,

$$(5) \qquad |f(x, u, u')| \leqq j(|u'|)$$

where $j(s)$ is a positive and continuous function on $[0, \infty)$ such that there exists a positive constant N, for which

$$(6) \qquad \int_\lambda^N \frac{sds}{j(s)} \max_{x \in I} v_0(x) - \min_{x \in I} u_0(x) \stackrel{\text{def}}{=} \Delta ,$$

where

$$(7) \qquad \lambda = \max(|u_0(0) - v_0(1)|, |u_0(1) - v_0(0)|) .$$

(H_4) $f(x, u, u')$ is continuously differentiable in u and u' on $I \times R \times R$.

REMARK 1. As a consequence of the Nagumo condition in (H_3), there exists a positive number N such that $|u'(x)| \leqq N$ for $x \in I$, where N is defined in (6) and $u'' = f(x, u, u')$ [8]. Notice that N depends only on u_0, v_0 and j.

REMARK 2. In view of (H_3) and (H_4), there exist positive numbers N, $\gamma(N)$, $\gamma'(N)$ such that $|f_u| \leqq \gamma$, $|f_{u'}| \leqq \gamma'$ for $x \in I$, $u \in \langle u_0, v_0 \rangle$ and $|u'(x)| \leqq N$.

REMARK 3. The assumption that $f(x, u, u')$ is continuously differentiable in u, u' on $I \times R \times R$ may be relaxed by requiring only that f_u, $f_{u'}$ exist and are bounded for $x \in I$, $u \in \langle u_0, v_0 \rangle$ and $|u'(x)| \leqq N$.

3. **Basic lemmas.** For $x \in I$, $z \in \langle u_0, v_0 \rangle$, define $F(x, u, u'; z) = f(x, z(x), u') + \gamma u - \gamma z$, where γ is defined in Remark 2 of § 2. For simplicity, we will always write $F(x, u, u'; z) = F(x, u, u')$. Clearly, F is continuous on $I \times R \times R \to R$, $F_u = \gamma > 0$, and $F_{u'} = f_{u'}$.

LEMMA 1. *Let* (H_1), (H_2), *and* (H_4) *hold. Then* u_0 *and* v_0 *are respectively lower and upper solutions for the* BVP

$$(8) \qquad u'' = F(x, u, u') , \quad 0 < x < 1$$

$$(9) \qquad B^i u = b_i , \quad i = 0, 1 .$$

Proof. Consider the case of a lower solution. We need only to show that $f(x, u_0, u_0') \geqq F(x, u_0, u_0')$ for $x \in I$. To see this, note that

$$\begin{aligned} f(x, u_0, u_0') &- F(x, u_0, u_0') \\ &= f(x, u_0, u_0') - f(x, z(x), u_0') - \gamma(u_0 - z) \\ &= [f_u(x, \tilde{u}_0, u_0') - \gamma](u_0 - z) \geqq 0 \end{aligned}$$

where $\tilde{u}_0 \in \langle u_0, z \rangle$ and we pick N large enough so that $|u_0'(x)|, |v_0'(x)| \leqq N$ for $x \in I$. Thus the above inequality holds since $f_u \leqq \gamma$ and $u_0 \leqq z$. Similarly, v_0 can be seen to be an upper solution.

LEMMA 2. *Let the assumptions of Lemma* 1 *hold. Further, suppose* (H_3) *is satisfied. Then F satisfies a Nagumo condition with respect to u and v provided*

$$\frac{j(s)}{s^2} \text{ is finite for } s \longrightarrow \infty , \tag{10}$$

and

$$\frac{\gamma(N)}{N^2} \longrightarrow 0 \quad as \quad N \longrightarrow \infty . \tag{11}$$

