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ON THE RETRACTABILITY OF SOME ONE-RELATOR GROUPS

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Recently the concept of a retractable group has been introduced. This class of groups contains the class of lattice-ordered groups as a proper subclass, and, in particular, contains the class of all torsion-free abelian groups. Retractable groups enjoy many of the properties of lattice-ordered groups; in fact, most results concerning lattice-ordered groups have immediate extensions to this wider class. In this note, we investigate the retractability of certain two-generator one-relator groups.

1. Introduction. There has been an abundance of literature on the class of groups presented by a single defining relation and, in particular, on the groups given by the presentation

$$\langle a, c \mid a^{-1}c^{m}a = c^{n}\rangle ,$$

where m and n are integers. In [1] the concept of a retractable group was introduced and in this note we attempt to determine which of this latter class of groups are retractable.

In Theorem 3.3 we show that the groups $\langle a, c \mid a^{-1}ca = c^{m}\rangle$, where m is a positive integer, are retractable and each admits at least a countably infinite number of retractions that satisfy condition (δ). (Definitions will be given in §§2 and 3.) It was shown in [5] that the group $\langle a, c \mid a^{-1}ca = c^{2}\rangle$ admits exactly four full orders. Each of these induces a retraction on this group. We show in Theorem 3.5 that each of the groups $\langle a, c \mid a^{-1}ca = c^{m}\rangle$, where $m > 1$, admits exactly four lattice-orders and each of these is a full order. In Theorem 3.6 we show that the groups $\langle a, c \mid a^{-1}ca = c^{m}\rangle$, where $m < 0$, admit retractions if and only if 2 is a factor of m, and in this case, none of these groups admit lattice-orders. In Theorem 3.1 we show that if G is a retractable group and $g^{n} = h^{n}$, for some $g, h \in G$ and some natural number n, then g and h are conjugate. As a corollary to this theorem, we are able to show that the groups $\langle a, c \mid a^{n} = c^{n}\rangle$, where n is a natural number and $n > 1$, and

$$\langle a, c \mid a^{-1}c^{m}a = c^{n}\rangle ,$$

where m and n are distinct integers and $gcd(m, n) > 1$, are not retractable.

2. Preliminaries. In this section we shall recall some definitions and establish some notation that we shall use for the remainder of this note. For a group G, let $F(G)$ denote the collection of all finite, nonempty subsets of G. Then $F(G)$ is a join monoid, where the join operation is set union and the binary operation is multiplication of complexes. A homomorphism σ of the monoid $F(G)$ into G such that $\{g\}\sigma = g$ for every g in G will be called a *retraction* of G. We denote by Ret G the set of all retractions of G. If Ret G is nonempty, then G is said to be a *retractable* group. It follows immediately from the definition of a retraction that if $\sigma \in \operatorname{Ret} G$, $A \in F(G)$, and $x, y \in G$, then $x(A\sigma)y = (xAy)\sigma$. If $\sigma \in \operatorname{Ret} G$ and H is a subgroup of G such that $\sigma | F(H) \in \operatorname{Ret} H$, then H is said to be a *σ-subgroup*.

Let G be a lattice-ordered group and define σ from $F(G)$ into G by $A\sigma = \bigvee A$ for every $A \in F(G)$. Then $\sigma \in \operatorname{Ret} G$ [1, Theorem 2.1] and σ is called the *retraction of G induced by the lattice-ordering of G*. Thus, the class of lattice-ordered groups is a subclass of the class of retractable groups and, as indicated in the introduction, it is a proper subclass. If $\sigma \in \operatorname{Ret} G$ and $\{A | A \in F(G) \text{ and } A\sigma = \iota\}$, where ι denotes the identity of G, is a convex subsemilattice of $F(G)$, then there is a lattice-ordering of G such that $A\sigma = \bigvee A$ for every $A \in F(G)$ [1, Theorem 3.2]. The class of retractable groups is a proper subclass of the class of torsion free groups [1, Theorem 2.2 and Example 2.7].

