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SQUARE INTEGRABLE REPRESENTATIONS AND THE FOURIER ALGEBRA OF A UNIMODULAR GROUP

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Let G be a unimodular group, and let λ_d be the subrepresentation of the left regular representation λ, which is the sum of the square integrable representations. The purpose of this paper is to study the representation λ_d with special emphasis on the closed subspace $A_d(G)$ of the Fourier algebra $A(G)$ of the group which is generated by the coefficients of λ_d. In the last part of the paper we study in detail a particular noncompact group for which $\lambda = \lambda_d$.

We denote, as in [4], by $A(G)$ the algebra of the coefficients of λ, that is the algebra of continuous functions on G of the type $(\lambda(x)f, g)$, with $f, g \in L^2(G)$. The first section contains results of a general nature: we show that $A_d(G)$ is the dual space of a C^*-algebra contained in $VN(G)$, the von Neumann algebra generated by the operators $\lambda(x)$, $x \in G$, and that its unit ball is the weak closure of the extreme points of the unit ball of $A(G)$. We also show that $A_d(G) \subset L^2(G)$ if and only if the formal degrees of the square integrable representations of G are bounded away from zero.

In the last section we make a closer study of an example due to J. Fell of a noncompact group G for which $A_d(G) = A(G)$. We show that the traces of the square integrable representations of this group are bounded measures and we construct a kind of Dirichlet kernels, which also turn out to be bounded measures.

We prove that summation with respect to these kernels converges in $L^p(G)$ for $1 < p < \infty$, but not for $L^1(G)$.

We conclude the paper with some remarks on the Wiener-Pitt phenomenon for bounded measures on this group.

1. We refer the reader to [4] for the definitions and the properties of the Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra $B(G)$ of a locally compact group G, and to [3] for the basic facts about C^*-algebras, von Neumann algebras and square integrable representations of unimodular groups. Throughout the paper "group" will always mean "locally compact unimodular group" and "representation" will mean "unitary continuous representation."

Following Arsac [1], given a representation π of G on a Hilbert space H_π, we denote by A_π the closed subspace of $B(G)$ spanned by the coefficients of π, i.e., the functions $(\pi(x)\mu | \nu)$, $x \in G$, $\mu, \nu \in H_\pi$. Let λ_d be the subrepresentation of the left regular representation of G

which is the sum of all the irreducible square integrable representations of G. Then $A_d(G) = A_{\lambda_d}$ is a closed subspace of $A(G)$. From the results of [1] it follows easily that there exists a closed subspace $A_c(G) \subset A(G)$ such that $A(G) = A_d(G) \oplus A_c(G)$. Moreover $A_d(G)$ itself is the direct sum $\bigoplus_{\pi \in \hat{G}_d} A_\pi$, where $\hat{G}_d$ denotes the family of all equivalence classes of irreducible square integrable representations of G.

Now, given a representation π of G on the Hilbert H_π, we denote by $\bar{\pi}$ its conjugate representation on $\bar{H}_\pi$ the Hilbert space conjugate to H_π. We remember that $A(G)$ is endowed with a structure of left $VN(G)$-module [3, Prop. 3.17]. For $\pi \in \hat{G}_d$ let P_π and K_π denote respectively the minimal central projection and the minimal biinvariant subspace corresponding to π. Then the following facts are a more or less immediate consequence of [3, Ch. 14]. For every $\pi \in \hat{G}_d$ $A_\pi = P_{\bar{\pi}} A(G)$ is contained in K_π. Moreover the mapping $u \longrightarrow d_\pi \pi(u)$, where d_π is the formal degree of π and $\pi(u) = \int_G \pi(x)u(x)dx$, is an isometric isomorphism of $A_{\bar{\pi}}$ onto the Banach space $TC(H_\pi)$ of all trace class operators on H_π. Any function $u \in A_d(G)$ is the sum of its Fourier series:

$$u(x) = \sum_{\pi \in \hat{G}_d} d_\pi \operatorname{tr} (\pi(x^{-1})\pi(u))$$

where the series converges absolutely as well as in $A(G)$.

If G is a compact abelian group it is well known that its dual group $\hat{G}$ is a discrete measure space. Therefore $A(G)$, being isometric to $l^1(\hat{G})$, can be identified with the dual of the Banach space $c_0(\hat{G})$ of all bounded complex functions on $\hat{G}$, vanishing at infinity. The following lemma shows that for G nonabelian a similar result holds for $A_d(G)$.

