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ON THE INTEGRAL MEANS OF UNIVALENT, MEROMORPHIC FUNCTIONS

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We consider two classes of functions, univalent and meromorphic in the unit disk Δ. The first class is normalized by requiring that the functions be nonzero in Δ with $f(0)=1$ and a pole at a fixed point, p, $0<p<1$. In the second class the functions are allowed to have a zero with fixed magnitude. Theorems concerning the integral means of functions in both classes are proven and consequences of these theorems are considered.

1. **Introduction.** Let $\Sigma(p)$, $0<p<1$, be the class of functions $f(z)$, univalent and meromorphic in $\Delta=\{z: |z|<1\}$, with a simple pole at $z=p$ and such that $f(z)\neq 0$ for z in Δ and $f(0)=1$. Also, if $0<p<1$ and $0<q<1$, we let $\Sigma(p, q)$ be the class of functions $f(z)$, univalent and meromorphic in Δ, with a simple pole at $z=p$ such that $f(z_0)=0$ for some z_0 with $|z_0|=q$ and $f(0)=1$. Recently Libera and the author [4] and the author [5] have studied a subclass of $\Sigma(p)$, namely the class of weakly starlike meromorphic functions $\Lambda^*(p)$ which have the representation

$$f(z)=\frac{z}{\left(1-\dfrac{z}{p}\right)(1-pz)}g(z)$$

where $g(z)$ is in Σ^*, the class of normalized meromorphic starlike functions. In this paper we will extend many of the results obtained for $\Lambda^*(p)$ to the class $\Sigma(p)$. In particular it was proven in [5] that if f is in $\Lambda^*(p)$ and $F(z)=(1+z)^2/(1-z/p)(1-pz)$, then

$$\int_{-\pi}^{\pi}|f(re^{i\theta})|^{\lambda}\,d\theta \leqq \int_{-\pi}^{\pi}|F(re^{i\theta})|^{\lambda}\,d\theta$$

for $0<r<1$ and $\lambda>0$. Using a powerful method of Baernstein [1], we will extend and generalize this result to the class $\Sigma(p)$. Similar results are also obtained for the class $\Sigma(p, q)$.

2. **The class $\Sigma(p)$.** The proof of the theorem concerning the integral means of a function in $\Sigma(p)$ follows the proof given by Kirwan and Schober [3] who consider the class $S(p)$ of functions $f(z)$, univalent and meromorphic in Δ, with a simple pole at $z=p$ and such that $f(0)=0$ and $f'(0)=1$. The proof relies on results of Baernstein [1] which we now state.

For this purpose we need to introduce some notation. If g is a measurable, extended real valued function on $[-\pi, \pi]$, then we define

$$g^*(\theta) = \sup_E \int_E g(\theta)d\theta$$

where the supremum is taken over all Lebesque measurable sets $E \subset [-\pi, \pi]$ with measure $m(E) = 2\theta$. In particular, if $u(re^{i\theta})$ is defined in an annulus $r_1 < |z| < r_2$ and the $*$ operation is performed in the θ variable, then $u^*(re^{i\theta})$ is defined in $\{re^{i\theta}: r_1 < r < r_2, 0 \leqq \theta \leqq \pi\}$. Baernstein [1] has proven the following.

PROPOSITION 1 ([1, Theorems A and A′ and Proposition 5]).

(i) *Let D be a domain containing $r_0 > 0$ and having a classical Green's function. Let u be the Green's function of D with pole at r_0.* (It is assumed here that u is defined on the extended plane by defining it to be zero on the complement of D.) *Then*

$$u^\sharp(re^{i\theta}) = u^*(re^{i\theta}) + 2\pi \log^+ \frac{r}{r_0}$$

is subharmonic in the upper half-plane.

(ii) *Let D and u be as in* (i) *and suppose further that D is circularly symmetric. Let $D^+ = D \cap \{z: \operatorname{Im} z > 0\}$. Then $u^\sharp(re^{i\theta})$ is harmonic in D^+.*

PROPOSITION 2 ([1, Proposition 2]). *For $g \in L^1[-\pi, \pi]$,*

$$g^*(\theta) = \int_{-\theta}^{\theta} G(x)dx\,, \qquad 0 \leqq \theta \leqq \pi\,,$$

where $G(x)$ is the symmetric nonincreasing rearrangement of g. (For the definition of $G(x)$ see [1] and [2].)