Proof. Define $l(N) = \gamma(N)\|u_0 - v_0\|$. Then, clearly,

$$|F(x, u, u')| \leqq j(|u'|) + l(N)$$

for $x \in I$ and $u \in \langle u_0, v_0 \rangle$. We want to pick N so large that

$$\int_\lambda^N \frac{sds}{j(s) + l(N)} \geqq \Delta ,$$

where λ is given by (7). From (10) there exists a $\tau > 0$, $\rho > 0$ such that $j(s) \leqq \rho s^2$ for $s > \tau$ and from (11) there exists a function $K(m)$ such that $K(m) \to 0$ as $m \to \infty$ and $l(N) \leqq K(m)N^2$ whenever $N \geqq m$. For some arbitrary (for the moment) positive number t pick $N \geqq tm$. Then, since we can assume $m > \lambda$ and $m > \tau$

$$\begin{aligned} \int_\lambda^N \frac{sds}{j(s) + l(N)} &> \int_m^N \frac{sds}{j(s) + l(N)} \geqq \int_m^N \frac{sds}{(\rho s^2 + K(m)N^2)} \\ &= \frac{1}{2} \ln \left[\frac{\rho N^2 + K(m)N^2}{\rho m^2 + K(m)N^2} \right] \\ &\geqq \frac{1}{2} \ln t^2 + \frac{1}{2} \ln \left[\frac{\rho + K(m)}{\rho + t^2 K(m)} \right] \\ &\geqq \Delta , \end{aligned}$$

provided we choose $t = e^{2\Delta}$ and m such that $K(m) = \rho e^{-2\Delta}$. Now Lemma 2 is established by picking $N \geqq e^{2\Delta} m$.

REMARK 4. As a consequence of Lemma 2 and Nagumo's lemma, we have that if $u \in \langle u_0, v_0 \rangle$ satisfies (8), then $|u'(x)| \leqq N$, for $x \in I$, where

$$N \geqq e^{2\Delta} K^{-1}(\rho e^{-2\Delta}) . \tag{12}$$

Here $K(\cdot)$ and ρ are defined as above. We can assume without loss of generality that $K(\cdot)$ is a decreasing function.

REMARK 5. The conditions (10) and (11) cannot be weakened much for the following reasons: If we allow $\gamma(N) = O(N^2)$, then since

$$\int_\lambda^N \frac{sds}{j(s) + N^2} \leqq N \int_\lambda^N \frac{ds}{j(s) + N^2} \leqq \frac{1}{N} \int_\lambda^N ds < 1 ,$$

F may not satisfy a Nagumo condition unless Δ happens to be sufficiently small.

On the other hand if we assume $j(s)$ only satisfies (6) and do not require (10), then by defining $j(s) = sh(s)$, where $h(s)$ is given in Example 2.3 in [3] and assuming $\gamma(N) = O(N)$, we have

$$\int_\lambda^N \frac{sds}{j(s) + N} < \int_\lambda^N \frac{sds}{j(s) + s} = \int_\lambda^N \frac{ds}{h(s) + 1} < \int_\lambda^\infty \frac{ds}{h(s) + 1} < \infty .$$

Thus, in general a compatibility condition between $j(s)$ and $\gamma(N)$ is needed to insure that F satisfies a Nagumo condition. Clearly, (10) and (11) are satisfied in [4], because both j and γ are linear there.

We shall now use the maximum principle to assert that there is at most one solution of the BVP (8)-(9) contained in $\langle u_0, v_0 \rangle$. Since we will be making much use of the maximum principle [10], we state it here for completeness as:

LEMMA A. *Let $q(x)$, $r(x)$ be real-valued functions on I with $r(x) \geqq 0$, $x \in I$. Suppose* (H_1) *holds and $\phi \in C'(I)$ satisfies*

$$\phi'' + q(x)\phi' - r(x)\phi \leqq 0 \tag{13}$$

$$\alpha_i \phi(i) + \beta_i \phi'(i) \geqq 0 , \quad i = 0, 1. \tag{14}$$

Then $\phi(x) \geqq 0$ for $x \in I$. If the inequalities in (13) *and* (14) *are reversed then $\phi(x) \leqq 0$.*