If $\sigma \in \operatorname{Ret} G$ and H is a subgroup of G, then H is said to be a *ρ-σ-subgroup* (resp., *λ-σ-subgroup*) if $A = \{a_1, \cdots, a_n\} \in F(G)$ and $h_1, \cdots, h_n \in H$ implies that $H(A\sigma) = H(\{h_1a_1, \cdots, h_na_n\}\sigma)$ (resp., $(A\sigma)H = (\{a_1h_1, \cdots, a_nh_n\}\sigma)H$). In [1] and [2] a ρ-σ-subgroup was called a c-σ-subgroup. A subgroup which is both a ρ-σ-subgroup and a λ-σ-subgroup will be called a *solid σ-subgroup*. Clearly a normal ρ-σ-subgroup is a solid σ-subgroup. It was shown in [1, Theorem 4.2] that the collection of all ρ-σ-subgroups of G is a complete sublattice of the lattice of all subgroups of G. If H is a ρ-σ-subgroup, then H satisfies the purity condition in the sense that $g^n \in H$, for some $g \in G$ and some natural number n, implies $g \in H$ [1, Corollary 4.10]. If G is a lattice-ordered group, σ is the retraction of G induced by the lattice-ordering of G, and H is a subgroup of G, then H is a ρ-σ-subgroup if and only if H is a convex l-subgroup of G [1, Theorem 4.7].

We shall denote the natural numbers, integers, and rational numbers by N, Z, and Q respectively. If X and Y are sets, $X\backslash Y$ will denote the set of elements in X but not in Y.

If $m \in Z\backslash\{0\}$, let $Q_m = \{nm^{-k} | n, k \in Z\}$ and $G_m = Q_m \times Z$. For $(a, x), (b, y) \in G_m$, define $(a, x) + (b, y) = (m^y a + b, x + y)$. Then G_m is a group and is a splitting extension of Q_m by Z. If $H_m = Q_m \times \{0\}$, then H_m is a normal subgroup of G_m. If $G = \langle a, c \mid a^{-1}ca = c^m \rangle$, then each g in G can be expressed as $g = a^r c^s a^t$, where $r, s, t \in Z$ and $t < 0$. It is easily established that the mapping that sends $a^r c^s a^t$ onto $(0, r) + (s, 0) + (0, t) = (m^t s, r + t)$ is an isomorphism of G onto G_m. In [2], Ret Q_m was classified and for this reason, we shall identity G with G_m in §3.

3. **One relator groups.** We begin this section by showing that the class of retractable groups is a subclass of the class of power conjugate groups.

THEOREM 3.1. *If G is a retractable group and $g, h \in G$ such that $g^n = h^n$ for some $n \in N$, then g and h are conjugate.*

Proof. Let $\sigma \in \operatorname{Ret} G$ and $\{g^{n-1}, g^{n-2}h, \cdots, gh^{n-2}, h^{n-1}\}\sigma = a$. Then

$$\begin{aligned} ga &= g(\{g^{n-1}, g^{n-2}h, \cdots, gh^{n-2}, h^{n-1}\}\sigma) = \{g^n, g^{n-1}h, \cdots, g^2h^{n-2}, gh^{n-1}\}\sigma \\ &= \{h^n, g^{n-1}h, \cdots, g^2h^{n-2}, gh^{n-1}\}\sigma = (\{h^{n-1}, g^{n-1}, \cdots, g^2h^{n-3}, gh^{n-2}\}\sigma)h \\ &= ah\ . \end{aligned}$$

Therefore, $g = aha^{-1}$.

COROLLARY 3.2. (i) *Let $m, n \in Z$ with $m \neq n$ and $gcd(m, n) = d > 1$. If $G = \langle a, c \mid a^{-1}c^m a = c^n \rangle$, then G is not retractable.*

(ii) *If $n \in N$ with $n > 1$ and $G = \langle a, c \mid a^n = c^n \rangle$, then G is not retractable.*

Proof. (i) Let F be the free group on the generators a and c, and suppose that $n > m$. Let $t = n - m$, Z_t denote the group of integers modulo t, and ϕ be the homomorphism of F into Z_t given by $a\phi = 0$ and $c\phi = 1$. Then $(a^{-1}c^m a)\phi = m$ and $(c^n)\phi = n$ and since $m - n = 0$ in Z_t, $a^{-1}c^m a c^{-n} \in \operatorname{Ker} \phi$. It follows that Z_t is a homomorphic image of G. Let $r, s \in Z$ so that $rd = m$ and $sd = n$. Then $(a^{-1}c^r a)^d = (c^s)^d$, $gcd(r, s) = 1$, and since $m \neq n$, $r \neq s$. Consequently, r and s are distinct elements in Z_t and hence, can not be conjugate in Z_t. Therefore, $a^{-1}c^r a$ and c^s are not conjugate in G. By the theorem, G is not retractable.

(ii) It is readily verified that a and c are not conjugate in G and hence, G is not retractable.

