A NOTE ON THE JACOBI-PERRON ALGORITHM

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In this paper we give a simple geometric description of the Jacobi-Perron algorithm, based on the matrix-theoretic approach to the algorithm. An important advantage of the geometric description is that it may be used as an aid to intuition as well as a practical tool. As an illustration, we prove convergence for a special case.

The theory and procedures of the JPA have been studied from many different viewpoints (cf. [1] and [3]). We consider here some of the linear algebra and geometry naturally associated with the matrix theory of the JPA.

In particular, for $n = 2, 3$ the procedures of the JPA may be represented concretely in the Euclidean spaces R^2 and R^3, and the notion of convergence of the JPA takes on a fairly simple geometrical meaning.

In §1 we summarize briefly the JPA as described by Bernstein in [1] (throughout, we use [1] as the standard reference). In §2, we restate the definitions given in §1 in matrix-theoretic terms. Based on this, we give a general description of the geometrical meaning which may be attached to the notion of convergence of the JPA. In §3 we consider in detail the geometry associated with the JPA for $n = 3$. The behavior of a JPA may be represented graphically in this case. (This may be done similarly for $n = 2$, but the case $n = 3$ is far more representative of the general case.) We conclude with a straightforward, elementary proof of the convergence of any JPA for a T-function whose values are positive integers, for the case $n = 3$.

1. **The Jacobi-Perron algorithm (JPA).** In this section, we briefly recall the description of the JPA given in [1], and throughout this section we use the notation of [1].

Let n be fixed, and let k be a nonnegative integer. The vector $a^{(k)}$ in R^{n-1} is defined by

(1.1) $$a^{(k)} = (a_1^{(k)}, a_2^{(k)}, \cdots, a_{n-1}^{(k)}) .$$

A transformation T of R^{n-1} to itself is defined as follows: suppose f is a (vector) function on R^{n-1} such that

(1.2) $$f(a^{(k)}) = b^{(k)} = (b_1^{(k)}, b_2^{(k)}, \cdots, b_{n-1}^{(k)})$$

and suppose also that $a_1^{(k)} \neq b_1^{(k)}$. Then put

(1.3) $T(a^{(k)}) = (a_1^{(k)} - b_1^{(k)})^{-1}(a_2^{(k)} - b_2^{(k)}, \cdots, a_{n-1}^{(k)} - b_{n-1}^{(k)}, 1)$

for $k = 0, 1, 2, \cdots$.

A sequence $\langle a^{(k)} \rangle = \langle a^{(0)}, a^{(1)}, a^{(2)}, \cdots \rangle$ of vectors in R^{n-1} is called a Jacobi-Perron algorithm of the vector $a^{(0)}$, provided there exists a transformation T of R^{n-1} defined as above such that

(1.4) $T(a^{(k)}) = a^{(k+1)} \quad (k = 0, 1, 2, \cdots),$

that is, if we have $\langle a^{(k)} \rangle = \langle T^k(a^{(0)}) \rangle$.

With a given JPA $\langle a^{(k)} \rangle$ we have the associated functions T and f, and also families of matrices defined as follows:

$A^{(0)}$ is the $n \times n$ identity matrix,

$$A_i^{(n+v)} = \sum_{j=0}^{n-1} b_i^{(v)} A_i^{(v+j)} (b_0^{(v)} = a_0^{(v)} = 1,$$

$$i = 0, \cdots, n-1,$$
$$v = 0, 1, \cdots)$$

$$B^{(v)} = \begin{bmatrix} 0 & \cdots & \cdots & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & \cdots & b^{(v)} \\ 0 & 1 & 0 & \cdots & 0 & b_2^{(v)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 1 & b_{n-1}^{(v)} \end{bmatrix}.$$

These matrices are related by:

$A^{(v+1)} = A^{(v)} B^{(v)} = (A_i^{(v+j)}) \quad (v = 0, 1, \cdots, i, j = 0, 1, \cdots, n-1).$

In [1] the properties of the matrices $A^{(k)}$ and $B^{(k)}$ are studied in some detail, and it is shown how they are related to the JPA. In the next section we consider some of the geometry which is naturally associated with these matrices.

2. **Matrix theory of the JPA.** In this section we use the notation of the last section for the vectors and matrices of a JPA. We also employ the following standard notation: If A is an $n \times n$ matrix and $v \in R^n$, then, regarding v as a column vector, Av is the usual matrix product. For emphasis, we may write col (v), indicating that v is to be regarded as a column vector.