PROPOSITION 3 ([1, Proposition 3]). *For $g, h \in L^1[-\pi, \pi]$ the following are equivalent.*

(a) *For every convex nondecreasing function Φ on $(-\infty, \infty)$,*

$$\int_{-\pi}^{\pi} \Phi(g(\theta))d\theta \leqq \int_{-\pi}^{\pi} \Phi(h(\theta))d\theta\,.$$

(b) *For every $t \in (-\infty, \infty)$,*

$$\int_{-\pi}^{\pi} [g(\theta) - t]^+ d\theta \leqq \int_{-\pi}^{\pi} [h(\theta) - t]^+ d\theta\,.$$

(c) $g^*(\theta) \leqq h^*(\theta)$, $0 \leqq \theta \leqq \pi$.

We can now state and prove the following theorem.

THEOREM 1. *Let Φ be a convex nondecreasing function on $(-\infty, \infty)$. Then for all $f \in \Sigma(p)$ and $0 < r < 1$,*

$$\int_{-\pi}^{\pi} \Phi(\pm \log |f(re^{i\theta})|)d\theta \leqq \int_{-\pi}^{\pi} \Phi(\pm \log |F_p(re^{i\theta})|)d\theta$$

where

$$F_p(z) = \frac{(1+z)^2}{\left(1 - \dfrac{z}{p}\right)(1-pz)} .$$

Proof. We first consider the inequality.

$$\int_{-\pi}^{\pi} \Phi(\log |f(re^{i\theta})|)d\theta \leqq \int_{-\pi}^{\pi} \Phi(\log |F_p(re^{i\theta})|)d\theta . \tag{2.1}$$

With f^{-1} denoting the inverse function of f we define

$$u(w) = \begin{cases} -\log |f^{-1}(w)| , & w \in f(\Delta) \\ 0 , & \text{otherwise} \end{cases} \tag{2.2}$$

and

$$v(w) = \begin{cases} -\log |F_p^{-1}(w)| , & w \in F_p(\Delta) \\ 0 , & \text{otherwise .} \end{cases}$$

According to Proposition 1(i) the function $u^\#(re^{i\theta}) = u^*(re^{i\theta}) + 2\pi \log^+ r$ is subharmonic in the upper half-plane and by ([1, Theorem A′]) is continuous on the real line with 0 deleted. The function F_p maps Δ onto the extended plane slit along the interval $[-4p/(1-p)^2, 0]$. Thus $F_p(\Delta)$ is circularly symmetric and according to Proposition 1(ii) the function $v^\#(re^{i\theta}) = v^*(re^{i\theta}) + 2\pi \log^+ r$ is harmonic in the upper half-plane and by [1, Theorem A′] is continuous on the real line with 0 deleted. It follows then that $u^* - v^* = u^\# - v^\#$ is subharmonic in the upper half-plane and continuous on the real line with 0 deleted.

The inequality (2.1) will follow from Proposition 3(b $\Rightarrow$ a) if it can be proven that for $f \in \Sigma(p)$, $0 < r < 1$ and $0 < \rho < \infty$,

$$\int_{-\pi}^{\pi} \log^+ \frac{|f(re^{i\theta})|}{\rho} d\theta \leqq \int_{-\pi}^{\pi} \log^+ \frac{|F_p(re^{i\theta})|}{\rho} d\theta . \tag{2.4}$$

At this point we have need of a lemma analogous to Proposition 4 in [1] and one which appears in [3].

LEMMA. *Let $f \in \Sigma(p)$, $0 < r < 1$ and $0 < \rho < \infty$. Then,*

$$\int_{-\pi}^{\pi} \log^+ \frac{|f(re^{i\phi})|}{\rho} d\phi + 2\pi \log^+ \frac{r}{p} = \int_{-\pi}^{\pi} [u(\rho e^{i\phi}) + \log r]^+ d\phi + 2\pi \log^+ \frac{1}{\rho} .$$

Because of this Lemma, we see that (2.4) is equivalent to the inequality

$$(2.5) \qquad \int_{-\pi}^{\pi} [u(\rho e^{i\phi}) + \log r]^+ d\phi \leqq \int_{-\pi}^{\pi} [v(\rho e^{i\phi}) + \log r]^+ d\phi .$$

However, applying Proposition 3(c $\Rightarrow$ b) we see that (2.5) will hold provided

$$(2.6) \qquad (u^* - v^*)(\rho e^{i\phi}) \leqq 0 , \qquad 0 < \rho < \infty , \qquad 0 \leqq \theta \leqq \pi .$$