LEMMA 1.1. *Let $c_0(\hat{G}_d)$ be the direct sum, in the C^*-algebra theoretical sense, of the algebras $C^*_\pi = \pi(C^*(G))$ for all $\pi \in \hat{G}_d$. Then $A_d(G)$ can be isometrically identified with the dual space of $c_0(\hat{G}_d)$ via the following pairing:*

$$\langle T, u \rangle = \sum_{\pi \in \hat{G}_d} d_\pi \operatorname{tr} (\pi(T)\pi(u))$$

for $T \in c_0(\hat{G}_d)$, $u \in A_d(G)$.

Proof. For every $\pi \in \hat{G}_d$, C^*_π is isometric to the C^*-algebra $LC(H_\pi)$ of all compact operators on H_π [3, 4.1.11, and 18.4.1]. Since $A_{\bar{\pi}}$ is isometric to $TC(H_\pi)$, which is the dual of $LC(H_\pi)$, the lemma easily follows.

Now, to find the extreme points of the unit ball of $A(G)$, we need the following lemma.

LEMMA 1.2. *Let T be an operator in the unit ball of the space $TC(H)$ of trace class operators on a Hilbert space H. Then these are equivalent*:

(i) *T is an extreme point.*

(ii) *$|T|$ is a projection of rank one.*

(iii) *There exist $\mu, \nu \in H$, $|\mu| = |\nu| = 1$ such that $T\phi = (\phi|\nu)\mu$ for every $\phi \in H$.*

Proof. Let us denote by $TC_1(H)$ the unit ball of the space $TC(H)$. It is obvious that (ii) and (iii) are equivalent. We prove only the equivalence between (i) and (ii). Let T be an extreme point in $TC_1(H)$ and suppose that there exist R, S in $TC_1(H)$ such that $|T| = 1/2(R + S)$; then $1 = 1/2(\mathrm{tr}\,(R) + \mathrm{tr}\,(S))$.

It follows that $\mathrm{tr}\,(R) = \mathrm{tr}\,(S) = 1$, so R and S are positive operators.

Since $|T| = 1/2(R + S)$ and since T is an extreme point we get: $T = UR = US$. Then $|T| = U^*UR = R$ and $T = U^*US = S$ because U^*U is greater than or equal to the supports of R and S. So $|T|$ is an extreme point of $TC_1(H)$. Since $TC(H)$ is the dual space of the C^*-algebra $LC(H)$ [3, 4.1.2], the extreme points in its positive unit ball are just zero and the pure states, i.e., the positive operators P in $TC_1(H)$ such that $0 \leqq P' \leqq P$ implies $P' = \lambda P$, $0 \leqq \lambda \leqq 1$ [3, 2.5.5]. This proves that $|T|$ must be a projection of rank one. So (i) implies (ii).

To show that (ii) implies (i) we shall prove that if P is a projection of rank one and U is a partial isometry such that $U^*U = P$, then UP is an extreme point in $TC_1(H)$. Since the final projection of U is one-dimensional there is an isometry W which coincides with U on its support. Therefore $UP = WP$. Now let R, S be in $CT_1(H)$ such that $WP = 1/2(R + S)$; then $P = 1/2(W^*R + W^*S)$ and W^*R, $W^*S \in TC_1(H)$. Since P is one-dimensional, P defines a pure state on $TC(H)$. Therefore P is an extreme point in $TC_1(H)$ and $P = W^*R = W^*S$.

So $UP = WP = R = S$ and UP is extreme in $TC_1(H)$.

THEOREM 1.1. *Let u be in the unit ball of $A(G)$. Then u is an extreme point if and only if there exist $\pi \in \hat{G}_d$ and vectors $\mu, \nu \in H_\pi$, $|\mu| = |\nu| = 1$, such that $u(x) = (\pi(x)\mu|\nu)$. Moreover $A_d(G)$ is the closed subspace of $A(G)$ spanned by the extreme points of the unit ball of $A(G)$.*