The formula (1.4) may be restated in matrix theoretic terms as follows. Put

$$v^{(k)} = \text{col} (1, a_1^{(k)}, a_2^{(k)}, \cdots, a_{n-1}^{(k)})$$

$(k = 0, 1, \cdots)$. Then

$$(2.1) \qquad v^{(k+1)} = \frac{1}{a_1^{(k)} - b_1^{(k)}} (B^{(k)})^{-1} \cdot v^{(k)} .$$

It is clear that we also have

$$v^{(0)} = \prod_{j=0}^{k} (a_1^{(j)} - b_1^{(j)}) \times B^{(0)} B^{(1)} \cdots B^{(k)} \cdot v^{(k+1)}$$

$$= \prod_{j=0}^{k} (a_1^{(j)} - b_1^{(j)}) \times A^{(k+1)} \cdot v^{(k+1)} .$$

The convergence of a JPA may be described in terms of the behavior of the column vectors of the matrices $A^{(k+1)}$. We first require some notation.

Let X be a vector in R^n, say $X = (x_1, x_2, \cdots, x_n)$. If X is projected onto the $x_1 - x_i$ plane, its image is the vector (x_1, x_i), which has polar coordinates (r_i, θ_i) where

$$r_i = \sqrt{x_1^2 + x_i^2}$$

$$\tan \theta_i = \frac{x_i}{x_1} .$$

It is well known that the vector X is completely determined by its length and the $n - 1$ angles $\theta_2, \theta_3, \cdots, \theta_n$. We shall refer to these angles as the *direction angles* of X.

The convergence of the JPA may be viewed as convergence of the direction angles of the column vectors of the matrices $A^{(k)}$ of the given JPA. To be precise: given the sequence $\{A^{(k)}\}$ of a JPA, for each matrix $A^{(k)}$, let $X_1^{(k)}, X_2^{(k)}, \cdots, X_n^{(k)}$ be its column vectors. Let the direction angles of $X_i^{(k)}$ be denoted by

$$\theta_{2,i}^{(k)}, \theta_{3,i}^{(k)}, \cdots, \theta_{n,i}^{(k)} ,$$

and let the direction angles of $v^{(0)}$ be $\theta_2, \cdots, \theta_n$. Then the JPA converges provided that, as $k \to \infty$, we have

$$\{\theta_{2,i}^{(k)}\} \longrightarrow \theta_2 \quad (i = 1, 2, \cdots, n)$$
$$\{\theta_{3,i}^{(k)}\} \longrightarrow \theta_3 \quad (i = 1, 2, \cdots, n)$$
$$\vdots$$
$$\{\theta_{n,i}^{(k)}\} \longrightarrow \theta_n \quad (i = 1, 2, \cdots, n)$$

that is, corresponding direction angles of the column vectors of $A^{(k)}$ must converge to a common value as $k \to \infty$.

(The actual definition of convergence given in [1] uses $\tan \theta$ rather than θ; it is required that

$$\{\tan \theta_{2,i}^{(k)}\} \longrightarrow \tan \theta_2$$

and so on. This is evidently equivalent to the previous description.)

The convergence of the JPA may also be viewed in the following way. We think of the matrices $B^{(k)}$ as linear transformations of R^n. Suppose for example that all of the $b_i^{(k)}$ are nonnegative, and put

$$S = \{(x_1, x_2, \cdots, x_n) \,|\, x_i \geqq 0\} \,.$$

Then $B^{(k)}(S) \subset S$ and we always have

$$A^{(k)}(S) \supset A^{(k+1)}(S) \,.$$

If the original vector $v^{(0)} = (1, a_1^{(0)}, \cdots, a_{n-1}^{(0)})$ lies in S, and if it should happen that

$$\bigcap_{k=0}^{\infty} A^{(k)}(S) = L \,, \qquad \text{a line in} \quad S,$$

then we must have the direction angles of L equal to the limiting angles $\theta_2, \cdots, \theta_n$ described above for a convergent JPA, that is, the JPA for $v^{(0)}$ converges and L is the line along $v^{(0)}$.

Conversely, if the JPA for $v^{(0)}$ converges, for $v^{(0)} \in S$ (and $B^{(k)}$ as described above), then we must have $\bigcap_{k=0}^{\infty} A^{(k)}(S) = L$, where L is the line along $v^{(0)}$.