As we have already noted $u^* - v^*$ is subharmonic in the upper half-plane and continuous on the real line with 0 deleted. In a neighborhood of $w = 0$ both $u(w)$ and $v(w)$ are continuous with $u(0) = v(0) = 0$. Thus given $\varepsilon > 0$ there exists $\delta > 0$ such that $|u(w)| < \varepsilon/2\pi$ if $|w| < \delta$. Thus if $|w| < \delta$, $w = \rho e^{i\phi}(0 \leqq \phi \leqq \pi)$ and $m(E) = 2\phi$ we have

$$\int_E u(\rho e^{i\theta}) d\theta < \frac{\varepsilon}{2\pi} m(E) \leqq \varepsilon .$$

Therefore

$$u^*(\rho e^{i\phi}) = \sup_E \int_E u(\rho e^{i\theta}) d\theta \leqq \varepsilon .$$

It follows then that $u^*(w)$ approaches 0 as w approaches 0. A similar statement holds for $v^*(w)$. Thus

$$(2.7) \qquad \lim_{w \to 0} (u^* - v^*)(w) = 0 .$$

We also have

$$\lim_{w \to \infty} u(w) = \lim_{w \to \infty} v(w) = -\log p .$$

Thus given $\varepsilon > 0$ there exists $\delta > 0$ such that $|u(w) + \log p| < \varepsilon/2\pi$ and $|v(w) + \log p| < \varepsilon/2\pi$ if $|w| > \delta$. Thus if $|w| > \delta$, $w = \rho e^{i\phi}(0 \leqq \phi \leqq \pi)$ and $m(E) = 2\phi$,

$$\left| \int_E (u(\rho e^{i\theta}) + \log p) d\theta \right| < \frac{\varepsilon}{2\pi} m(E) \leqq \varepsilon .$$

It follows that

$$-\varepsilon \leqq u^*(\rho e^{i\phi}) + 2\phi \log p \leqq \varepsilon .$$

Similarly, we have

$$-\varepsilon \leqq v^*(\rho e^{i\phi}) + 2\phi \log p \leqq \varepsilon .$$

Thus

$$-2\varepsilon \leqq (u^* - v^*)(\rho e^{i\phi}) \leqq 2\varepsilon .$$

It follows then that

$$\lim_{w\to\infty} (u^* - v^*)(w) = 0 . \tag{2.8}$$

From (2.8) and previous remarks it follows that the subharmonic function $u^* - v^*$ is bounded in the upper half-plane. Thus, by the maximum principle, it is enough to prove that $(u^* - v^*)(s) \leqq 0$ for s on the real axis R.

For this purpose we let

$$D_f = \sup_{w \notin f(\Delta)} |w|$$

and divide the real line into 3 intervals,

$$R = (-\infty, -D_f) \cup [-D_f, 0) \cup [0, +\infty) .$$

Case (i). $s \in [0, +\infty)$. Because of (2.7) we need only consider $s \in (0 + \infty)$. But then $u^*(s) = v^*(s) = 0$ by definition, if $s > 0$.

Case (ii). $s \in (-\infty, -D_f)$. We first note that $u(w)$ is harmonic for $\max\{1, D_f\} < |w| \leqq \infty$ and $v(w)$ is subharmonic in the same region. Thus $(u - v)(w)$ is superharmonic in $\max\{1, D_f\} < |w| \leqq \infty$. In general, $u(w) + \log|w - 1|$ is harmonic in $|w| > D_f$ and $v(w) + \log|w - 1|$ is subharmonic in $|w| > D_f$. It follows that $(u - v)(w)$ is superharmonic for $D_f < |w| \leqq \infty$. Thus we have

$$(u^* - v^*)(s) = \int_{-\pi}^{\pi} (u - v)(|s|e^{i\theta})d\theta \leqq 2\pi(u - v)(\infty) = 0 .$$

Case (iii). $s \in [-D_f, 0)$. Following Kirwan and Schober [3], for a given $\varepsilon > 0$ we introduce the subharmonic function

$$Q(\rho e^{i\phi}) = (u^* - v^*)(\rho e^{i\phi}) - \varepsilon\phi \qquad (0 \leqq \rho < \infty, 0 \leqq \phi \leqq \pi) .$$

