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T^n-ACTIONS ON SIMPLY CONNECTED $(n+2)$-MANIFOLDS

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In this paper we show that, for each $n \geqq 2$, there is a unique, closed, compact, connected, simply connected $(n+2)$-manifold, M_{n+2}, admitting an action of T^n satisfying the following condition: there are exactly n T^1-stability groups $T_1, \cdots, T_n$ with each $F(T_i, M_{n+2})$ connected. In this case we have $T^n \cong T_1 \times \cdots \times T_n$. Any other action (T^n, M^{n+2}), M^{n+2} simply connected, can be obtained from an action (T^n, M_{n+2}) by equivariantly replacing copies of $D^4 \times T^{n-2}$ with copies of $S^3 \times D^2 \times T^{n-3}$. As an application, we classify all actions of T^n on simply connected $(n+2)$-manifolds for $n = 3, 4$.

Several results have been obtained about T^n-actions on $(n+2)$-manifolds. Orlik and Raymond have obtained various classification theorems for the cases $n = 1, 2$ (see [11], [12] and [14]). Various general results have been obtained in [4] and [5] for $n > 2$. This paper is a continuation of the work done in [4]. We also obtain classification theorems similar to those of [12] for $n = 3, 4$.

In [4] it was shown that, for each n, there exist actions of T^n on simply connected $(n+2)$-manifolds. Here we prove the following.

THEOREM. *For each n, there is a unique closed, compact, connected, simply connected $(n+2)$-manifold M_{n+2} admitting an action of T^n satisfying the following conditions*:

(i) *There are exactly n T^1-stability groups $T_1, \cdots, T_n$.*

(ii) *Each $F(T_i, M_{n+2})$ is connected.*

Furthermore, $T^n \cong T_1 \times \cdots \times T_n$.

We then show that any action (T^n, M^{n+2}), M^{n+2} a closed, compact, connected, simply connected $(n+2)$-manifold, can be obtained from an action (T^n, M_{n+2}) by equivariantly replacing copies of $D^4 \times T^{n-2}$ with copies of $S^3 \times D^2 \times T^{n-3}$.

The above results are applied to two specific cases. We show that if T^3 acts on a simply connected 5-manifold, M, then M is $M_5 = S^5$ or a connected sum of copies of $S^2 \times S^3$. For T^4-actions on simply connected 6-manifolds, M, we show that M is $M_6 = S^3 \times S^3$ or M is a connected sum of copies of $S^2 \times S^4$ and $S^3 \times S^3$.

1. Preliminaries. We shall use standard terminology and notation throughout (e.g. see [2]). Unless otherwise stated, all manifolds are closed, connected and compact. All actions are assumed to be locally smooth and effective.

Let (G, M) and (G, N) be two G-actions. We shall use $(G, M) \cong_{eq} (G, N)$ or $M \cong_{eq} N$ to mean that M and N are equivariantly homeomorphic. Given actions (G, M) and (H, N), $(G \times H, M \times N)$ will indicate the obvious product action.

The n-dimensional torus $T^n = S^1 \times \cdots \times S^1$ (n factors) can parameterized as:

$$T^n = \{(e^{i\varphi_1}, \cdots, e^{i\varphi_n}) \mid 0 \leqq \varphi_i \leqq 2\pi\} .$$

We simplify this as $T^n = \{(\varphi_1, \cdots, \varphi_n) \mid 0 \leqq \varphi_i \leqq 2\pi\}$. Similarly, we write:

$$D^n = \{(r_1, \theta_1, \cdots, r_{[n+1/2]}, \theta_{[n+1/2]}) \mid \sum r_i^2 \leqq 1, 0 \leqq \theta \leqq 2\pi , \\ \theta_{[n+1/2]} = 0 \text{ if } n \text{ odd}\} .$$

Of course for S^n, we have $\sum r_i^2 = 1$.