Proof. Let u be an extreme point in the unit ball of $A(G)$. Let $P \in VN(G)$ be the central support of u, considered as an ultraweakly continuous form on $VN(G)$. We claim that P is a minimal central projection in $VN(G)$. Suppose to the contrary that there exists a central projection Q in $VN(G)$ such that $0 < Q < P$. Let $\alpha = \|Qu\|_A$ and $\beta = \|(P-Q)u\|_A$. Then $\alpha + \beta = 1$ and $\alpha > 0, \beta > 0$ because P is the minimal central projection such that $Pu = u$. Hence $u = \alpha u_1 + \beta u_2$, where $u_1 = Qu/\|Qu\|_A$ and $u_2 = (P-Q)u/\|(P-Q)u\|_A$ are in the unit ball of $A(G)$. But this contradicts the extremality of u. Hence there exists $\pi \in \hat{G}_d$ such that u is an extreme point in the unit ball of A_π. Therefore $d_\pi\bar{\pi}(u)$ is an extreme point in the unit ball of $TC(H_\pi)$. Then, by Lemma 1.2, there exist $\mu, \nu \in \bar{H}_\pi$, $|\mu| = |\nu| = 1$, such that $d_\pi\bar{\pi}(u)\phi = (\phi|\mu)\nu$ for every $\phi \in \bar{H}_\pi$. We may assume that $\bar{\pi}$ is a subrepresentation of the left regular representation λ of G. Then, denoting by $[\cdot|\cdot]$ the inner product in H_π and by $(\cdot|\cdot)$ the inner product in $\bar{H}_\pi$, and remembering that $(\phi|\psi) = [\psi|\phi]$, we have by [3, 14.3.3]:

$$\int_G u(x)(\bar{\pi}(x)\phi|\psi)dx = (\bar{\pi}(u)\phi|\psi) = d_\pi^{-1}(\phi|\mu)\overline{(\psi|\nu)}$$
$$\times \int_G (\pi(x)\phi|\psi)\overline{(\bar{\pi}(x)\mu|\nu)}dx = \int_G (\bar{\pi}(x)\phi|\psi)[\pi(x)\mu|\nu]dx$$

for every $\phi, \psi \in \bar{H}_\pi$. Since, by [3, 14.3.1], the functions $(\pi(x)\phi|\psi)$, $\phi, \psi \in \bar{H}_\pi$ are dense in $K_{\bar{\pi}}$, the identity $u(x) = [\pi(x)\mu|\nu]$ follows at once. The last assertion of the theorem follows by Lemma 1.1 and the Krein Milman theorem.

As we have seen in the introductory remarks, $A_\pi \subset L^2(G)$ for every $\pi \in \hat{G}_d$. It is natural to ask whether $A_d(G)$ is contained in $L^2(G)$ or not. Since we have that $\|u\|_A = \sum_\pi d_\pi \operatorname{tr}(|\pi(u)|)$ for every $u \in A_d(G)$ and $|f|_2^2 = \sum_\pi d_\pi \operatorname{tr}(\pi(f)^*\pi(f))$ for every $f \in \bigoplus K_\pi, \pi \in \hat{G}_d$, a comparison of the two formulas, together with the closed graph theorem, yields at once that $A_d(G)$ is contained in $L^2(G)$ if and only if the formal degrees of the square integrable representations of G are bounded away from zero.

It is well known that if G is a locally compact abelian group then $A(G) = A_d(G)$ if and only if G is compact. If G is not abelian the situation is more complicated, because there exist noncompact groups such that $A(G) = A_d(G)$. In the next section we shall study in detail an example of such groups. A more detailed discussion of the structure and properties of unimodular groups, whose regular representation is the direct sum of irreducible subrepresentations, will appear in a forthcoming paper of M. Picardello and the author [7]. Here we bound ourselves to the following few remarks.

REMARK 1. Let G be a noncompact unimodular group such that

$A(G) = A_d(G)$. Then $\inf\{d_\pi : \pi \in \hat{G}_d\} = 0$. This is an easy consequence of the remark following Theorem 1.1 and the fact $A(G)$ cannot be contained in $L^2(G)$, because G is noncompact [9].

REMARK 2. Let G be as before and let K be a compact normal subgroup of G. Then G/K is again a noncompact unimodular group whose regular representation is the direct sum of its irreducible components. Indeed G/K is clearly unimodular and noncompact and $A(G/K)$ is isometric to the biinvariant closed selfadjoint subalgebra of $A(G)$ of the functions which are constant on K-cosets [4]. It follows easily that $A(G/K) = A_d(G/K)$.

