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W_δ (T) IS CONVEX

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Stampfli introduced a generalization of the numerical range for any bounded linear operator T on a Hilbert space $\mathscr{H}$. This is denoted by $W_\delta(T)$ and is defined by

$$W_\delta(T) = \textbf{closure } \{\langle Tx, x\rangle : \|x\| = 1 \textbf{ and } \|Tx\| \geqq \delta\} .$$

Stampfli asked whether $W_\delta(T)$ is convex. In this short note we provide an affirmative answer to this question.

$\mathscr{L}(\mathscr{H})$ will denote the set of bounded linear operators on the Hilbert space $\mathscr{H}$.

LEMMA 1. *Suppose S and A belong to $\mathscr{L}(\mathscr{H})$, and that $S = S^*$. Then*

$$S(A, \delta) = \{x \in \mathscr{H}: \|x\| = 1 \quad \text{and} \quad \|Ax\| \geqq \delta \quad \text{and} \quad \langle Sx, x\rangle = 0\}$$

is path connected.

Proof. Suppose x and y belong to $S(A, \delta)$. We may assume that x and y are linearly independent. (If not, they both lie on an arc of

$$\{e^{i\theta}x: 0 \leqq \theta \leqq 2\pi\}$$

which lies in $S(A, \delta)$ if x does.)

Choose θ in $\boldsymbol{R}$ such that $e^{i\theta}\langle Sx, y\rangle$ is purely imaginary and let $a = e^{i\theta}x$.

Choose n such that $(-1)^n \boldsymbol{Re} \langle (A^*A - \delta^2 I)a, y\rangle$ is positive and let $b = (-1)^n y$. Then a and b may be joined by a path in $S(A, \delta)$ to x and y respectively. Thus we need only find a path connecting a to b. Let $y(t) = ta + (1 - t)b$ and let $x(t) = \|y(t)\|^{-1}y(t)$. Then $\langle Sx(t), x(t)\rangle = 0 \Leftrightarrow \langle Sy(t), y(t)\rangle = 0$ and

$$\begin{aligned}\langle Sy(t), y(t)\rangle &= t^2\langle Sa, a\rangle + (1 - t)^2\langle Sb, b\rangle \\ &\quad + 2\boldsymbol{Re}\, t(1 - t)\langle Sa, b\rangle \\ &= 2(-1)^n t(1 - t)\boldsymbol{Re}\, e^{i\theta}\langle Sx, y\rangle \\ &= 0 .\end{aligned}$$

Also

$$\begin{aligned}\|Ay(t)\|^2 &= \langle A^*Ay(t), y(t)\rangle \\ &= t^2 \|Aa\|^2 + (1 - t)^2 \|Ab\|^2 \\ &\quad + 2t(1 - t)\boldsymbol{Re}\langle A^*Aa, b\rangle\end{aligned}$$

$$\geqq \delta^2(t^2 + (1-t)^2 + 2\boldsymbol{Re}\, t(1-t)\langle a, b\rangle) + 2t(1-t)\boldsymbol{Re}\langle (A^*A - \delta^2 I)a, b\rangle$$
$$= \delta^2 ||y(t)||^2 + 2t(1-t)(-1)^n \boldsymbol{Re}\langle (A^*A - \delta^2 I)a, y\rangle$$
$$\geqq \delta^2 ||y(t)||^2 .$$

Hence $||Ax(t)|| \geqq \delta$ and so $t \rightarrow x(t)$ is a path connecting a to b in $S(A, \delta)$ as required.

LEMMA 2. *Suppose H and K are self-adjoint elements in $\mathscr{L}(\mathscr{H})$. Let*

$$V(A, \delta) = \{(\langle Hx, x\rangle, \langle Kx, x\rangle): ||x|| = 1 \quad \text{and} \quad ||Ax|| \geqq \delta\} .$$

Then $V(A, \delta)$ is a convex subset of $\boldsymbol{R}^2$.

Proof. We need only show that $V(A, \delta) \cap L$ is connected whenever L is a straight line in $\boldsymbol{R}^2$. Suppose L is given by

$$\alpha\xi + \beta\eta + \gamma = 0 .$$

Let

$$S = \alpha H + \beta K + \gamma I .$$

Then the mapping π, given by

$$\pi(x) = (\langle Hx, x\rangle, \langle Kx, x\rangle) \text{ is continuous, and}$$
$$S(A, \delta) = \{x: ||x|| = 1; ||Ax|| \geqq \delta \text{ and } \pi(x) \in L\} .$$

Thus $V(A, \delta) \cap L = \pi(S(A, \delta))$ is connected.

THEOREM 3. *Suppose T and A are in $\mathscr{L}(\mathscr{H})$. Then*

$$V(T; A, \delta) = \{\langle Tx, x\rangle: ||x|| = 1 \text{ and } ||Ax|| \geqq \delta\}$$

is convex.

Proof. Suppose $T = H + iK$ with H and K both self-adjoint. Then

$$V(T; A, \delta) = \{\xi + i\eta: (\xi, \eta) \in V(A, \delta)\} .$$

Hence $V(T; A, \delta)$ is convex.

COROLLARY 4. *$W_\delta(T)$ is convex.*

Proof. Take $A = T$. Indeed we have shown that

$$\{\langle Tx, x\rangle : ||x|| = 1 \text{ and } ||Tx|| \geqq \delta\}$$

is convex.

REMARK. It will be noticed that the ideas here are improvements on basic ideas in **1**.

REFERENCES

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