EXAMPLE 1.1. We have an action of T^n on $D^4 \times T^{n-2}$ defined as follows. If $t = (\varphi_1, \cdots, \varphi_n) \in T^n$ and $z = ((r_1, \theta_1, r_2, \theta_2), (\theta_3, \cdots, \theta_n)) \in D^4 \times T^{n-2}$, let

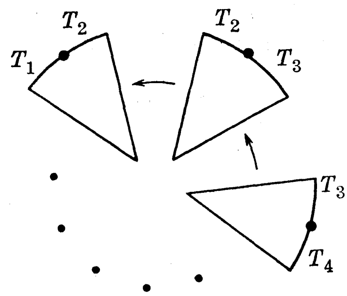
$$tz = ((r_1, \theta_1 + a_{11}\varphi_1 + \cdots + a_{1n}\varphi_n, r_2, \theta_2 + a_{21}\varphi_1 + \cdots + a_{2n}\varphi_n) , \\ (\theta_3 + a_{31}\varphi_1 + \cdots + a_{2n}\varphi_n, \cdots, \theta_n + a_{n1}\varphi_1 + \cdots + a_{nn}\varphi_n)) .$$

This action defines a matrix $A = (a_{ij})$. For the action to be effective, we must have $\det(A) \neq 0$. We shall frequently define such an action by giving the matrix A.

We shall often use the following (see [8]). Suppose M is an m-manifold with boundary and $G \cong T^n$ acts on M with $m > n$. If M^* is a closed cone with vertex x_0^* and $G_{x_0} \cong T^k$, $0 \leqq k \leqq n$ ($T^0 = \text{id}$), then $(T^n, M) \cong_{eq} (T^{n-k} \times T^k, T^{n-k} \times D^{m-n+k})$.

Suppose T^n acts on a simply connected $(n+2)$-manifold M. It was shown in [4] that the orbit space, M^*, will be D^2, with points on the boundary corresponding to singular orbits and interior points corresponding to principal orbits. Isolated points on the boundary correspond to orbits of type T^{n-2} and the remaining boundary points correspond to orbits of type T^{n-1}. The result mentioned above shows that an invariant tubular neighborhood of an orbit of type T^{n-2} will be $D^4 \times T^{n-2}$.

It was also shown in [4] that, for all n, actions of T^n on simply connected $(n+2)$-manifolds exist. The following picture shows how such actions can be constructed.



M^*

Each sector of the disk $D^2 \cong M^*$ represents an invariant tubular neighborhood of an orbit of type T^{n-2} which, as mentioned above, must be $D^4 \times T^{n-2}$. These are attached to one another along subspaces of the boundary homeomorphic to $D^2 \times T^{n-1}$. Another result of [4] is that the circle stability groups of the action must span T^n. Hence, we must have at least n copies of $D^4 \times T^{n-2}$.

We shall say that (T_1, T_2) is an adjacent pair of T^1-stability groups for an action (T^n, M^{n+2}), if there is an invariant $D^4 \times T^{n-2}$ so that the induced action $(T^n, D^4 \times T^{n-2})$ has stability groups T_1, T_2 and $T_1 \times T_2$. (T_1, T_2, T_3) will be called an adjacent triple of T_1-stability groups if (T_1, T_2) and (T_2, T_3) are adjacent pairs, with invariant copies of $D^4 \times T^{n-2}$, $(D^4 \times T^{n-2})_1$ and $(D^4 \times T^{n-2})_2$, respectively, such that $(D^4 \times T^{n-2})_1 \cap (D^4 \times T^{n-2})_2 \cong D^2 \times T^{n-1}$ and $0 \times T^{n-1} \subseteqq F(T_2, M^{n+2})$. In this case $(D^4 \times T^{n-2})_1$ and $(D^4 \times T^{n-2})_2$ are said to be adjacent.

2. Orbit structure.

Suppose T^n acts on a simply connected $(n+2)$-manifold M. As mentioned above, we know that the T^1-stability groups span T^n. In this section we show that, in certain cases, T^n is the direct product of the T^1-stability groups. If G is a group and $S \subseteqq G$ is a subset let $\langle S \rangle$ denote the subgroup spanned by S.