We conclude this section with a result which is related to the contents of the last remark but not to the main theme of the paper. If G is *any* locally compact group and K is a compact normal subgroup, the functions of $A(G)$ which are constant on the cosets of K form a closed biinvariant selfadjoint subalgebra of $A(G)$. The fact that viceversa every closed biinvariant selfadjoint subalgebra of $A(G)$ is of this type is a special case of a result of M. Takesaki and N. Tatsuuma [Duality and subgroups, Annals of Math., v. 93 (1971) 344-364, Theorem 9]. It can also be deduced from the following theorem which is a slight improvement of a result of [1]. We believe that our proof, shorter than that of [1] can also shed light on the result of Takesaki and Tatsuuma.

Let $\mathfrak{A}$ be a nonzero right invariant closed selfadjoint subalgebra of $A(G)$. Then by [11] there exists a projection $P \in VN(G)$ such that $\mathfrak{A} = PA(G)$. Let $H_P = P(L^2(G))$ be the corresponding subspace of $L^2(G)$.

THEOREM 1.2. *The space H_P is closed under multiplication by functions of $\mathfrak{A}$. If $\mathfrak{A}$ separates the points of G then $\mathfrak{A} = A(G)$.*

Proof. First we claim that $\mathfrak{A} \cap H_P$ is dense in H_P. Indeed let f be any function in H_P and let ϕ_α be an approximated identity for the convolution of continuous functions with compact support in G. Then $\phi_\alpha * f$ is in $A(G) \cap L^2(G)$ and $\lim \phi_\alpha * f = f$ in $L^2(G)$. Since $P(\phi_\alpha * f) \in \mathfrak{A} \cap H_P$, the claim is proved. Now let $u \in \mathfrak{A}$, $f \in H_P$ and let $\{f_\alpha\}$ be a net in $\mathfrak{A} \cap H_P$ converging to f in $L^2(G)$. Then $uf_\alpha \in \mathfrak{A} \cap H_P$, because u is a bounded function and $\mathfrak{A}$ is an algebra. Since $\lim uf_\alpha = uf$, we have $uf \in H_P$.

To prove the last assertion of the theorem, observe that if $\mathfrak{A}$ separates the points of G, by the Stone-Weierstrass theorem, $\mathfrak{A}$ is uniformly dense in the space $C_0(G)$ of continuous functions on G vanishing at infinity. Therefore H_P is also closed under the multiplication by functions in $C_0(G)$. Let g be any nonzero continuous function in H_P. Such function actually exists, because $\mathfrak{A} \cap H_P$ is dense in H_P. Let $\mathscr{U}$ be any open set on which g is bounded away from zero. Now let f be any continuous function with compact support in $\mathscr{U}$ and denote by h the function so defined: $h(x) = f(x)/g(x)$ for $x \in \mathscr{U}$, $h(x) = 0$ for $x \notin \mathscr{U}$. Then $h \in C_0(G)$ and $f = hg$ is in H_P. Hence H_P contains the space $C_c(\mathscr{U})$ of the continuous functions with compact support in $\mathscr{U}$. Applying the translation invariance of H_P and a simple partition of unity argument, it is easy to see that $C_c(G) \subset H_P$. Therefore $H_P = L^2(G)$. Hence $P = I$ and $\mathfrak{A} = A(G)$.

2. Fell's example. In [2] L. Baggett describes the following example, due to Fell of group G such that $A(G) = A_d(G)$.

Let p be a prime number, N the p-adic numbers field, K the subset of p-adic numbers k whose valuation $|k|_p$ is one. K is a compact abelian group w.r.t. multiplication. For $n \in N$, $k \in K$ set $k(n) = kn$. Then K acts as group of automorphisms of the additive group N. The orbits of N under the action of K are $\{0\}$ and $N_j = \{n: n \in N, |n|_p = p^{-j}\}$, $j \in \boldsymbol{Z}$. Let $G = K \circ N$ be the semidirect product of K and N. Then G is a regular semidirect product because $\hat{N} = N = (\bigcup_{j \in Z} N_j) \cup \{0\}$. Therefore using the representation theory of group extensions [6], we can describe the irreducible representations of G.