From previous cases we have,

$$\limsup_{w\to s} Q(w) \leqq 0 \tag{2.9}$$

for all $s \in \{R - [-D_f, 0)\} \cup \{\infty\}$. Suppose $\sup_{\mathrm{Im}\, w>0} Q(w) = M > 0$. Then as in [3] we have by the maximum principle and (2.9) the existence of some $\hat{s} \in [-D_f, 0)$ such that

$$Q(\hat{s}) \geqq Q(|\hat{s}|e^{i\phi}) , \qquad 0 \leqq \phi \leqq \pi . \tag{2.10}$$

Thus,

$$\begin{aligned}(2.11)\qquad 0 &\leqq \lim_{\phi\to\pi}\frac{Q(|\hat{s}|e^{i\phi})-Q(\hat{s})}{\phi-\pi}\\ &= \lim_{\phi\to\pi}\frac{u^*(|\hat{s}|e^{i\phi})-u^*(\hat{s})}{\phi-\pi}-\lim_{\phi\to\pi}\frac{v^*(|\hat{s}|e^{i\phi})-v^*(\hat{s})}{\phi-\pi}-\varepsilon\,.\end{aligned}$$

From Proposition 2 and the definition of $G(x)$ [1] it follows that

$$(2.12)\qquad \lim_{\phi\to\pi}\frac{u^*(|\hat{s}|e^{i\phi})-u^*(\hat{s})}{\phi-\pi}=2\min_{0\leqq\varphi\leqq\pi}u(|\hat{s}|e^{i\phi})\,.$$

A similar equality holds for v^*. Combining (2.11) and (2.12) we obtain

$$(2.13)\qquad 0\leqq 2\min_{0\leqq\varphi\leqq\pi}u(|\hat{s}|e^{i\phi})-2\min_{0\leqq\phi\leqq\pi}v(|\hat{s}|e^{i\phi})-\varepsilon\leqq-\varepsilon\,.$$

Inequality (2.13) follows since the circle $|w|=|\hat{s}|$ intersects the complement of $f(\varDelta)$ and thus $u(|\hat{s}|e^{i\phi})=0$ for some ϕ and since $v(|\hat{s}|e^{i\phi})\geqq 0$ for all ϕ.

However (2.13) is obviously contradictory and thus we must have $\sup_{\mathrm{Im}\,w>0}Q(w)\leqq 0$. Letting $\varepsilon\to 0$ we obtain $(u^*-v^*)(s)\leqq 0$ for all $s\in[-D_f,0)$. This then completes the proof of (2.6) and hence (2.1).

The proof that

$$\int_{-\pi}^{\pi}\Phi(-\log|f(re^{i\theta})|)d\theta\leqq\int_{-\pi}^{\pi}\Phi(-\log|F_p(re^{i\theta})|)d\theta$$

follows the proofs given in [1] and [3]. The only difference is that (52) of [1] is replaced by

$$\int_{-\pi}^{\pi}\log^+(\rho|f(re^{i\theta})|)d\theta=2\pi\Big(\log\rho-\log^+\frac{r}{p}\Big)+\int_{-\pi}^{\pi}\log^+\frac{1}{\rho|f(re^{i\theta})|}d\theta\,.$$

This then completes the proof of Theorem 1.

We have the following theorem as an immediate consequence of Theorem 1.

THEOREM 2. *Let $f\in\Sigma(p)$, then for all λ, $-\infty<\lambda<\infty$, and $0<r<1$,*

$$(2.14)\qquad \int_{-\pi}^{\pi}|f(re^{i\theta})|^{\lambda}d\theta\leqq\int_{-\pi}^{\pi}|F_p(re^{i\theta})|^{\lambda}d\theta\,.$$

3. Applications of Theorem 2.

THEOREM 3. *Let $f\in\Sigma(p)$ and $0<r<1$, then for $|z|=r$.*

$$(3.1)\qquad F_p(-r)\leqq|f(z)|\leqq|F_p(r)|\,.$$

REMARK. Inequality (3.1) was obtained earlier by Libera and the author [4] for the class $\Lambda^*(p) \subset \Sigma(p)$.