LEMMA 2.1. *If T^n acts on a simply connected $(n+2)$-manifold M, there exists an adjacent triple (T_1, T_2, T_3) such that $\langle T_1 \cup T_2 \cup T_3 \rangle \cong T_1 \times T_2 \times T_3$.*

Proof. Let (T_1, T_2) be an adjacent pair so that we have an invariant $(D^4 \times T^{n-2})_1$ with stability groups T_1, T_2 and $T_1 \times T_2$. Write $T^n = T_1 \times T_2 \times T^{n-2}$ and parameterize so that the action $(T^n, (D^4 \times T^{n-2})_1)$ is defined by the matrix I (see 1.1).

Consider an adjacent invariant $(D^4 \times T^{n-2})_2$ with T^1-stability groups T_1 and C so that (T_2, T_1, C) is an adjacent triple. The action $(T^n, (D^4 \times T^{n-2})_2)$ will be determined by a matrix of the form

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix} .$$

We may consider $N = (D^4 \times T^{n-2})_1 \cup_f (D^4 \times T^{n-2})_2$, where $f: (D^2 \times T^{n-1})_1 \rightarrow (D^2 \times T^{n-1})_2$ is an equivariant attaching homeomorphism, as an invariant subspace of M. f is determined by A in the following manner. If

$$z = ((r_1, \theta_1, r_2, \theta_2), (\theta_3, \cdots, \theta_n)) \in (D^2 \times T^{n-1})_1 \subseteqq (D^4 \times T^{n-2})_1$$

then $f(z) = ((r_1, \theta_1', r_2, \theta_2'), (\theta_3', \cdots, \theta_n'))$ where

$$\bar{\theta}' = (\theta_1' \cdots \theta_n')^t = A(\theta_1 \cdots \theta_n)^t = A\bar{\theta} .$$

To show that f is equivariant, we ignore the r's for convenience. Let $\alpha = (\varphi_1 \cdots \varphi_n)^t = \bar{\varphi} \in T^n$. Then

$$\begin{aligned} \alpha(f(z)) &= \alpha A\bar{\theta} \\ &= A\bar{\theta} + A\bar{\varphi} \\ &= A(\bar{\theta} + \bar{\varphi}) = f(\alpha z) . \end{aligned}$$

Now note that if, for each j, there exists an $i > 2$ with $a_{ij} \neq 0$, then $f_\#: \pi_1((D^2 \times T^{n-1})_1) \rightarrow \pi_1((D^2 \times T^{n-1})_2)$ is injective. In this case, it follows that $\pi_1(N) \cong \boldsymbol{Z}^{n-1}$. Since M is obtained, as described above, by attaching successive copies of $D^4 \times T^{n-2}$, some attaching map must kill an element of some $\pi_1(D^2 \times T^{n-1})$. Hence, let us assume that $a_{in} = 0$ for all $i > 2$.

It now follows that the stability group C is defined by the following system of equations:

$$\begin{aligned} \varphi_1 + a_{12}\varphi_2 + \cdots + a_{1n}\varphi_n &\equiv 0(2\pi) \\ a_{32}\varphi_2 + \cdots + a_{1,n-1}\varphi_{n-1} &\equiv 0(2\pi) \\ &\vdots \\ a_{n2}\varphi_2 + \cdots + a_{n,n-1}\varphi_{n-1} &\equiv 0(2\pi) . \end{aligned}$$

Since the action is effective, it follows that $\varphi_2 = \cdots = \varphi_{n-1} = 0$. Hence, $C = \{(-a_{1n}\varphi_n, 0, \cdots, 0, \varphi_n) \mid 0 \leqq \varphi_n \leqq 2\pi\}$. It is easy to see that $\langle T_1 \cup T_2 \cup C \rangle \cong T_1 \times T_2 \times C$.