In the next section, we employ a slight modification of this viewpoint to give a graphical representation of the behavior of a convergent JPA for the case $n = 3$.

3. **The geometry of a JPA for $n = 3$.** When $n = 2$ or 3, the situation can be pictured in R^2 or R^3; R^3 is the more illuminating. To keep our pictures simple, we shall assume that the T-function yields only positive integer values, and that the starting vector $v^{(0)}$ has all nonnegative entries.

First define six 3×3 matrices $\{E_{ij} \,|\, i \neq j; \, i, j = 1, 2, 3\}$ by:

$$(E_{ij})_{rs} = \begin{cases} 1, & r = s \\ 1, & r = j, s = i \\ 0 & \text{otherwise}\,, \end{cases}$$

that is, E_{ij} is the result of performing the elementary operation "add row i to row j" on the identity matrix I_3. We also define the 3×3 matrix T by

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \,.$$

One checks easily that the matrices $B^{(k)}$ factor as:

$$B^{(k)} = E_{12}^{r_k} E_{13}^{s_k} T$$

(where r_k, s_k are the positive integers $b_1^{(k)}$, $b_2^{(k)}$). One checks that

(3.1) $E_{13}T = TE_{32}, \; E_{12}T = TE_{31}, \; E_{13}T^2 = T^2 E_{21}, \; E_{12}T^2 = T^2 E_{23}$

and so we may write

(3.2) $E_{12}^{r_1} E_{13}^{s_1} T E_{12}^{r_2} E_{13}^{s_2} T E_{12}^{r_3} E_{13}^{s_3} T = E_{12}^{r_1} E_{13}^{s_1} E_{23}^{r_2} E_{21}^{s_2} E_{31}^{r_3} E_{32}^{s_3}$.

For our purposes we use, instead of the sequence $\{B^{(k)}\}$, the equivalent sequence $\{\widetilde{B}^{(k)}\}$ defined by: for $B^{(k)} = E_{12}^{r_k} E_{13}^{s_k} T$, put

$$\widetilde{B}^{(k)} = \begin{cases} E_{12}^{r_k} E_{13}^{s_k} & \text{if} \quad k = 3m \\ E_{23}^{r_k} E_{21}^{s_k} & \text{if} \quad k = 3m+1 \\ E_{31}^{r_k} E_{32}^{s_k} & \text{if} \quad k = 3m+2 \end{cases}$$
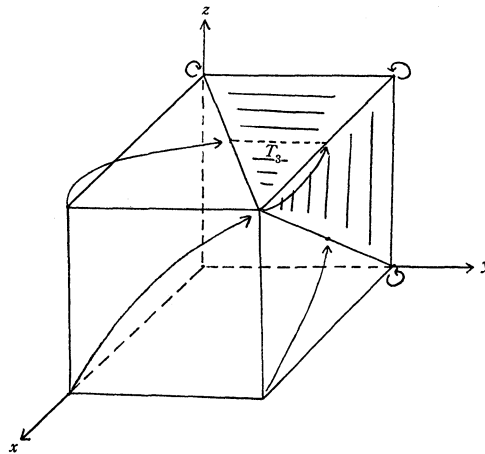
for $k = 0, 1, 2, \cdots$.

Let

$$F_1 = \{(1, y, z) \mid 0 \le y, z \le 1\}$$
$$F_2 = \{(x, 1, z) \mid 0 \le x, z \le 1\}$$
$$F_3 = \{(x, y, 1) \mid 0 \le x, y \le 1\}$$
$$\mathscr{O} = \{(x, y, z) \mid x, y, z \ge 0\} \,,$$

so that $\mathscr{O}$ is the first octant and the F_i are the three faces of the unit cube in $\mathscr{O}$.

$$A = E_{12}E_{13}$$
$$A(1, 0, 0) = (1, 1, 1)$$
$$A(1, 0, 1) = (1, 1, 2) \longrightarrow (\tfrac{1}{2}, \tfrac{1}{2}, 1)$$
$$A(1, 1, 0) = (1, 2, 1) \longrightarrow (\tfrac{1}{2}, 1, \tfrac{1}{2})$$
$$A(1, 1, 1) = (1, 2, 2) \longrightarrow (\tfrac{1}{2}, 1, 1)$$