One verifies that $\hat{G}$ is the union of two sets $\hat{G}_1 = \{\pi_j: j \in \boldsymbol{Z}\}$, $\hat{G}_2 = \{\pi_\theta: \theta \in \hat{K}\}$. The representations in $\hat{G}_1$ can be realized on the Hilbert space $L^2(K)$, while the representations in $\hat{G}_2$ are one-dimensional.

If $\pi_j \in \hat{G}_1$ and $f \in L^2(K)$, then:

$$[\pi_j(l, m)f](k) = \exp(2\pi i p^j k m) f(kl)$$

for $(l, m) \in G$, $k \in K$. Here the exponential of a p-adic number

$$n = p^j \sum_{i \geq 0} n_i p^i, \ 0 \leqq n_i < p ,$$

is defined as follows:

$$\exp(2\pi i n) = \begin{cases} \exp(2\pi i p^j \sum_{i+j<0} n_i p^i) & \text{for} \quad j < 0 \\ 1 & \text{for} \quad j \geqq 0 . \end{cases}$$

If $\pi_\theta \in \hat{G}_2$, $(l, m) \in G$, then $\pi_\theta(l, m) = \theta(l)$.

Figà-Talamanca in [5] proved that when $A(G) \neq A_d(G)$ there exist positive definite continuous functions which vanish at infinity but are not in $A(G)$. He also asked whether or not unimodularity alone is sufficient to prove the existence of such functions for G noncompact.

The following corollary answers in the negative to this question. Recall first that the Fourier-Stieltjes algebra $B(G)$ is the algebra, under pointwise operations of all linear combinations of continuous positive definite functions on G [4]. $B(G)$ is also the Banach involution algebra of the coefficients $u(x) = (\pi(x)\xi \mid \eta)$, $\xi, \eta \in H_\pi$ of all unitary continuous representations of G, normed thus:

$$\|u\|_B = \min\{|\xi|\,|\eta| : u(x) = (\pi(x)\xi \mid \eta)\} .$$

COROLLARY 2.1. *$B(G)$ is the direct sum $A(G) \oplus AP(G)$, where $AP(G)$ is the Banach involution algebra of the almost periodic functions on G. In particular a function $u \in B(G)$ vanishes at infinity if and only if $u \in A(G)$.*

Proof. Since $\hat{G}$ is countable an arbitrary unitary representation π of G is the direct sum (rather than the direct integral) $\pi = (\bigoplus_{j \in Z} n_j \pi_j) \oplus (\bigoplus_{\theta \in \hat{K}} n_\theta \pi_\theta)$. Therefore if u is any coefficient of π, u decomposes into the sum $u = u_1 + u_2$, $u_1 \in A(G)$ and $u_2 = \sum_{\theta \in \hat{K}} n_\theta \theta$, where the series converges in $B(G)$, and hence uniformly. Thus u_2 is an almost periodic function [3, 16.2.1. (v)]. If u vanishes at infinity u_2, $\bar{u}_2$ and hence $|u_2|^2$ vanish at infinity. Since $AP(G)$ is a Banach involution algebra with respect to pointwise operations and complex conjugation, $|u_2|^2$ is an almost periodic function whose mean is zero. Hence $u_2 = 0$ [3, 16.3].

We shall now evaluate the "diagonal" coefficients of the representation $\pi_j \in \hat{G}_d$, with respect to the orthonormal basis $\hat{K}$ in $L^2(K)$. This will enable us to compute the formal degrees of the square integrable representations of G and to study the convergence of the Fourier series for functions in $L^p(G)$, $1 \leqq p < +\infty$.

LEMMA 2.2. *For $\pi_j \in \hat{G}_d$ denote by $\phi_{\theta\theta}^{(j)}(l, m) = (\pi_j(l, m)\theta \mid \theta)$ the coefficient of π_j corresponding to $\theta \in \hat{K}$. Then $\phi_{\theta\theta}^{(j)}(l, m) = \phi^{(j)}(m)\theta(l)$, where:*

$$\phi^{(j)}(m) = \begin{cases} 1 & \text{for } |m|_p \leqq p^j \\ \dfrac{1}{1-p} & \text{for } |m|_p = p^{j+1} \\ 0 & \text{for } |m|_p > p^{j+1} . \end{cases}$$

The formal degree of π_j is $d_j = p^{-j}[(p-1)/p]^2$.