COROLLARY 2.2. *If T^n acts on a simply connected $(n+2)$-manifold, M, there exists an invariant $D^2 \times S^3 \times T^{n-3}$ with the* *standard product action* $(T^1 \times T^2 \times T^{n-3}, D^2 \times S^3 \times T^{n-3})$.

Proof. By the lemma, one can find an adjacent triple of T^1-stability groups (T_2, T_1, T_3) so that $T^n = T_1 \times T_2 \times T_3 \times T^{n-3}$. Let $(D^4 \times T^{n-2})_1$ and $(D^4 \times T^{n-2})_2$ be the adjacent copies of $D^4 \times T^{n-2}$ corresponding to the adjacent pairs (T_2, T_1) and (T_1, T_3), respectively. Let $N = (D^4 \times T^{n-2})_1 \cup_f (D^4 \times T^{n-2})_2$ as in the proof of 2.1 so that we have the action (T^n, N). We have the standard action

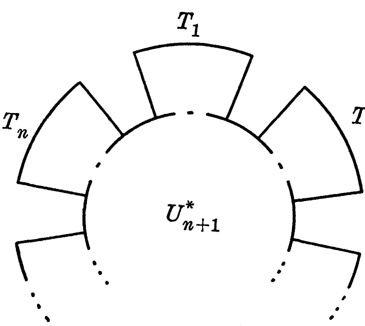
$$(T^n, D^2 \times S^3 \times T^{n-3}) = (T_1 \times (T_2 \times T_3) \times T^{n-3}, D^2 \times S^3 \times T^{n-3})$$

with weighted orbit space equivalent to N^*. It follows from standard techniques that $N \cong_{eq} D^2 \times S^3 \times T^{n-3}$.

In case there are only n T^1-stability groups we have the following much stronger result.

THEOREM 2.3. *Suppose T^n acts on a simply connected $(n + 2)$-manifold, M, so that there are exactly n T^1-stability groups $T_1, \cdots, T_n$ with each $F(T_i, M)$ connected. Then $T^n \cong T_1 \times \cdots \times T_n$.*

Proof. First remove nonintersecting neighborhoods $D^4 \times T^{n-2}$ of each orbit of type T^{n-2}. We obtain a T^n-manifold with boundary, N, with N^* as shown below.



N^*

Using the Seifert-Van Kampen theorem, it is easy to see that $\pi_1(N) \cong 0$. Let $U_{n+1} \cong T^n \times D^2$ be as shown. For each i, $1 \leqq i \leqq n$, choose $U_i \cong (D^3 \times T^{n-1})$, so that $F(T_i, N) \subseteqq U_i$ and $U_i \cap U_j = U_{n+1}$, $1 \leqq i < j \leqq n$.

Each inclusion $\pi_1(U_{n+1}) \to \pi_1(U_i)$ has kernel isomorphic to $\boldsymbol{Z}$, generated by an element $z_i \in \pi_1(U_{n+1})$ corresponding to $T_i \subseteqq T^n$. Let $K = \langle z_1, \cdots, z_n \rangle$. For each i, we have the following commutative diagram.

$$\begin{array}{ccc} \pi_1(U_{n+1}) & & \\ \Big\downarrow & \searrow^{\rho_{n+1}} & \\ & & \pi_1(U_{n+1})/K\,. \\ & \nearrow_{\rho_i} & \\ \pi_1(U_i) & & \end{array}$$

The vertical map is the inclusion, ρ_{n+1} is the natural projection and ρ_i is defined to make the diagram commute. By the Seifert-Van Kampen theorem, we then have the commutative diagram.

$$\begin{array}{ccc} & & \pi_1(N) \cong 0 \\ & \nearrow & \Big\downarrow \\ \pi_1(U_{n+1}) & & \\ & \searrow_{\rho_{n+1}} & \\ & & \pi_1(U_{n+1})/K\,. \end{array}$$

Therefore $\rho_{n+1} \equiv 0$ and $K = \pi_1(U_{n+1})$.