Proof. The right side of (3.1) follows upon taking the λth root of both sides of (2.14) and letting $\lambda \to +\infty$. To obtain the left side of (3.1) we note that 2.1 gives for $\lambda > 0$

$$\int_{-\pi}^{\pi} \left|\frac{1}{f(re^{i\theta})}\right|^{\lambda} d\theta \leqq \int_{-\pi}^{\pi} \left|\frac{1}{F_p(re^{i\theta})}\right|^{\lambda} d\theta .$$

Taking the λth root in the last inequality and letting $\lambda \to +\infty$ we obtain

$$\frac{1}{|f(z)|} \leqq \max_{|z|=r} \frac{1}{|f(z)|} \leqq \max_{|z|=r} \frac{1}{|F_p(z)|} = \frac{\left(1+\frac{r}{p}\right)(1+pr)}{(1-r)^2} = \frac{1}{F_p(-r)} .$$

The last inequality is equivalent to the left side of (3.1).

Let $f \in \Sigma(p)$ and $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ for $|z| < p$. It has been proven [4] that if $f \in \Lambda^*(p) \subset \Sigma(p)$, then

$$\frac{(1-p)^2}{p} \leqq |a_1| \leqq \frac{(1+p)^2}{p} .$$

The inequality $|a_1| \leqq (1+p)^2/p$ can be obtained for the class $\Sigma(p)$ by considering the case $\lambda = 2$ in Theorem 2 and letting $r \to 0$. However, making use of some results of Kirwan and Schober [3] we can obtain both the upper and lower bounds on $|a_1|$.

THEOREM 4. *Let $f \in \Sigma(p)$ and $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ for $|z| < p$, then*

$$\frac{(1-p)^2}{p} \leqq |a_1| \leqq \frac{(1+p)^2}{p} . \tag{3.2}$$

The inequalities are sharp.

Proof. It is easily seen that if $f \in \Sigma(p)$ with $f'(0) = a_1$, then we can write $f(z) = a_1 g(z) + 1$ where $g \in S(p)$. According to Kirwan and Schober [3], $g(\Delta)$ contains $\{w: |w| < p/(1+p)^2\}$ and $\{w: |w| > p/(1-p)^2\}$. It follows that $f(\Delta)$ contains

$$\{w: |w-1| < p|a_1|/(1+p)^2\}$$

and

$$\{w: |w-1| > p|a_1|/(1-p)^2\} .$$

Since $0 \notin f(\Delta)$ we must have $1 \geqq p|a_1|/(1+p)^2$ and $1 \leqq p|a_1|/(1-p)^2$, which gives (3.2).

The function $F_p(z) = (1+z)^2/(1-z/p)(1-pz)$ gives equality on the right side of (3.2) and $f(z) = (1-z)^2/(1-z/p)(1-pz)$ gives equality on the left side of (3.2).

REMARK. Using Theorem 4 and the representation $f(z) = a_1 g(z) + 1$ where $g \in S(p)$, estimates $|a_n|$ similar to those given in [3] may be obtained. Estimates may also be obtained by using Theorem 2 directly.

In [4] sharp estimates on the quantity $|f'(z)/f(z)|$ were obtained for $f \in \Delta^*(p)$. Making use of Theorem 4, we can now extend the results to the class $\Sigma(p)$.

THEOREM 5. *Let $f \in \Sigma(p)$ and $w \in \Delta$, $w \neq p$, then*

$$\frac{1}{(1-|w|^2)}\frac{(1-|a|)^2}{|a|} \leqq \left|\frac{f'(w)}{f(w)}\right| \leqq \frac{1}{(1-|w|^2)}\frac{(1+|a|)^2}{|a|} \tag{3.3}$$

where $a = (p-w)/(1-p\bar{w})$.

Moreover, given $w \in \Delta$, $w \neq p$, there exists a function $f \in \Sigma(p)$ for which equality is obtained on the right side of (3.3) *and similarly for the left side of* (3.3).

Proof. Let $f \in \Sigma(p)$ and $w \in \Delta$, $w \neq p$, and let

$$g(z) = \frac{1}{f(w)} f\left(\frac{e^{i\theta}z + w}{1 + \bar{w}e^{i\theta}z}\right)$$

where $\theta = \arg(p-w)/(1-p\bar{w})$. Obviously g is univalent in Δ with $g(0) = 1$ and letting $a = (p-w)/(1-p\bar{w})$ we see that g has a simple pole at $z = |a|$. Thus $g \in \Sigma(|a|)$. Therefore by Theorem 4 we have

$$\frac{(1-|a|)^2}{|a|} \leqq |g'(0)| \leqq \frac{(1+|a|)^2}{|a|}.$$

A straightforward computation now gives (3.3).