LEMMA 3. *Let the assumptions of Lemma 2 hold. In addition, assume N satisfies* (12) *and*

$$N \geqq \max(\|u_0'\|, \|v_0'\|) . \tag{15}$$

Then the BVP (8)-(9) *has at most one solution in $\langle u_0, v_0 \rangle$.*

Proof. Suppose u_1 and u_2 are two solutions of BVP (8)-(9) in $\langle u_0, v_0\rangle$. Then from Lemma 2 and Remark 4, we conclude that for $x \in I$, $|u_i'(x)| \leqq N$, $i = 1, 2$. Set $\phi = u_1 - u_2$. Then using the mean-value theorem, we obtain

$$\begin{aligned}\phi'' &= f(x, z, u_1') - f(x, z, u_2') + \gamma(u_1 - u_2)\\ &= f_{u'}(x, z, \psi(x))\phi' + \gamma\phi\end{aligned}$$

$$B^i\phi = 0$$

where $|\psi(x)| \leqq N$ for $x \in I$. An application of Lemma A then concludes the proof of Lemma 3.

We are now in a position to use a result in [5] to obtain the existence of a solution of the BVP (8)-(9) in $\langle u_0, v_0\rangle$.

LEMMA 4. *Assume* (H_1)-(H_4), (10) *and* (11) *hold and let* N *satisfy* (12) *and* (15). *Then there exists a solution of* $u(x)$ *of the* BVP (8)-(9) *such that* $u \in \langle u_0, v_0\rangle$ *and* $|u'(x)| \leqq N$ *for* $x \in I$.

Proof. First observe that from Lemma 1, u_0, v_0 are lower and upper solutions respectively of the BVP (8)-(9), and from Lemma 2, F satisfies a Nagumo condition with respect to u_0 and v_0. Since $\beta_0 \leqq 0$ and $\beta_1 \geqq 0$, the result in [5] together with Remark 4, establishes Lemma 4. We should remark that although in [5], it is assumed that the strict inequalities $u_0(0) < v_0(0)$ and $u_0(1) < v_0(1)$ are satisfied, these can be relaxed. For instance, using well known approximation arguments [2, 6] the result in [5] is valid for $u_0(0) \leqq v_0(0)$ and $u_0(1) \leqq v_0(1)$.

Thus, from Lemmas 3 and 4, we conclude that the BVP (8)-(9) is uniquely solvable in $\langle u_0, v_0\rangle$.

4. Minimal and maximal solution. For each function $z(x) \in C'(I) \cap \langle u_0, v_0\rangle$, define the image $w(x)$ of the mapping A to be the solution of the nonlinear BVP (8)-(9), that is, $w = Az$ if and only if $w(x)$ satisfies (8) and (9). From the previous section, $w(x)$ is uniquely defined for each $z(x) \in C'(I) \cap \langle u_0, v_0\rangle$, is contained in $C'(I) \cap \langle u_0, v_0\rangle$ and satisfies $|w'(x)| \leqq N$ for $x \in I$.

LEMMA 5. *Assume* (H_1)-(H_4), (10) *and* (11) *hold and let* N *satisfy* (12) *and* (15). *Then,*

(i) $Au_0 \geqq u_0$, $Av_0 \leqq v_0$

(ii) *A is monotone on* $\langle u_0, v_0\rangle$, *that is, if* $z_1, z_2 \in C'(I) \cap \langle u_0, v_0\rangle$, *and* $z_1 \leqq z_2$ *then* $Az_1 \leqq Az_2$.