A retraction σ of G is said to satisfy (δ) provided that $\{g_1, \cdots, g_m\} \in F(G)$ and $n \in N$ implies $\{g_1^n, \cdots, g_m^n\}\sigma = (\{g_1, \cdots, g_m\}\sigma)^n$. If G is abelian and $\sigma \in \operatorname{Ret} G$, then σ satisfies (δ) [2, Corollary 3.4]. If G is a lattice-ordered group and σ is the retraction induced by the lattice-ordering of G, then σ satisfies (δ) if and only if the lattice-ordering is representable (see [4, Theorem 1.8]).

Let H be a subgroup of Q and $H^{\perp} = \{r \mid r \in Q$ and $rh \in H$ for every $h \in H\}$. It was shown in [2, Corollary 3.12] that if H is a nonzero subgroup of Q, then $\operatorname{Ret} H = \{\sigma_r \mid r \in H^{\perp}\}$, where

$$A\sigma_r = (r+1)\max A - r\min A$$

for all $A \in F(H)$. Since $Z \subseteq H^{\perp}$, $\operatorname{Ret} H$ is countably infinite and hence so is $\operatorname{Ret} H_m$, $m \in N$. A corollary of the next result is that $\operatorname{Ret} G_m$ is at least countably infinite.

THEOREM 3.3. *If $\tau \in \operatorname{Ret} H_m$, where $m \in N$, then τ has an extension to a retraction σ of G_m that satisfies (δ).*

Proof. If $m = 1$, then G_m is abelian, H_m is a direct summand of G_m, and the verification of the theorem is routine. Thus, we assume that $m > 1$. Let $\tau \in \operatorname{Ret} H_m$ and $A = \{(a_1, 0), \cdots, (a_t, 0)\} \in F(H_m)$, where $a_1 < \cdots < a_t$. Then, by the above remarks, $A\tau = ((r+1)a_t - ra_1, 0)$ for some $r \in Q_m^{\perp}$. Next we show that if $g \in G_m$, $-g + A\tau + g = (-g + A + g)\tau$ and to prove this, it suffices to take $g = (0, x)$, where $x \in Z$. Now

$$-(0, x) + A\tau + (0, x) = (m^x((r+1)a_t - ra_1), 0) .$$

Since $m^x > 0$, $m^x a_1 < \cdots < m^x a_t$ and hence, $(-(0, x) + A + (0, x))\tau = ((r+1)m^x a_t - rm^x a_1, 0)$. Since G_m/H_m can be fully ordered, we have that τ has an extension to a retraction σ of G_m and that H_m is a ρ-σ-subgroup of G_m [2, Theorem 3.18]. We explicitly describe the construction of σ below, so that it may be proven that σ satisfies (δ).

Let $A = \{(a_1, x_1), \cdots, (a_t, x_t)\} \in F(G_m)$, where $x_1 \leqq \cdots \leqq x_{p-1} < x_p = \cdots = x_t$ and $a_p < \cdots < a_t$, $B = \{(a_p, x_p), \cdots, (a_t, x_t)\} - (a_t, x_t)$, $s \in N$, and $C = \{s(a_1, x_1), \cdots, s(a_t, x_t)\}$. It was shown in the proof of [2, Theorem 3.18] that $A\sigma = B\tau + (a_t, x_t) = ((r+1)a_t - ra_p, x_t)$. Consequently, $s(A\sigma) = (\sum_{j=1}^{s} m^{(j-1)x_t}((r+1)a_t - ra_p), sx_t)$. On the other hand, $C\sigma = ((r+1)(\sum_{j=1}^{s} m^{(j-1)x_t}a_t) - r(\sum_{j=1}^{s} m^{(j-1)x_t}a_p), sx_t)$ and it follows that σ satisfies (δ).

If $\sigma \in \operatorname{Ret} G$, H is a normal ρ-σ-subgroup of G, $X = \{Hg_1, \cdots, Hg_t\} \in F(G/H)$, and $X\sigma^* = H(\{g_1, \cdots, g_t\}\sigma)$, then $\sigma^* \in \operatorname{Ret} G/H$ [1, Theorem 4.3]. The following example shows that if $\sigma \in \operatorname{Ret} G$, H is a normal ρ-σ-subgroup of G, $\sigma \mid F(H)$ and σ^* satisfy (δ), then σ need not satisfy (δ).