Proof. We have for $(l, m) \in G$:

$$\phi_{\theta,\theta}^{(j)}(l, m) = (\pi_j(l, m)\theta \mid \theta) = \int_K \exp(2\pi i p^j k m)\theta(kl)\theta(k)d_k(k)$$

where $d_k(k)$ denotes the Haar measure on K which coincides with the Haar measure $d_N(k)$ on N, since $d_N(m, k) = |m|_p d_N(k)$ for every $m \in N$.

Then:

$$\phi^{(j)}_{\theta,\theta}(l, m) = \theta(l)\int_K \exp(2\pi i p^j km) d_N(k) \qquad (l, m) \in G .$$

By a change of variable, setting $p^j km = t$ and $|m|_p = p^\mu$, the integral

$$I = \int_K \exp(2\, i\pi p^j km) d_N(k) \text{ becomes:}$$

$$I = p^{j-\mu}\int_{N_{j-\mu}} \exp(2\pi i t) d_N(t) .$$

So we need to compute, for every relative integer s, the integral:

$$I_s = \int_{N_s} \exp(2\pi i t) d_N(t) .$$

Let t be in N_s; then $t = p^s \sum_{n=0}^{+\infty} t_n p^n$ where $0 \leqq t_n < p$ for $n \in N$, and $t_0 \neq 0$. Then:

$$\exp(2\pi i t) = \begin{cases} \exp(2\pi i p^s \sum_{0 \leqq i < -s} t_n p^n) & \text{for} \quad s < 0 \\ 1 & \text{for} \quad s \geqq 0 . \end{cases}$$

Therefore for $s \geqq 0$, I_s is just the measure of $N_s = p^s K$, i.e., $I_s = p^{-s}$. For $s = -1$ we have:

$$I_{-1} = \sum_{j=1}^{p-1}\int_{N_{-1,j}} \exp(2\pi i j p^{-1}) d_N(t) = \sum_{j=1}^{p-1} \exp(2\pi i j p^{-1}) \int_{N_{-1,j}} d_N(t)$$

where $N_{-1,j} = \{t : t \in N_{-1}, t_0 = j\}$. Since N_{-1} is the disjoint union of the $N_{-1,j}$ for $j = 1, \cdots, p-1$ and $N_{-1,j} = N_{-1,i} + j - i$ for $i, j = 1, \cdots, p-1$:

$$\int_{N_{-1,j}} d_N(t) = \frac{1}{p-1}\int_{N_{-1}} d_N(t) = \frac{p}{p-1}$$

for every $j = 1, \cdots, p-1$.
So

$$I_{-1} = \frac{p}{p-1}\sum_{j=1}^{p-1} \exp(2\pi i j p^{-1}) = \frac{-p}{p-1} .$$

Now for $s \leqq -2$, let J be the set of multiindices $j = (j_0, j_1, \cdots, j_{1-s})$ s.t. $0 \leqq j_l < p$ for $1 = 0, \cdots, 1-s$ and $j_0 \neq 0$. Setting $\langle j, p\rangle = \sum_{l=0}^{1-s} j_l p^l$ we have:

$$I_s = \sum_{j \in J} \exp(2\pi i p^s \langle j, p\rangle) \int_{N_{s,j}} d_N(t)$$

where $N_{s,j} = \{t : t \in N_s, t_0 = j_0, t_1 = j_1, \cdots, t_{1-s} = j_{1-s}\}$. Since N_s is the disjoint union of the $N_{s,j}$, $j \in J$ and $N_{s,j} = N_{s,j'} + \langle j, p\rangle - \langle j', p\rangle$, j, $j' \in J$, then

$$\int_{N_{s,j}} d_N(t) = \frac{p^{-s}}{p^{1-s}(p-1)}$$

for every $j \in J$. Therefore:

$$I_s = \frac{1}{p(p-1)} \sum_{j \in J} \exp(2\pi i p^s \langle j, p\rangle) = 0$$

because

$$\sum_{j \in J} \exp(2\pi i p^s \langle j, p\rangle) = \sum_{k=0}^{p^{-s}-1} \exp(2\pi i p^s k) - \sum_{k=0}^{p^{1-s}-1} \exp(2\pi i\, p^{s-1} k)$$

and

$$\sum_{k=0}^{\alpha-1} \exp(2\pi i \alpha^{-1} k) = 0$$

for every positive integer α.