Label the T_i's so that $T_1 \times \cdots \times T_k$ is a direct product and k is a maximum. Suppose $k<n$. For each $i>k$ we have $\langle T_1 \cup \cdots \cup T_i \rangle \cong T^i$. For $1 \leqq i \leqq k$ let $C_i = T_i$ and for $i > k$ let $C_i \cong T^1$ be such that $\langle T_1 \cup \cdots \cup T_i \rangle \cong C_1 \times \cdots \times C_i$ and $T_i \not\subseteq C_1 \times \cdots \times C_{i-1}$. Parameterize $T^n \cong C_1 \times \cdots \times C_n$ in the obvious manner. For $1 \leqq i \leqq k$, $T_i = \{(0, \cdots, 0, \varphi_i, 0, \cdots, 0) \mid 0 \leqq \varphi_i \leqq 2\pi\}$. For $i > k$, we have $T_i = \{(c_{1i}\varphi_i, \cdots, c_{ii}\varphi_i, 0, \cdots, 0) \mid 0 \leqq \varphi_i \leqq 2\pi\}$. Let δ_{ij} be the Kronecker delta. If we write $\pi_1(U_{n+1}) \cong \pi_1(C_1) \times \cdots \times \pi_1(C_n)$, then for $1 \leqq i \leqq k$, $z_i = (\delta_{i1}, \cdots, \delta_{in})$ and for $i > k$, $z_i = (c_{1i}, \cdots, c_{ii}, 0, \cdots, 0)$. Since $T_i \cap (C_1 \times \cdots \times C_k) \neq \mathrm{id}$ and $T_i \not\subseteq C_1 \times \cdots \times C_{i-1}$ for $i > k$, we have $c_{ii} > 1$. Therefore,

$$\det \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & 0 & & c_{k+1,k+1} & \\ & & & & \ddots \\ & & & & & c_{nn} \end{pmatrix} \neq 1\,.$$

This would imply that $K \neq \pi_1(U_{n+1})$, a contradiction. Therefore $k = n$ and $T^n \cong T_1 \times \cdots \times T_n$.

3. The manifolds M_{n+2} and the construction of actions (T^n, M^{n+2}). In this section we show the existence of basic simply connected $(n+2)$-manifolds admitting actions of T^n. We then show

how any action of T^n on a simply connected $(n+2)$-manifold can be obtained from some action (T^n, M_{n+2}).

THEOREM 3.1. *For each $n = 2$ there exists a unique manifold M_{n+2} admitting an action of T^n satisfying the following condition: there are exactly n T^1-stability groups with each $F(T_i, M_{n+2})$ connected.*

Proof. By the construction in [4] there exists a simply connected $(n+2)$-manifold M and an action $\theta: T^n \times M \to M$ with T^1-stability groups $T_1, \cdots, T_n$ satisfying the stated conditions. Let $\varphi: T^n \times N \to N$ be another such action with T^1-stability groups $C_1, \cdots, C_n$. We assume the T_i's and C_i's are labeled in a clockwise direction going around the orbit spaces. We must show that $M \cong N$.

By 2.3, $T^n \cong T_1 \times \cdots \times T_n = C_1 \times \cdots C_n$. We have the obvious isomorphism $f: T^n \to T^n$ with $f(C_i) = T_i$. Define an action $\theta': T^n \times M \to M$ by $\theta'(t, m) = \theta(f(t(, m)$. It is easy to see that the weighted orbit space of this action is equivalent to that for φ. By the equivariant classification theorem of [4], $M \cong N$.

While it is not true that all actions of T^n on M_{n+2} are equivalent, the above proof shows the following

COROLLARY 3.2. *Any two actions of T^n on M_{n+2} are weakly equivalent.*

The standard actions (T^2, S^4), (T^3, S^5) and $(T^5, S^3 \times S^3)$ show that $M_4 = S^4$, $M_5 = S^5$ and $M_6 = S^3 \times S^3$. The manifolds M_{n+2}, $n > 4$, have not been identified at this time.

The manifolds M_{n+2} provide a starting point for the construction of T^n-actions on simply connected $(n+2)$-manifolds.