EXAMPLE 3.4. Let $K = Z \times Z \times Z$ and define

$$(a, b, c) + (x, y, z) = \begin{cases} (a+x, b+y, c+z) & \text{if } z \text{ is even} , \\ (b+x, a+y, c+z) & \text{if } z \text{ is odd} . \end{cases}$$

Then K is a group and a splitting extension of $Z \times Z$ by Z. Define $(a, b, c) \geqq (0, 0, 0)$ if $c > 0$, or if $c = 0$, $a \geqq 0$, and $b \geqq 0$. Then K is a nonrepresentable lattice-ordered group and hence, if σ is the retraction induced by the lattice-ordering of K, then σ does not satisfy (δ). If $J = Z \times Z \times \{0\}$, then J is an abelian normal ρ-σ-subgroup of K and K/J is abelian. Thus, $\sigma | F(J)$ and σ^* satisfy (δ).

We note in passing that it is not difficult to give an example of a group G, distinct retractions σ and τ of G, and a normal subgroup H of G such that H is both a ρ-σ-subgroup and a ρ-τ-subgroup, $\sigma | F(H) = \tau | F(H)$, and $\sigma^* = \tau^*$.

In [5, Theorem 2] Fuchs and Sasiada showed that G_2 is an 0^*-group with exactly four different full orders and that H_2 is a convex subgroup in each of these orders. If $m \in N$ with $m > 1$, then H_m is an abelian normal subgroup of G_m such that G_m/H_m is abelian and such that for each $a \in H_m$ and $b \in G_m \backslash H_m$, there exist distinct positive integers s and t so that $-b + sa + b = ta$. Kargapolov [6, p. 17] proved that a group that satisfied these conditions had the property that each full order for any subgroup can be extended to a full order of the group. A group satisfying this property is an 0^*-group (see [7] or [9]). Although Theorem 3.3 (and the discussion preceding it) showed that G_m has at least a countably infinite number of retractions, our next theorem shows that G_m has exactly four lattice-orders and each of these is a full ordering.

THEOREM 3.5. *Let $m \in Z$ and $\sigma \in \operatorname{Ret} G_m$.*

(i) *If $m \neq 1$ and H is a proper solid σ-subgroup of G_m, then $H = H_m$.*

(ii) *If $m > 1$, then each lattice-ordering of G_m is a full ordering and G_m has exactly four full orders.*

Proof. (i) First we show that G_m can have at most one proper solid σ-subgroup. Let H and J be distinct nonzero solid σ-subgroups of G_m.

Case 1. J is properly contained in H. If $(a, 0) \in J$ for some $a \neq 0$, then by the purity of J, $(1, 0) \in J$. Since J is properly contained in H and H is pure, it follows that $(0, 1) \in H$. Therefore, $G_m = H$. Consequently, we may assume that for each $(a, b) \in J \backslash \{(0, 0)\}$, $b \neq 0$. If $(c, 0) \in H$ for some $c \neq 0$, then by the preceding argument, we again have $G_m = H$. Thus, we may further assume that for each $(c, d) \in H \backslash J$, $d \neq 0$. Let $(a, b) \in J \backslash \{(0, 0)\}$ and $(c, d) \in H \backslash J$. By the purity of J, $b(c, d) \neq d(a, b)$. But then $(0, 0) \neq (h, 0) = b(c, d) - d(a, b) \in H$, a contradiction.

Case 2. J is not properly contained in H. If $H \cap J \neq \{(0, 0)\}$, then $H \cap J$ is a nonzero solid σ-subgroup properly contained in J. By Case 1, $J = G_m$. Suppose (by way of contradiction) that $H \cap J = \{(0, 0)\}$. Then the subgroup K generated by H and J is their direct sum [3, Corollary 4.3]. Since G_m is subdirectly irreducible, K is a proper solid σ-subgroup of G_m and K properly contains J. But by Case 1, $K = G_m$, a contradiction.

Therefore, G_m can have at most one proper solid σ-subgroup and necessarily, this subgroup must be normal in G_m. The only proper subgroup with these properties is H_m. We note for $m \neq 0, 1$ that H_m is the centralizer of $(1, 0)$ in G_m. Hence, by [1, Theorem 2.14], H_m is always a σ-subgroup of G_m.

(ii) Let σ be the retraction induced by a lattice-ordering of G_m. Then each convex l-subgroup is a solid σ-subgroup. By (i) the convex l-subgroups that contain $\{(0, 0)\}$ form a chain. Consequently, G_m is a fully ordered group [4, Theorem 1.7].