We have thus that:

$$\phi_{\theta\theta}^{(j)}(l, m) = \theta(1) p^{j-\mu} I_{j-\mu} = \begin{cases} \theta(l) & \text{for } |m|_p \leqq p^j \\ \dfrac{1}{1-p}\theta(l) & \text{for } |m|_p = p^{j+1} \\ 0 & \text{for } |m|_p > p^{j+1} . \end{cases}$$

To compute the formal degree d_j of π_j it is sufficient to observe that, since $\phi^{(j)}$ is a positive definition function, whose value in the identity is one $d_j^{-1} = |\phi^{(j)}|^2 = p^j[p/p-1]^2$ [3, 14.4.3].

Let γ_j be the positive definite central measure on G, defined by $\gamma_j(f * g^*) = \mathrm{tr}\,(\pi_j(f)\pi_j(g)^*)$, for $f, g \in C_c(G)$ and $\pi_j \in \hat{G}_1$. If δ_1 denotes the Dirac measure at 1 on K then $\gamma_j = \phi^{(j)} \otimes \delta_1$ (here we have identified $\phi^{(j)}$ with the measure $\phi^{(j)}(x) d_N(x)$). Indeed it is easy to verify by means of Lemma 2.2 that for every $\psi \in C_c(N)$ and $\theta \in \hat{K}$

$$(\phi^{(j)} \otimes \delta_1)(\psi \otimes \theta) = \gamma_j(\psi \otimes \theta) .$$

The measure γ_j is called the "character measure" of the representation π_j [3, 17.2.4].

DEFINITION 1. We define the *Dirichlet Kernel* $\{D_n\}$ by:

$$D_n = \sum_{j=-n}^{n} d_j \gamma_j$$

for every positive integer n.

An easy but lengthy computation shows that, for every n, D_n

is a measure whose total variation $\|D_n\|_M$ is bounded by the constant 2. Therefore the convolution operator $\lambda(D_n)f = D_n * f$ is a bounded operator on $L^p(G)$ for $1 \leqq p \leqq +\infty$. Since $\gamma_i * \gamma_j = d_j^{-1}\delta_{ij}\gamma_j$, for $i, j \in \boldsymbol{Z}$, $\lambda(D_n)$ is a projection, which for $p = 2$ coincides with the sum $\sum_{j=-n}^{n} P_{\pi_j}$ of the minimal central projections associated with the square integrable representations π_j, $j = -n, \cdots, n$.

Therefore for $f \in L^2(G)$:

$$f = \sum_{j \in \boldsymbol{Z}} d_j(\gamma_j * f)$$

where the series converges in $L^2(G)$.

DEFINITION 2. For every function $f \in L^p(G)$, $1 \leqq p \leqq \infty$ we call $\sum_{j \in Z} d_j(\gamma_j * f)$ the *formal Fourier series* of f. We say that a function $f \in L^p(G)$ is a *trigonometric polynomial* in $L^p(G)$ if

$$D_n * f = \sum_{j=-n}^{n} d_j(\chi_j * f) = f$$

for some n. The following theorem shows that the Dirichlet kernel is a *summability kernel* for $L^p(G)$ $1 < p < \infty$.

THEOREM 2.3. *If $f \in L^p(G)$, $1 < p < \infty$, then:*

$$f = \lim_{n \to +\infty} D_n * f = \lim_{n \to +\infty} \sum_{j=-n}^{n} d_j(\gamma_j * f)$$

in the $L^p(G)$ norm.

Proof. Assume that trigonometric polynomials are dense in $L^p(G)$ for $1 < p < \infty$. Let $f \in L^p(G)$ $\varepsilon > 0$ and let P be a trigonometric polynomial in $L^p(G)$, satisfying $|f - P|_p < \varepsilon/4$. For n large enough we have $D_n * P = P$ and hence:

$$|D_n f - f|_p \leqq |D_n * (f - P)|_p + |P - f|_p < \varepsilon .$$

Therefore it remains only to show that trigonometric polynomials are actually dense in $L^p(G)$, $1 < p < \infty$. Let g be a continuous function with compact support in G. Then $g \in L^2(G)$ and $\lim_{n \to +\infty} D_n * g = g$ in the $L^2(G)$ norm. On the other hand, since $|D_n * g|_p \leqq 2|g|_p$ and $1 < p < \infty$, there exists a subsequence $\{D_{n_k} * g\}$ which converges in the weak topology of $L^p(G)$ to some limit h. By the Banach-Saks theorem, there is a sequence of convex combinations of the $D_{n_k} * g$, which converges to h in the norm of $L^p(G)$. Taking a subsequence which converges almost everwhere, we have $g = h$ a.e. Thus trigonometric polynomials are dense in $L^p(G)$, $1 < p < \infty$.