THEOREM 3.3. *Suppose T^n acts on a simply connected $(n+2)$-manifold M. Then the action (T^n, M) can be obtained from an action (T^n, M_{n+2}) by equivariantly replacing copies of $D^4 \times T^{n-2}$ with copies of $S^3 \times D^2 \times T^{n-3}$.*

Proof. Consider the action (T^n, M). By 2.2, M contains an invariant $S^3 \times D^2 \times T^{n-3}$. When this is replaced equivariantly with a $D^4 \times T^{n-2}$, the number of T^{n-2}-orbits is decreased by one. If this process is continued, M_{n+2} will be obtained. Reversing the process proves the theorem.

From [4] we know that if T^n acts on a simply connected $(n+2)$-

manifold, M, then the T^1-stability groups span T^n. We now have the following.

COROLLARY 3.4. *Suppose T^n acts on a simply connected $(n+2)$-manifold M. Then there are T^1-stability groups $T_1, \cdots, T_n$ such that $T^n \cong T_1 \times \cdots \times T_n$.*

Proof. Obtain M_{n+2} from M as in the proof of 3.3. Then $T^n \cong T_1 \times \cdots \times T_n$ where $T_1, \cdots, T_n$ are the T^1-stability groups of the resulting action (T^n, M_{n+2}). However these will also be T^1-stability groups for the original action (T^n, M).

4. The cases $n = 3, 4$. It was noted that $M_4 = S^4$, $M_5 = S^5$ and $M_6 = S^3 \times S^3$. These are the only M_{n+2}'s identified. In fact no explicit actions of T^n on simply connected $(n+2)$-manifolds have been identified for $n > 4$.

In [12], Orlik and Raymond classify actions of T^2 on simply connected 4-manifolds. In this section we use results of Wall, [16], and Barden, [1], to classify actions of T^3 and T^4 on simply connected 5- and 6-manifolds, respectively.

Recall that the orbit space, D^2, of an action (T^n, M^{n+2}) has isolated points on the boundary, each corresponding to an orbit of type T^{n-2}.

THEOREM 4.1. *Suppose T^3 acts on a simply connected 5-manifold M so that there are k distinct orbits of type T^1. If $k = 3$, $M \cong S^5$. If $k > 3$, M is a connected sum of $k - 3$ copies of $S^2 \times S^3$.*

Proof. If $k = 3$, then $M \cong M_5 = S^5$. Suppose the theorem is true for some $k \geqq 3$. Let T^3 act on M with $k+1$ orbits of type T^1. M is obtained from a manifold N by equivariantly replacing an $S^1 \times D^4$ with a $D^2 \times S^3$. Since N has k orbits of type T^1, N is a connected sum of $k - 3$ copies of $S^2 \times S^3$ or S^5 if $k - 3 = 0$. By the Mayer-Vietoris sequence

$$H^p(M) \cong \begin{cases} Z & p = 0, 5 \\ Z^{k-2} & p = 2, 3 \\ 0 & \text{otherwise} . \end{cases}$$

By results in [1], the above construction can be done in $\boldsymbol{R}^7$ so M embeds in $\boldsymbol{R}^7$. It follows that $\omega_k(\nu^2) = 0$ for all $k \geqq 1$, where ν^2 is the normal bundle of M and ω_k is the k^{th} Stiefel-Whitney class. By Whitney Duality, $\omega_2(M) = 0$. Therefore, by [1], M is a connected sum of $k - 2$ copies of $S^2 \times S^3$.

It is worthwhile to note that M will not be an equivariant connected sum. In fact, equivariant connected sums of codimension two actions cannot exist for $n \geqq 3$ since T^n cannot act on S^{n+1} for $n \geqq 3$.

It was noted that all T^n-actions on M_{n+2} are weakly equivalent. The following example shows that this is not true for T^n-actions on other simply connected $(n+2)$-manifolds.