As in [5], G_m has at least the four full orders corresponding to the cases $(0, 0) < (1, 0) < (0, 1)$, $(0, 0) < (-1, 0) < (0, 1)$, $(0, 0) < (1, 0) < (0, -1)$, and $(0, 0) < (-1, 0) < (0, -1)$. Since $m > 1$, G_m is nonabelian. Hence, by Hölder's theorem (see [4, p. 0.24]), G_m must have a proper convex subgroup in any full ordering. By (i), this subgroup must be H_m. It follows that G_m can have no more than the four full orders given above.

We shall show below that if m is negative, then G_m admits no lattice-orders. It was noted in [5] that G_1 has continuously many full orders. Since G_1 is the direct product of two copies of Z, it also admits retractions σ for which there are two proper ρ-σ-subgroups.

It was shown in [1, Theorem 5.1] that if ϕ is an automorphism or an anti-automorphism of G and $\sigma \in \operatorname{Ret} G$, then $\phi\sigma\phi^{-1} \in \operatorname{Ret} G$. If ϕ is the anti-automorphism of G given by $g\phi = g^{-1}$, then the retraction $\sigma' = \phi\sigma\phi^{-1}$ is called the *dual* of σ. If G is a lattice-ordered group and σ is the retraction induced by the lattice-ordering of G, then σ' induces a lattice-ordering of G which is the dual of the given lattice-ordering [1, Corollary 5.2]. If $\sigma = \sigma'$, then we say that σ is *self dual*. If H is a two-divisible subgroup of Q, then, as noted earlier, $\sigma_{-1/2} \in \operatorname{Ret} H$ and it is easily verified that $\sigma_{-1/2}$ is self dual. In fact, $\sigma_{-1/2}$ is the only self dual retraction of H. (Any two-divisible torsion free abelian group admits a self dual retraction.)

THEOREM 3.6. *Let m be a negative integer. Then G_m is retractable if and only if 2 divides m. Moreover,*

(i) *if $\sigma \in \operatorname{Ret} G_m$, then H_m is a σ-subgroup of G_m and $\sigma | F(H_m)$ is self dual;*

(ii) *G_m is not lattice-orderable;*

(iii) *if 2 divides m and τ is the self dual retraction of H_m, then τ has an extension to a retraction σ of G_m that satisfies (δ) and H_m is a proper ρ-σ-subgroup of G_m.*

Proof. Suppose that G_m is retractable and let $\sigma \in \operatorname{Ret} G_m$. We observed in the proof of (i) of Theorem 3.5 that H_m is a σ-subgroup of G. Hence, $\{(0, 0), (1, 0)\}\sigma = (nm^{-k}, 0)$ for some $n, k \in Z$ with $k \geqq 0$. Thus,

$$\begin{aligned}(nm^{-k+1}, 0) &= -(0, 1) + (nm^{-k}, 0) + (0, 1) \\ &= \{(0, 0), -(0, 1) + (1, 0) + (0, 1)\}\sigma \\ &= \{(0, 0), (m, 0)\}\sigma .\end{aligned}$$

By [1, Theorem 2.4],

$$\{(0, 0), (m, 0)\}\sigma = (-m)(nm^{-k}, 0) + (m, 0) = (m - nm^{-k+1}, 0) .$$

Therefore, $2n = m^k$ and it follows that 2 divides m. Also, we have $(1/2, 0) = \{(0, 0), (1, 0)\}\sigma$. It was shown in [1, Example 5.7] that the image of $\{(0, 0), (1, 0)\}$ under σ completely determines $\sigma | F(H_m)$. Consequently, $\sigma | F(H_m)$ is self dual and we have proven (i).

If G_m is lattice-orderable and σ is the retraction of G_m induced by the lattice-ordering, then $(1/2, 0) = \{(0, 0), (1, 0)\}\sigma$ must be positive. Since $m < 0$, $(-m/2, 0)$ would also be positive. But $(m/2, 0) = -(0, 1) + (1/2, 0) + (0, 1)$ and hence, the positive cone of G contains a nonzero element and its inverse, which is impossible. Therefore, we have proven (ii).

Next suppose that 2 divides m. For $A = \{(a_1, 0), \cdots, (a_t, 0)\} \in F(H_m)$, define $A\tau = (\{a_1, \cdots, a_t\}\sigma_{-1/2}, 0)$. Then τ is a self dual retraction of H_m. To show that for every $g \in G_m$,

$$-g + (A\tau) + g = (-g + A + g)\tau ,$$

it suffices to take $g = (0, x)$, where $x \in Z$. Now $(0, -x) + A\tau + (0, x) = (m^x(\{a_1, \cdots, a_t\}\sigma_{-1/2}), 0) = (\{m^x a_1, \cdots, m^x a_t\}\sigma_{-1/2}, 0) = ((0, -x) + A + (0, x))\tau$. Since G_m/H_m can be linearly ordered, τ has an extension σ to G_m and H_m is a ρ-σ-subgroup of G_m [2, Theorem 3.18]. The proof that the extension demonstrated in [2] satisfies (δ) is similar to the proof of Theorem 3.3 and will be omitted.