Proof. (i) Suppose $Au_0 = w$. Set $\phi = w - u_0$. Then exactly as in the proof of Lemma 3 with $z = u_0$

$$\phi'' - f_{u'}(x, u_0, \tilde{u}_0')\phi' - \gamma\phi \leqq 0$$

and

$$B^i\phi \geqq 0\ , \quad i = 0, 1$$

where $|\tilde{u}_0'(x)| \leqq N$ for $x \in I$. Therefore, from Lemma A, we conclude that $w \geqq u_0$. Similarly we can show that $Av_0 \leqq v_0$. This proves (i). (ii) Suppose $z_1, z_2 \in C'(I) \cap \langle u_0, v_0\rangle$ and $z_1 \leqq z_2$. Let $Az_i = w_i$, $i = 1, 2$. Then setting $\phi = w_2 - w_1$, and using the same techniques as in Lemma 3, ϕ satisfies

$$\phi'' - f_{u'}(x, z_2, w')\phi' - \gamma\phi = [f_u(x, \tilde{z}, w_1') - \gamma](z_2 - z_1) \leqq 0$$

where $\tilde{z} \in \langle u_0, v_0\rangle$ and $|w'(x)| \leqq N$. The above inequality follows from the fact that $f_u \leqq \gamma$ for $x \in I$, $u \in \langle u_0, v_0\rangle$ and $|u'(x)| \leqq N$. Also, $B^i\phi = 0$. Again from the generalized maximum principle we conclude that $w_1 \leqq w_2$. This completes the proof.

From Lemma 5, we see that A is monotone on $\langle u_0, v_0\rangle$ and maps this closed, bounded and convex set into itself. Thus, using the mapping $Az = w$ defined by BVP (8)-(9), we introduce the sequences $\{u_n\}$ and $\{v_n\}$ by means of

$$u_n = Au_{n-1} \text{ where } u_0 \text{ is given in } (H_2)\ ,$$

$$v_n = Av_{n-1} \text{ where } v_0 \text{ is given in } (H_2)\ .$$

THEOREM. *Let* (H_1)-(H_4), (10) *and* (11) *hold and assume N satisfies* (12) *and* (15). *Let $\{u_n\}$, $\{v_n\}$ be defined as above. Then $\{u_n\}$ and $\{v_n\}$ converge uniformly and monotonically to minimal and maximal solutions $u_{\min}$, $v_{\max}$, respectively, of* BVP (1)-(2) *on $\langle u_0, v_0\rangle$: that is. if w is any solution of* BVP (1)-(2) *in $\langle u_0, v_0\rangle$, then*

$$(16)\quad u_0 \leqq u_1 \leqq \cdots \leqq u_n \leqq \cdots \leqq u_{\min} \leqq w \leqq v_{\max} \cdots \leqq v_n \leqq \cdots \leqq v_1 \leqq v_0.$$

Proof. In view of Lemma 5, the proof follows essentially the same arguments as given in the proof of Theorem 1 in [4]. We only outline it here. By Lemma 5, $u_{n-1} \leqq u_n$ for $n = 0, 1, 2, \cdots$. If w is any solution of (1)-(2) in $\langle u_0, v_0\rangle$, then $u_0 \leqq w$ and $Au_0 \leqq Aw = w$. This implies that $u_n \leqq w$. Since $u_0 \leqq v_0$, then by Lemma 5, $u_n \leqq v_n$, and $v_{n+1} \leqq v_n$ by the above arguments. Thus (16) follows where $u_{\min}$ and $v_{\max}$ denote limits of the monotone bounded sequences $\{u_n\}$, $\{v_n\}$ respectively.

It remains only to show that $u_{\min}$ is a solution of the BVP (1)-(2) (with a similar argument for $v_{\max}$). If $u_{\min}$ is a solution, then it is the minimal solution in $\langle u_0, v_0\rangle$, since $u_n \leqq w$ for all n and any solution w of (1) and (2) in $\langle u_0, v_0\rangle$. It is easy to see that the sequence $\{u_n\}$ is uniformly bounded and equicontinuous and thus converges (the full sequence by monotonicity) on I. By considering the integral equation which is equivalent to the BVP (8)-(9) and using the fact that $\lim u_n = \lim u_{n-1}$, it follows that $\lim u_n = u_{\min}$ is a solution of the BVP (1)-(2).

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Received September 14, 1975.

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