FIGURE 1

Since $B^{(k)}$ has nonnegative entries, we have $B^{(k)}(\mathscr{O}) \subset \mathscr{O}$. We may visualize $B^{(k)}(\mathscr{O})$ more easily by looking at the intersection of $B^{(k)}(\mathscr{O})$ with the set $F = F_1 \cup F_2 \cup F_3$. The effect of T is just a rotation, and for geometric simplicity we prefer to use the matrices $\tilde{B}^{(k)}$ instead of the matrices $B^{(k)}$. In Figure 1 we show the effect of $E_{12}E_{13}$ on $\mathscr{O}$: it is a sweep towards the $y - z$ plane (resp. for $E_{21}E_{23}$, the $x - z$ plane; for $E_{31}E_{32}$, the $x - y$ plane). Figures 2, 3 show the picture for $E_{31}^r E_{32}^s$ according to whether $r < s$ or $r > s$; if $r = s$, the picture is symmetric about the line $x = y = 1$.

Next, put

$$D_1 = E_{12}E_{13}(\mathscr{O}) \cap F$$
$$D_2 = E_{21}E_{23}(\mathscr{O}) \cap F$$
$$D_3 = E_{31}E_{32}(\mathscr{O}) \cap F .$$
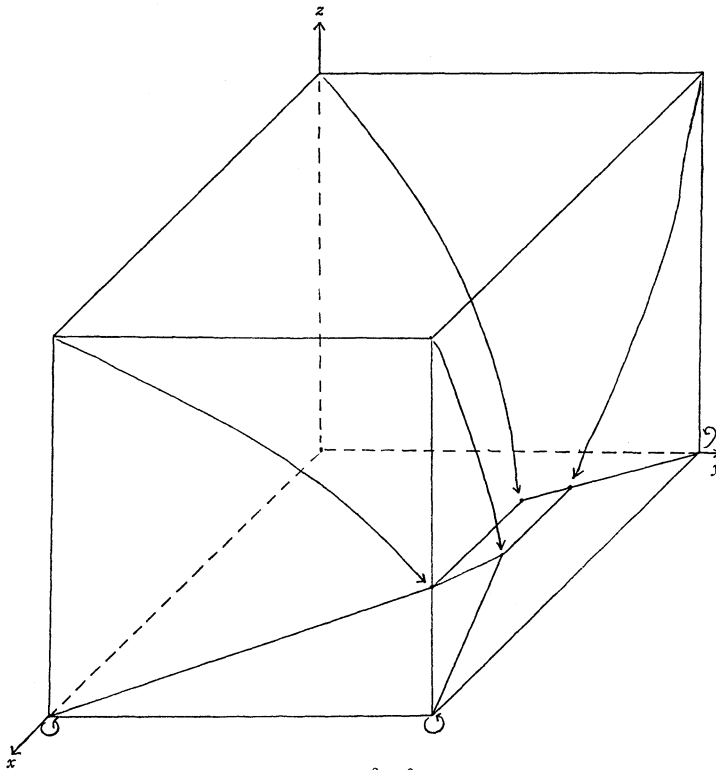
$$A = \tilde{E}_{31}^2 \tilde{E}_{32}^3$$

$A(1, 0, 0) = (1, 0, 0)$

$A(1, 0, 1) = (3, 3, 1) \longrightarrow (1, 1, \tfrac{1}{3})$
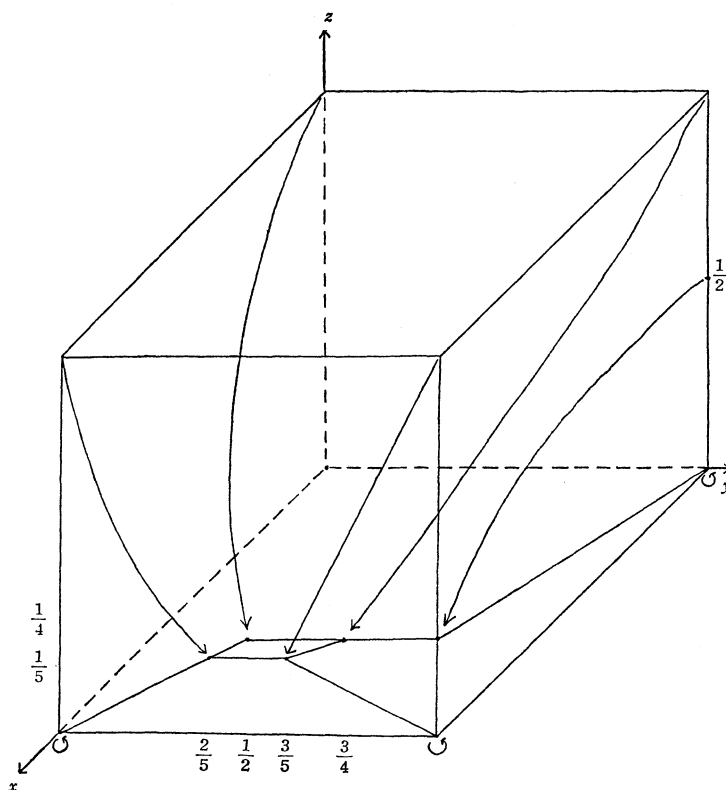
$A(1, 1, 0) = (1, 1, 0)$