Suppose we are given $w \in \Delta$, $w \neq p$. Let $a = (p-w)/(1-p\bar{w})$ and $\theta = \arg a$. For $z \in \Delta$, let

$$f(z) = \frac{\left(1 + \frac{we^{-i\theta}}{|a|}\right)(1 + |a|we^{-i\theta})(1 + A(z))^2}{(1 - we^{-i\theta})^2\left(1 - \frac{A(z)}{|a|}\right)(1 - |a|A(z))}$$

where

$$A(z) = \frac{z - w}{e^{i\theta}(1 - \bar{w}z)} .$$

The function $f(z)$ is univalent in Δ, different from 0 and $f(0) = 1$. Moreover, f has a pole at that value of z for which $A(z) = |a|$. That is, when $z = p$. Thus $f \in \Sigma(p)$ and a straightforward computation gives equality on the right side of (3.3).

To obtain sharpness on the left side of (3.3) for a given $w \neq p$, we set

$$f(z) = \frac{\left(1 + \frac{we^{-i\theta}}{|a|}\right)(1 + |a|we^{-i\theta})(1 - A(z))^2}{(1 + we^{-i\theta})^2\left(1 - \frac{A(z)}{|a|}\right)(1 - |a|A(z))}$$

where a, θ, and $A(z)$ have the same meaning as before. Again it is easily seen that $f \in \Sigma(p)$ and that equality is obtained on the left side of (3.3).

4. The class $\Sigma(p, q)$. In this section we extend the previous results to the class $\Sigma(p, q)$ where the functions now take on the value 0. Here the function playing the role of $F_p(z)$ is the function

$$\underset{(p,q)}{G}(z) = \frac{\left(1 + \frac{z}{q}\right)(1 + qz)}{\left(1 - \frac{z}{p}\right)(1 - pz)} .$$

It is easily seen that $G_{(p,q)} \in \Sigma(p, q)$ and maps Δ onto the extended plane slit along the interval

$$[-p(1 + q)^2/q(1 - p)^2,\ -p(1 - q)^2/q(1 + p)^2] .$$

THEOREM 6. *Let Φ be a convex nondecreasing function on $(-\infty, \infty)$. Then for all $f \in \Sigma(p, q)$ and $0 < r < 1$,*

$$\int_{-\pi}^{\pi} \Phi(\pm\log|f(re^{i\theta})|)d\theta \leqq \int_{-\pi}^{\pi} \Phi(\pm\log|G_{(p,q)}(re^{i\theta})|)d\theta .$$

Proof. We first consider the inequality

$$\int_{-\pi}^{\pi} \Phi(\log|f(re^{i\theta})|)d\theta \leqq \int_{-\pi}^{\pi} \Phi(\log|G_{(p,q)}(re^{i\theta})|)d\theta . \tag{4.1}$$

Let

$$u(w) = \begin{cases} -\log|f^{-1}(w)| , & w \in f(\Delta) \\ 0 , & \text{otherwise} \end{cases}$$

and

$$v(w) = \begin{cases} -\log |G^{-1}_{(p,q)}(w)| , & w \in G_{(p,q)}(\Delta) \\ 0 , & \text{otherwise} . \end{cases}$$

Arguing as in Theorem 1, inequality (4.1) will be proven if we can prove that

$$(u^* - v^*)(s) \leqq 0 , \qquad s \in R . \tag{4.2}$$

For this purpose we let

$$d_f = \inf_{w \notin f(\Delta)} |w| \qquad \text{and} \qquad D_f = \sup_{w \notin f(\Delta)} |w|$$

and

$$R = (-\infty, -D_f) \cup [-D_f, -d_f] \cup (-d_f, 0) \cup [0, +\infty) .$$

Case (i). $s \in [0, +\infty)$. This case is exactly as in Theorem 1.

Case (ii). $s \in (-\infty, -D_f)$. The argument is the same as the corresponding case in Theorem 1.

Case (iii). $s \in (-d_f, 0)$. Since $\{w: |w| < d_f\} \subset f(\Delta)$, we have that $u(w) + \log|w - 1|$ is harmonic in $|w| < d_f$ and $v(w) + \log|w - 1|$ is subharmonic in $|w| < d_f$. (The term $\log|w - 1|$ is only necessary when $1 < d$.) It follows that $(u - v)$ is superharmonic for $|w| < d_f$ and therefore

$$(u^* - v^*)(s) = \int_{-\pi}^{\pi} (u - v)(|s|e^{i\theta})d\theta \leqq 2\pi(u - v)(0) = 0 .$$

Case (iv). $s[-D_f - d_f]$. The argument in this case is the same as the argument given in case (iii) of the proof of Theorem 1.