REMARK. The Dirichlet kernel $\{D_n\}$ is not a summability kernel for $L^1(G)$. In fact let $f \in L^1(G)$ be a function such that

$$\pi_{\theta_0}(f) = \int_N \int_K f(1, m)\theta_0(1) d_N(1) d_N(m) \neq 0$$

for some $\theta_0 \in \hat{K}$. For every function $h \in L^1(G)$ and for every $\theta \in \hat{K}$, we have $\lim_{j \to -\infty} (\pi_j(h)\theta | \theta) = \pi_\theta(h)$, by Lemma 2.2 and the dominated converge theorem. Since $\pi_j(D_n * h) = 0$ for $|j| > n$, then $\pi_\theta(D_n * h) = 0$ for every positive integer n.

Now, for every $n \in \boldsymbol{N}$:

$$|D_n * f - f|_1 \geqq ||\lambda(D_n * f - f)|| = \sup_{j \in Z} ||\pi_j(D_n * f - f)|| \geqq |\pi_{\theta_0}(f)|$$

since $||\pi_j(D_n * f - f)|| \geqq |(\pi_j(D_n * f - f)\theta_0 | \theta_0)|$, $j \in \boldsymbol{Z}$ and

$$\lim_{j \to -\infty} |(\pi_j(D_n * f - f)\theta_0 | \theta_0)| = |\pi_{\theta_0}(f)| \ .$$

This proves that $D_n * f$ cannot converge to f in $L^1(G)$, if $\pi_{\theta_0}(f) \neq 0$.

We conclude now our study of the group G with some final remarks on the algebra $M(G)$ of all complex measures of bounded variation on G.

REMARK. There exists a measure $\mu \in M(G)$ such that the operator $\lambda(\mu)f = \mu * f$ for $f \in L^2(G)$ has inverse and yet μ^{-1} does not exist, as an element of the algebra $M(G)$. This phenomenon was first discovered in $M(\boldsymbol{R})$ by Wiener and Pitt [8], who showed that there exists a measure $\mu \in M(\boldsymbol{R})$ such that 0 is in the spectrum of μ but the Fourier-Stieltjes transform $\hat{\mu}$ of μ is bounded away from zero. Since K is a compact abelian group by [10, Th. 6.4.1], there exists a measure $\nu \in M(K)$ such that $0 \in \mathrm{sp}(\nu)$ and $\hat{\nu}(\theta) \geqq 1$ for every $\theta \in \hat{K}$.

Then it is straightforward to check that if $\mu = \delta_e - d_c\phi^{(0)} \otimes (\delta_1 - \nu)$, where δ_e is the Dirac measure at the identity $e = (1, 0)$ of G, $0 \in \mathrm{sp}(\mu)$. Moreover, for $j \neq 0$, $\pi_j(\mu)$ is the identity on $L^2(K)$ and $\pi_0(\mu)$ is the operator whose matrix representation with respect to the basis $\hat{K}$ in $L^2(K)$ is given by the diagonal matrix whose eigenvalues are the $\hat{\nu}(\theta)$, $\theta \in \hat{K}$. Therefore $\lambda(\mu)^{-1}$ exists.

REMARK. The same construction as above, together with [10, Th. 6.4.1] can be used to show that the spectral radius of a measure μ in $M(G)$ is much larger than the spectral radius of the operator $\lambda(\mu)$. Actually, given any complex number z_0 there is a measure $\mu \in M(G)$ such that $z_0 \in \mathrm{sp}(\mu)$ and $||\lambda(\mu)|| \leqq 1$. (Take $\nu \in M(K)$ such that $z \in \mathrm{sp}(\nu)$ and $|\hat{\nu}(\theta)| \leqq 1$ for every $\theta \in \hat{K}$, and set $\mu = d_0\phi^{(0)} \otimes \nu$.)

After this paper was completed we learned that Corollary 2.1 was also proved independently by M. E. Walter [12].

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