$A(1, 1, 1) = (3, 4, 1) \longrightarrow (\tfrac{3}{4}, 1, \tfrac{1}{4})$

$A(0, 0, 1) = (2, 3, 1) \longrightarrow (\tfrac{2}{3}, 1, \tfrac{1}{3})$

$A(0, 1, 1) = (2, 4, 1) \longrightarrow (\tfrac{1}{2}, 1, \tfrac{1}{4})$

$A(0, 1, 0) = (0, 1, 0)$

FIGURE 2

$$A = E_{31}^4 E_{32}^2$$

$A(1, 0, 0) = (1, 0, 0)$

$A(1, 0, 1) = (5, 2, 1) \longrightarrow (1, \frac{2}{5}, \frac{1}{5})$

$A(1, 1, 0) = (1, 1, 0)$

$A(1, 1, 1) = (5, 3, 1) \longrightarrow (1, \frac{3}{5}, \frac{1}{5})$

$A(0, 0, 1) = (4, 2, 1) \longrightarrow (1, \frac{1}{2}, \frac{1}{4})$

$A(0, 1, 1) = (4, 3, 1) \longrightarrow (1, \frac{3}{4}, \frac{1}{4})$

$A(0, 1, 0) = (0, 1, 0)$

$A(0, 1, \frac{1}{2}) = (1, 1, \frac{1}{4})$

FIGURE 3

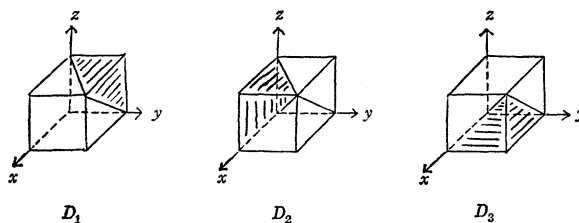The sets D_i cover F and have disjoint interior.

FIGURE 4

In Figure 4 we have sketched in the sets D_1, D_2, D_3. One sees in Figure 2, that $E_{31}E_{32}$ acting on D_1 produces a plane set in F_2 which

is a good deal smaller in its plane dimensions than D_1; similarly for $E_{21}E_{23}$ acting on D_3, and $E_{12}E_{13}$ acting on D_2.

This observation may be made precise, and used to yield a proof of convergence of any JPA in this setting (starting vector $v^{(0)}$ in $\mathscr{O}$; T-function gives positive integers; Theorem 3.5). We indicate below the motivation for our proof.

Referring to Figure 1 again, we have sketched in a triangle

$$T_3 = \left\{ (x, y, 1) \in D_1 \,|\, x \geqq \frac{1}{2} \right\}$$

and in fact, $E_{12}E_{13}(D_2) \cap F = T_3$. Now $E_{23}E_{21}(D_3) \cap F$ is contained in a similar triangle T_1 on F_1, so that

$$R = E_{12}E_{13}(T_1) \cap F \subset T_3;$$

we sketch this set in Figure 5.

If we continue to iterate this process, we expect to get nested sets with diameter decreasing to zero.