This then proves (4.2) and hence (4.1).

The inequality

$$\int_{-\pi}^{\pi} \Phi(-\log |f(re^{i\theta})|)d\theta \leqq \int_{-\pi}^{\pi} \Phi(-\log |G_{(p,q)}(re^{i\theta})|)d\theta$$

is obtained as in Theorem 1 except that (52) of [1] is now replaced by

$$\int_{-\pi}^{\pi} \log^+ (\rho |f(re^{i\theta})|)d\theta = 2\pi\left[\log \rho + \log^+ \frac{r}{q} - \log^+ \frac{r}{p}\right] + \int_{-\pi}^{\pi} \log^+ \frac{1}{\rho |f(re^{i\theta})|} d\theta .$$

We have the following as an immediate consequence of Theorem 6.

THEOREM 7. *Let* $f \in \Sigma(p, q)$, $0 < r < 1$, $-\infty < \lambda < \infty$, *then*

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda} d\theta \leqq \int_{-\pi}^{\pi} |G_{(p,q)}(re^{i\theta})|^{\lambda} d\theta .$$

5. **Applications of Theorem 7.** Arguing as in Theorem 3 we obtain the following.

THEOREM 8. *Let* $f \in \Sigma(p, q)$, *then for* $|z| = r$.

$$|G_{(p,q)}(-r)| \leqq |f(z)| \leqq |G_{(p,q)}(r)| .$$

THEOREM 9. *Let* $f \in \Sigma(p, q)$ *and* $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$, $|z| < p$, *then*

$$\frac{|p - q|(1 - pq)}{pq} \leqq |a_1| \leqq \frac{(p + q)(1 + pq)}{pq} . \tag{5.1}$$

Both inequalities are sharp.

Proof. Let $f \in \Sigma(p, q)$ with $f(z_0) = 0$ where $|z_0| = q$. Let $g(z) = (f(z) - 1)/a_1$, then $g \in S(p)$. We therefore have [3]

$$\frac{|z_0|}{\left(1 + \frac{|z_0|}{p}\right)(1 + p|z_0|)} \leqq |g(z_0)| \leqq \frac{|z_0|}{\left|1 - \frac{|z_0|}{p}\right|(1 - p|z_0|)} .$$

Since $g(z_0) = -1/a_1$ and $|z_0| = q$, we immediately obtain (5.1).

Equality on the right side of (5.1) is attained by the function $G_{(p,q)}(z)$ and on the left side by the function

$$f(z) = (1 - z/q)(1 - qz)/(1 - z/p)(1 - pz) .$$

REMARK. The right side of (5.1) could also be obtained by considering the case $\lambda = 2$ of Theorem 7 and letting r approach 0.

REMARK. We may obtain estimates on $|a_n|$, $n \geqq 2$, by either using the case $\lambda = 1$ of Theorem 7 or by using Theorem 9 and the fact that $f(z) = a_1 g(z) + 1$ where $g \in S(p)$ and then using the estimate on the coefficients of a function in $S(p)$ [3].

As an application of Theorem 9 we obtain the following analogue of Theorem 5.

THEOREM 10. *Let* $f \in \Sigma(p, q)$ *with* $f(z_0) = 0$, $|z_0| = q$, *then for*

$w \in \Delta$, $w \neq z_0$, $w \neq p$,

$$\text{(5.2)} \qquad \frac{1}{1-|w|^2}\left[\frac{||a|-|b||(1-|a||b|)}{|a||b|}\right] \leqq \left|\frac{f'(w)}{f(w)}\right| \leqq \frac{1}{1-|w|^2}\left[\frac{(|a|+|b|)(1+|a||b|)}{|a||b|}\right]$$

where

$$|a| = \left|\frac{p-w}{1-p\bar{w}}\right| \quad \textit{and} \quad |b| = \left|\frac{z_0-w}{1-\bar{w}z_0}\right| .$$

The left hand side of (5.2) *is sharp and the right side is sharp at least for* $|w| < q$.