EXAMPLE 4.2. Let T^3 act on S^5 with T^1-stability groups T_1, T_2 and T_3 so that $T^3 = T_1 \times T_2 \times T_3$. Define an action $(T^3, S^3 \times D^2)$ as follows:

$$tz = ((r_1, \theta_1 + \varphi_1 - \varphi_2, r_2, \theta_2 + \varphi_1 - \varphi_3), (r_3, \theta_3 + \varphi_1)) .$$

This action has T^1-stability groups T_2, T_3 and

$$T_4 = \{(\varphi_1, \varphi_2, \varphi_3) \mid \varphi_1 = \varphi_2 = \varphi_3\} .$$

Replace the $D^4 \times S^1 \subseteqq S^5$ containing $F(T_2 \times T_3)$ with $S^3 \times D^2$ to obtain an action $(T^3, S^2 \times S^3)$ with T^1-stability groups T_1, T_2, T_3 and T_4. However, we have another action $(T^3, S^2 \times S^3)_2 = (T_1 \times (T_2 \times T_3), S^2 \times S^3)$ with T^1-stability groups T_1, T_2 and T_3 where $F(T_1, S^2 \times S^3)$ has two components. It is obvious that these actions are not weakly equivalent.

We now consider T^4-actions on 6-manifolds.

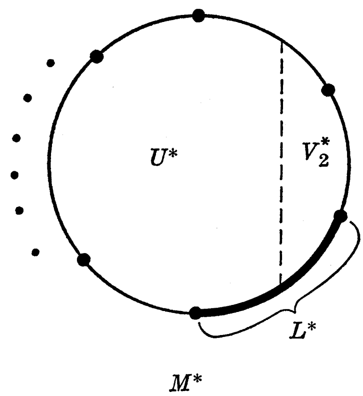
THEOREM 4.2. *Suppose T^4 acts on a simply connected 6-manifold, M, with k orbits of type T^2. Then M is a connected sum of $k-4$ copies of $S^2 \times S^4$ and $k-3$ copies of $S^3 \times S^3$.*

Proof. For $k = 4$, $M \cong M_6 = S^3 \times S^3$. Assume the theorem is true for some $k \geqq 4$, and let T^4 act on M with $k+1$ orbits of type T^2. M is obtained from a T^4-manifold N by equivariantly replacing $V_1 \cong D^4 \times T^2$ with $V_2 \cong S^3 \times D^2 \times T^1$. Since N has k orbits of type T^2, N is a connected sum of $k-4$ copies of $S^2 \times S^4$ and $k-3$ copies of $S^3 \times S^3$. We may assume $N = U \cup V_1$, $M = U \cup V_2$ and $V_i \cap U = S^3 \times T^2$. Applying the Mayer-Vietoris sequence, it is easy to see that $H^2(U) \cong \boldsymbol{Z}^{k-4}$. Also, by examining the pair $(U, \partial U)$, one can show that $H^2(\partial U) \to H^3(U, \partial U)$ is injective so that $H^2(U) \to H^2(\partial U) \cong H^2(T^2 \times S^3)$ is trivial. Therefore, the following sequence is exact.

$$0 \longrightarrow H^1(V_2) \longrightarrow H^1(U \cup V_2) \longrightarrow H^2(M) \longrightarrow H^2(U) \longrightarrow 0 .$$

It follows that $H^2(M) \cong \boldsymbol{Z}^{k-3} \cong H^4(M)$. Since there are no fixed points, $\chi(M) = 0$, so $H^3(M) \cong Z^{2k-4}$.

Since $\omega_2(N) = 0$, $\omega_2(U) = 0$, so if $\omega_2(M) \neq 0$, it must be the generator of $K = \ker(H^2(M) \to H^2(U))$ (with Z_2-coefficients). One can choose as a generator of K a two cochain vanishing off L where L^* is as shown below.



Now L is a closed, compact 4-manifold admitting an action of T^3 so, by [13], $L \cong L(p, q) \times S^1$. Since each factor of L is parallelizable, L is and $\omega_2(L) = 0$. Therefore $\omega_2(M) = 0$. The result follows from [16].

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