Define sets R_k by induction on k:

$$R_0 = \tilde{B}^{(0)}(\mathscr{O}) \cap F$$
$$R_k = (\tilde{B}^{(0)}\tilde{B}^{(1)} \cdots \tilde{B}^{(k)}(\mathscr{O})) \cap F \,.$$

Then clearly we have

(3.3) $R_0 \supset R_1 \supset R_2 \supset \cdots$

and we may state

THEOREM 3.4. *The JPA which gives the sequence $\{B^{(k)}\}$ for a vector $v^{(0)}$ in $\mathscr{O}$, converges if and only if*

$$\bigcap_{k=0}^{\infty} R_k = \{P\} = \{v^{(0)} \cap F\}$$

a single point in F, hence if and only if $\operatorname{diam} R_k \to 0$ *as* $k \to \infty$.

Proof. The set R_k is a triangle in F whose vertices, regarded as vectors in $\mathscr{O}$, are multiples of the column vectors of the matrix

$$\tilde{B}^{(0)} \cdots \tilde{B}^{(k)} = \tilde{A}^{(k)}$$

and when $k = 3m$ we have $\tilde{A}^{(k)} = A^{(k)}$, the kth matrix of the JPA. The first statement [follows from this, the second follows from the Cantor intersection theorem.

It is possible to show geometrically that $\operatorname{diam} R_k \to 0$ as $k \to \infty$,
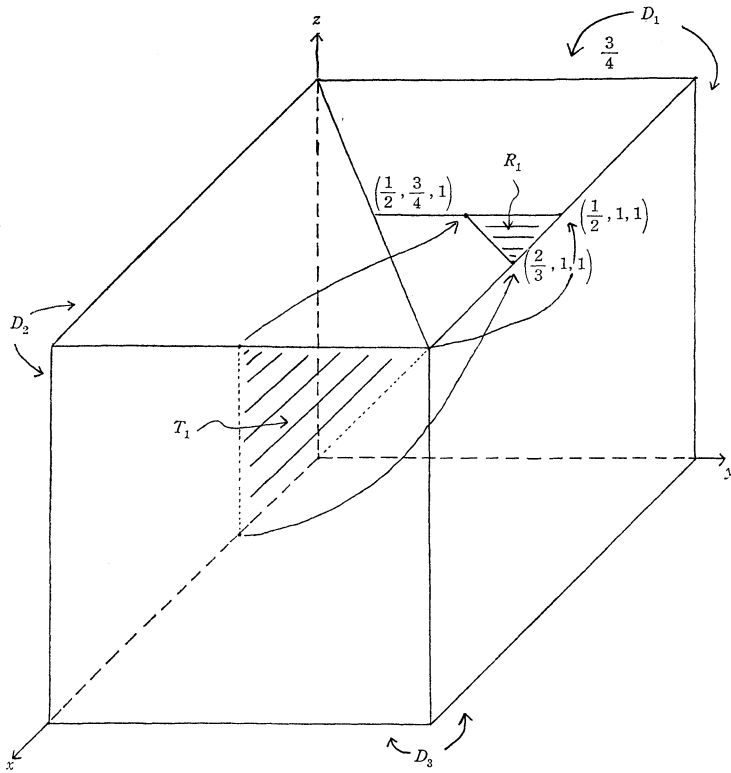
FIGURE 5

thus proving convergence for all such JPA. We indicate briefly here how such a proof may be carried out.

THEOREM 3.5. *For the sets R_k defined above, $\operatorname{diam} R_k \to 0$ as $k \to \infty$.*

Proof (Sketch). The matrix $\tilde{A}^{(k)}$ has non-negative integer entries, and if $k \geq 1$, R_k lies on one of the sets F_i. Write

$$\tilde{A}^{(k)} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

By the construction of the JPA, $\tilde{A}^{(k)}$ must have the form (say $k = 3m + 2$)

$$A^{(k)} = E_{12}^{r_0} E_{13}^{s_0} E_{23}^{r_1} E_{21}^{s_1} \cdots E_{23}^{r_{k-1}} E_{21}^{s_{k-1}} E_{31}^{r_k} E_{32}^{s_k}$$

and from this we have the inequalities

$$c_3 > c_2 > c_1, \, b_3 > b_2 > b_1, \, a_3 > a_2 > a_1, \, c_3 > a_3, \, b_3 > a_3.$$

The column vectors of $\tilde{A}^{(k)}$ intersected with F, give a triangle on F_3, whose vertices we label $\bar{a}$, $\bar{b}$, $\bar{c}$: $\bar{a} = (a_1/a_3,\ a_2/a_3,\ 1)$; $\bar{b} = (b_1/b_3,\ b_2/b_3,\ 1)$; $\bar{c} = (c_1/c_3,\ c_2/c_3,\ 1)$.

