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REAL REPRESENTATIONS OF GROUPS WITH A SINGLE INVOLUTION

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If G is a finite group containing just one involution and G has a faithful, absolutely irreducible real representation, then G has order 2.

This was proved by Jerry Malzan [2] using the classification of simple groups with dihedral Sylow 2-subgroups. The purpose of this note is to give a proof of Malzan's theorem which assumes nothing but some elementary character theory.

Let G have the unique involution z and assume $G > \langle z \rangle$. Let $\chi \in \mathrm{Irr}(G)$ be faithful and real valued (where $\mathrm{Irr}(G)$ is the set of complex irreducible characters of G). By the Frobenius-Schur theory (see Lemma 4.4 and Corollary 4.15 of [1]) it follows that in order to prove that χ is not afforded by a real representation, it suffices to show that

$$\sum_{g \in G} \chi(g^2) \neq |G| .$$

THEOREM. *In the above situation we have*

$$\sum_{g \in G} \chi(g^2) < |G| .$$

Proof. Each $g \in G$ may be uniquely factored as $g = \sigma c$ where σ has 2-power order and $c \in C(\sigma)$ has odd order. We write $\sigma = g_2$. For each cyclic 2-subgroup $U \subseteq G$ we set $Y(U) = \{g \in G \mid \langle g_2 \rangle = U\}$. Thus the sets $Y(U)$ partition G. We shall prove

$$(1) \qquad \sum_{g \in Y(1)} \chi(g^2) = \sum_{g \in Y(\langle z \rangle)} \chi(g^2) < |G|/2$$

$$(2) \qquad \sum_{g \in Y(U)} \chi(g^2) \leqq 0 \quad \text{if} \quad |U| = 4$$

$$(3) \qquad \sum_{g \in Y(U)} \chi(g^2) = 0 \quad \text{if} \quad |U| \geqq 8 .$$

The theorem will then follow.

Proof of (1). $Y(1)$ is the set of elements of G of odd order and since $z \in \mathbb{Z}(G)$, we have $Y(\langle z \rangle) = zY(1)$ and so $\sum_{Y(1)} \chi(g^2) = \sum_{Y(\langle z \rangle)} \chi(g^2)$. Since the map $g \mapsto g^2$ is a permutation of $Y(1)$, the common value of these sums is

$$s = \sum_{g \in Y(1)} \chi(g) .$$

If α is any automorphism of the field $\mathbb{Q}(\chi)$, then there exists an integer m with $(m, |G|) = 1$ such that $\chi(g)^\alpha = \chi(g^m)$ for all $g \in G$. Since the map $g \mapsto g^m$ is a permutation of $Y(1)$, it follows that $s^\alpha = s$ and thus s is rational.

Now let $\chi = \chi_1, \chi_2, \cdots, \chi_n$ be the distinct Galois conjugates of χ and let $\theta = \sum \chi_i$. Then θ is rational valued and hence $\theta(g) \in \mathbb{Z}$ and $\theta(g) \leqq \theta(g)^2$ for all $g \in G$. Furthermore, $s = \sum_{Y(1)} \chi_i(g)$ for all i since s is rational, and thus

$$ns = \sum_{g \in Y(1)} \theta(g) \leqq \sum_{g \in Y(1)} \theta(g)^2 .$$

Since $\chi(zg) = -\chi(g)$ for all $g \in G$, we have $\sum_{Y(1)} \theta(g)^2 = \sum_{Y(\langle z \rangle)} \theta(g)^2$ and so

$$\begin{aligned} 2ns &\leqq \sum_{g \in Y(1) \cup Y(\langle z \rangle)} \theta(g)^2 \\ &\leqq \sum_{g \in G} \theta(g)^2 = |G|[\theta, \theta] = n|G| . \end{aligned}$$

Therefore, $s \leqq |G|/2$. In fact, this inequality is strict since otherwise $\theta(1) = \theta(1)^2$ and hence $\chi(1) = 1$. Since χ is real-valued and faithful and $|G| > 2$, this is impossible and (1) follows.

Proof of (2). Let $|U| = 4$ with $\langle \sigma \rangle = U$. Since $C(\sigma)$ has a unique involution and a central element of order 4, it follows that $C(\sigma)$ has a cyclic Sylow 2-subgroup and therefore has a normal 2-complement N. Thus $Y(U) = \sigma N \cup \sigma^{-1} N$. Since $\sigma^2 = (\sigma^{-1})^2 = z$ and $\chi(zg) = -\chi(g)$ for all $g \in G$, we have

$$\begin{aligned} \sum_{g \in Y(U)} \chi(g^2) &= -2 \sum_{g \in N} \chi(g^2) \\ &= -2 \sum_{g \in N} \chi(g) = -2|N|[\chi_N, 1_N] \leqq 0 \end{aligned}$$

since $g \mapsto g^2$ is a permutation of N.

Proof of (3). Let $|U| \geqq 8$ and let V be the subgroup of order 4 in U. If $g \in Y(U)$ and $\tau \in V$, then $\tau g \in Y(U)$ and hence $Y(U)$ is a union of cosets of V of the form Vx with $x \in C(V)$. Now

$$\sum_{g \in Vx} \chi(g^2) = 2\chi(x^2) + 2\chi(zx^2) = 0 .$$

References

1. I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, 1976.
2. J. Malzan, *On groups with a single involution*, Pacific J. Math., **57** (1975), 481-489.
3. ———, Corrections to *On groups with a single involution*, Pacific J. Math., **67** (1976), 555.

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