Proof. Let $f \in \Sigma(p, q)$ with $f(z_0) = 0$, $|z_0| = q$. For $w \in \Delta$, $w \neq p$, $w \neq z_0$, let $a = (p-w)/(1-p\bar{w})$ and $\theta = \arg a$. Let

$$h(z) = \frac{1}{f(w)} f\left[\frac{e^{i\theta}z + w}{1 + \bar{w}e^{i\theta}z}\right] .$$

The function h is univalent and meromorphic in Δ with $h(0) = 1$. Moreover h has a pole at $z = |a|$ and $h(z) = 0$ when

$$z = (z_0 - w)/e^{i\theta}(1 - \bar{w}z_0) = b .$$

Thus $h \in \Sigma(|a|, |b|)$. By Theorem 9 we then have

$$\frac{||a|-|b||(1-|a||b|)}{|a||b|} \leqq |h'(0)| \leqq \frac{(|a|+|b|)(1+|a||b|)}{|a||b|}$$

which gives (5.2).

With p and q fixed let $w \neq p$ be such that $|w| < q$. Let $a = (p-w)/(1-p\bar{w})$ and $\theta = \arg a$. Choose z_0 with $|z_0| = q$ such that $(z_0 - w)/e^{i\theta}(1 - \bar{w}z_0) < 0$. Such a choice is possible since $|w| < q$. With this choice of z_0 let $b = (z_0 - w)/e^{i\theta}(1 - \bar{w}z_0)$ and define

$$f(z) = \frac{\left(1 + \frac{we^{-i\theta}}{|a|}\right)(1 + |a|w\,e^{-i\theta})\left(1 + \frac{A(z)}{|b|}\right)(1 + |b|\,A(z))}{\left(1 - \frac{we^{-i\theta}}{|b|}\right)(1 - |b|\,we^{-i\theta})\left(1 - \frac{A(z)}{|a|}\right)(1 - |a|\,A(z))}$$

where

$$A(z) = \frac{z - w}{e^{i\theta}(1 - \bar{w}z)} .$$

The function f is univalent and meromorphic in Δ with $f(0) = 1$.

Moreover, f has a pole when $A(z) = |a|$, that is when $z = p$. f has a zero when $A(z) = -|b|$. By the choice of z_0,

$$A(z_0) = (z_0 - w_0)/e^{i\theta}(1 - \bar{w}z_0) = -|b| .$$

Thus $f(z_0) = 0$ and $f \in \Sigma(p, q)$. A straightforward calculation gives equality on the right side of (5.2).

For equality on the left side of (5.2), let $|w| < q$, $w \neq p$ and a and θ be as before. Choose z_0 so that $(z_0 - w)/e^{i\theta}(1 - \bar{w}z_0) > 0$ and set $b = (z_0 - w)/e^{i\theta}(1 - \bar{w}z_0)$. With this choice of z_0, let

$$f(z) = \frac{\left(1 + \dfrac{we^{-i\theta}}{|a|}\right)(1 + |a|\,we^{-i\theta})\left(1 - \dfrac{A(z)}{|b|}\right)(1 - |b|\,A(z))}{\left(1 + \dfrac{we^{-i\theta}}{|b|}\right)(1 + |b|\,we^{-i\theta})\left(1 - \dfrac{A(z)}{|a|}\right)(1 - |a|\,A(z))} .$$

It is easily seen that $f \in \Sigma(p, q)$ and that we get equality on the left side of (5.2).

Suppose $q < r < p$. Let $a = (p - r)/(1 - pr)$ and $b = (q + r)/(1 + qr)$ and let

$$f(z) = \frac{\left(1 + \dfrac{r}{a}\right)(1 + ar)\left(1 + \dfrac{A(z)}{b}\right)(1 + bA(z))}{\left(1 - \dfrac{r}{b}\right)(1 - br)}$$

where

$$A(z) = \frac{z - r}{1 - rz} .$$

The function f has a pole at $z = p$ and a zero at $z = -q$. Thus $f \in \Sigma(p, q)$ and a straightforward computation gives equality on the right side of (5.2) when $w = r$.

Let p and q be fixed and $r > 0$. Let $a = (p + r)/(1 + pr)$ and $b = (q + r)/(1 + qr)$ and

$$f(z) = \frac{\left(1 - \dfrac{r}{a}\right)(1 - ar)\left(1 - \dfrac{A(z)}{b}\right)(1 - bA(z))}{\left(1 - \dfrac{r}{b}\right)(1 - br)\left(1 - \dfrac{A(z)}{a}\right)(1 - aA(z))}$$

where

$$A(z) = \frac{z + r}{1 + rz} .$$

The function f has a pole at $z = p$ and a zero at $z = q$. Thus $f \in \Sigma(p, q)$ and we get equality on the left side of (5.2) when $w = -r$.

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