Now $\tilde{A}^{(k+1)} = \tilde{A}^{(k)} E_{12}^r E_{13}^s$ for positive integers r, s; the new triangle has vertices $\bar{b}$, $\bar{c}$, and

$$\bar{v} = \left(\frac{a_1 + rb_1 + sc_1}{a_3 + rb_3 + sc_3},\ \frac{a_2 + rb_2 + sc_2}{a_3 + rb_3 + sc_3},\ 1 \right)$$
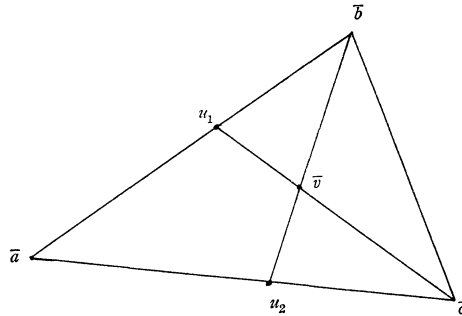

FIGURE 6

The points u_1, u_2 are defined by:

$$u_1 = \left(\frac{a_1 + rb_1}{a_3 + rb_3},\ \frac{a_2 + rb_2}{a_3 + rb_3},\ 1 \right)$$

$$u_2 = \left(\frac{a_1 + sc_1}{a_3 + sc_3},\ \frac{a_2 + sc_2}{a_3 + sc_3},\ 1 \right).$$

One finds that the ratios of the lengths of collinear segments are rational; we have:

(3.6) $$\frac{d(\bar{b}, u_1)}{d(\bar{b}, a)} = \frac{a_3}{a_3 + rb_3} = \frac{1}{1 + \dfrac{rb_3}{a_3}} < \frac{1}{2}$$

(since $b_3 > a_3$); and

(3.7) $$\frac{d(\bar{c}, u_2)}{d(\bar{c}, \bar{a})} = \frac{a_3}{a_3 + sc_3} = \frac{1}{1 + \dfrac{sc_3}{a_3}} < \frac{1}{2}$$

(since $a_3 < c_3$). Now the altitude from v to $\overline{\overline{bc}}$ is less than half the altitude from $\bar{a}$ to $\overline{\overline{bc}}$.

After two more steps, the set R^{k+3} is a triangle with altitudes respectively less than half the altitudes of $R^k = \Delta \bar{a}\bar{b}\bar{c}$. Hence diam $R^k \to 0$ as $k \to \infty$, and this completes the proof.

REMARK 1. At the kth step, the triangle R^{k+1} has a common side

with the triangle R^k; its other two sides are shorter. Thus conceivably we could have diam $R^k =$ diam R^{k+1}. The triangle R^{k+2} has a common side with R^{k+1}, but not the same as the common side of R^k and R^{k+1}; it is this "turning" effect that makes things work out. In fact, we do have diam $R^k <$ diam R^{k+2}; our proof could probably be strengthened along these lines.

In the general case, in R^n, we would have nested sets in R^{n-1} with diameter converging to zero. The preceding proof would require that the process be carried out to at least $n-1$ steps before one could say diam $R^k < c$ diam R^{k+n-1} (for some $c \in (0, 1)$).

REMARK 2. The proof outlined above is rather tedious (and the tedium varies directly with n). It does have the advantage over the usual proofs given for a JPA, of being easy, elementary, and involving only rational procedures. It also shows clearly how the rate of convergence is affected by the exponents r_k, s_k (formulas (3.6) and (3.7)).

REMARK 3. Concerning the use of integers in the T-function, we note that a careful examination of Figures 1, 2 and 3 suggests why a "greatest integer" type of T-function would be valuable.

For example, if we wish to write

$$v^{(k)} = E_{31}^r E_{32}^s v^{(k+1)} \, ,$$

it is obviously most efficient to select the "smallest" possible set like the ones sketched in Figure 2 or 3, containing $v^{(k)}$. One could say also that the triangles R^k are made as small as possible, consistent with other requirements, by the use of a "greatest integer" T-function. (In Figure 6, the point $v^{(0)} \cap F$ lies in $\triangle \overline{abc}$; one selects r, s so that the $\triangle \overline{vbc}$ is as small as possible and contains $v^{(0)} \cap F$.)

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