If $m < -1$, then a straightforward computation shows that G_m is an R-group, that is, $ng = nh$ for $n \in N$ implies $g = h$. Since every R-group is a power conjugate group (as defined in Theorem 3.1), the class of retractable groups is a proper subclass of the class of power conjugate groups.

THEOREM 3.8. *If F is a free group and $\sigma \in \operatorname{Ret} F$, then every cyclic subgroup of F is a σ-subgroup.*

Proof. By [1, Corollary 2.5], it suffices to show that $\{\iota, g\}\sigma = g^n$ for some n and every $g \in F$. First we assume that if $g = h^m$, then $m = \pm 1$. Then the centralizer of g in F is the subgroup generated by g [8, p. 42]. By [1, Theorem 2.4] $\{\iota, g\}\sigma$ belongs to the centralizer of g. Hence, the subgroup of F generated by g is a σ-subgroup. Since every element of F is a power of such a g, it follows from the description of $\operatorname{Ret} Z$ given prior to Theorem 3.3 that every cyclic subgroup is a σ-subgroup.

EXAMPLE 3.9. If $G = \langle a, c \mid a^{-1}c^n a = c^n \rangle$, where n is a positive even integer, then G is not retractable.

Suppose (by way of contradiction) that $\sigma \in \operatorname{Ret} G$ and let $x = \{a, c^{-m}ac^m\}\sigma$, where $m = n/2$. Then $c^{-m}xc^m = c^{-m}(\{a, c^{-m}ac^m\}\sigma)c^m = \{c^{-m}ac^m, a\}\sigma = x$. Thus, x commutes with c^m and it follows that $x = c^{-t}$ for some integer t. Then, $\iota = \{ac^t, c^m ac^{m+t}\}\sigma$ and by [1, Theorem 2.2], $(ac^t)(c^m ac^{m+t}) = (c^m ac^{m+t})(ac^t)$. From this relation, we obtain $(ac^{m+t}a)c^m = c^m(ac^{m+t}a)$ and hence, $ac^{m+t}a = c^s$, for some integer s. In the symmetric group of degree $n + 3$, the cycles $c_1 = (1, \cdots, n)$ and $a_1 = (n + 1, n + 2, n + 3)$ satisfy the relation $a_1^{-1}c_1^n a_1 = c_1^n$ and so the subgroup generated by a_1 and c_1 is a homomorphic image of G. But then $a_1 c_1^{m+t} a_1 = c_1^s$, which is impossible.

We note that Theorems 3.3 and 3.6 have generalizations to splitting extensions of certain torsion free abelian groups by the integers. We close with the following three questions.

(1) Is the group $\langle a, c \mid a^{-1}c^n a = c^n \rangle$, where n is an odd positive integer retractable ($n > 1$)?

(2) Is the group $\langle a, c \mid a^{-1}c^m a = c^n \rangle$, where $m, n \in Z \backslash \{-1, 0, 1\}$ and $gcd(m, n) = 1$, retractable?

(3) If σ is a retraction of G that satisfies (δ), is G an R-group?

REFERENCES

1. R. D. Byrd, J. T. Lloyd, R. A. Mena and J. R. Teller, *Retractable groups*, Acta Math. Acad. Sci. Hungar. (to appear).

2. ———, *Extensions of group retractions*, submitted.

3. ———, *The lattice of solid σ-subgroups of a retractable group*, submitted.

4. P. Conrad, *Lattice-ordered groups*, Lecture notes, Tulane University, 1970.

5. L. Fuchs and E. Sasiada, *Note on orderable groups*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., **7** (1964), 13–17.

6. M. I. Kargapolov, *Completely ordered groups* (Russian), Algebra; Logika, **1** (1962), 16-21.
7. A. I. Kokorin, *On the theory of completely orderable groups* (Russian), Ural. Gos. Univ. Mat. Zap., **4** (1963), 25-29.
8. W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, John Wiley & Sons, Inc., 1966.
9. D. P. Minassian, *Types of fully ordered groups*, Amer. Math. Monthly, **80** (1973), 159-169.

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