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INVOLUTIONS FIXING CODIMENSION TWO KNOTS

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1. Involution. An m-knot (Σ^{m+2}, M^m) consists of an $(m+2)$-homotopy sphere Σ^{m+2} and an m-homotopy sphere M^m differentiably (or piecewise linearly) embedded in it. A $(2n-1)$-knot is callsd simple if $\pi_j(\Sigma - M) = \pi_j(S^1)$ for $j < n$. It is well known that each knot cobordism class contains a simple knot, [5] or [7].

Associated to each $(2n-1)$-knot, we have Seifert matrices B, with $B + \varepsilon B'$ unimodular, where $\varepsilon = (-1)^n$ and B' denotes the transpose of B. For $n \geqq 2$, the isotopy class of simple knot is completely determined by its Seifert matrices [8].

In [1, §11], Cappell and Shaneson used their algebraic K-theoretic obstruction groups to determine which knot cobordism classes admit semifree Z_p actions fixing the knots. In §3 below, we will prove the following theorem from the viewpoint of [5] and [8].

THEOREM 1. *A simple knot $(\Sigma^{2n+1}, M^{2n-1})$, $n \geqq 3$, admits a p. 1. involution T fixing M^{2n-1} if and only if it has an associated Seifert matrix B of the form $B = A(A - \varepsilon A')^{-1}A$ for some matrix A with both $A + \varepsilon A'$ and $A - \varepsilon A'$ being unimodular.*

We will also discuss the differentiable case in the last section.

2. A technical lemma. Recall that $\varepsilon = (-1)^n$.

LEMMA 2. *Let A be an $(r \times r)$-matrix with both $A + \varepsilon A'$ and $A - \varepsilon A'$ being unimodular. Then the following system of equations has a unique solution for the pair of $(r \times r)$-matrices C_1 and C_2.*

$$C_1 A + \varepsilon C_2 A' = A + \varepsilon A' \tag{1}$$

$$\varepsilon C_1 A' + C_2 A = 0 . \tag{2}$$

Proof.

$$(1)+(2) \quad C_1(A + \varepsilon A') + C_2(A + \varepsilon A') = A + \varepsilon A' . \tag{3}$$

Since $A + \varepsilon A'$ is unimodular, (3) becomes

$$C_1 + C_2 = I, \text{ the identity} \tag{4}$$

$$(1)-(2) \quad C_1(A - \varepsilon A') - C_2(A - \varepsilon A') = A + \varepsilon A' . \tag{5}$$

Since $A - \varepsilon A'$ is unimodular, (5) becomes

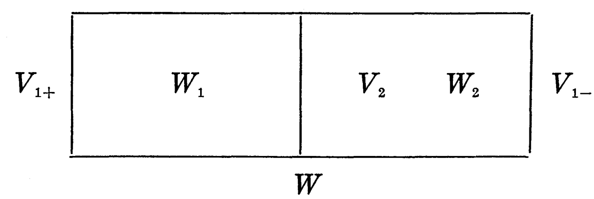
$$C_1 - C_2 = (A + \varepsilon A')(A - \varepsilon A')^{-1} . \tag{6}$$

From (4) and (6), we have

$$C_1 = A(A - \varepsilon A')^{-1} \quad \text{and} \quad C_2 = -\varepsilon A'(A - \varepsilon A')^{-1} .$$

3. Proof of Theorem 1. If a simple knot $(\Sigma^{2n+1}, M^{2n-1})$, $n \geqq 3$, admits a $p.$l. (or differentiable) involution T fixing M^{2n-1}, then it is easy to see that $\Sigma_1 = \Sigma/T$ is a $(2n+1)$-homotopy sphere, and (Σ_1, M) is again a simple knot. Let Y be the closure of $(\Sigma_1 - M \times D^2)$, and $V^{2n} \subseteqq Y$ be an $(n-1)$-connected Seifert manifold for (Σ_1, M) with $\partial V = M \times e^{i0}$, (we consider $S^1 = \{e^{i\theta}\}$), [5], [7]. Lifting V to Σ, we have two equivariant Seifert manifolds V_1 and V_2 with $TV_1 = V_2$, $\partial V_1 = M \times e^{i0}$, and $\partial V_2 = M \times e^{i\pi}$, [9]. We then cut $X =$ closure of $(\Sigma - M \times D^2)$ along V_1 to get a manifold W.

(*)



We see immediately that W_1 is the manifold obtained from Y (in Σ_1) by cutting it along V and $W_2 = TW_1$. Let $\{e_1, \cdots, e_r\}$ be a basis for $H_n V_{1+}$, and $\{f_1, \cdots, f_r\}$ a basis for $H_n W$ determined by the Alexander duality (in Σ). Similarly, viewing $\{e_i\}$ as a basis for $H_n V$, we have a basis $\{d_i\}$ for $H_n W_1$ by using the Alexander duality in Σ_1. Let A and B be the Seifert matrices associated to (Σ_1, M) and (Σ, M) respectively (with respect to the basis $\{e_i\}$) [5], [7].

From [5], we know that A represents the map $H_n V_{1+} \to H_n W_1$ with respect to the bases $\{e_i\}$ and $\{d_i\}$, also the map $H_n V_2 \to H_n W_2$ with respect to the bases $\{T_* e_i\}$ and $\{T_* d_i\}$. The matrix $-\varepsilon A'$ represents the map $H_n V_2 \to H_n W_1$ with respect to the bases $\{T_* e_i\}$ and $\{d_i\}$, also the map $H_n V_{1-} \to H_n W_2$ with respect to the bases $\{e_i\}$ and $\{T_* d_i\}$. The matrix B represents $H_n V_{1+} \to H_n W$ with respect to the bases $\{e_i\}$ and $\{f_i\}$, and $-\varepsilon B'$ represents $H_n V_{1-} \to H_n W$ with respect to the bases $\{e_i\}$ and $\{f_i\}$. All the maps here are induced by inclusions. Finally, let C_1 and C_2 denote the matrices represent the maps $H_n W_1 \to H_n W$ and $H_n W_2 \to H_n W$ with respect to the appropriate bases respectively. From (*), we have the following equation:

$$B = C_1 A, \ -\varepsilon B' = C_2(-\varepsilon A'), \ C_1(-\varepsilon A') = C_2 A .$$

These, together with the fact that $A + \varepsilon A' = B + \varepsilon B' =$ intersection form on $H_n V$, [5], give us the two equations in Lemma 2. Also, we have proved in [9] that both $A+\varepsilon A'$ and $A-\varepsilon A'$ are unimodular. Thus it follows from Lemma 2 that $B = C_1 A = A(A-\varepsilon A')^{-1}A$.

Conversely, given a knot $(\Sigma^{2n+1}, M^{2n-1})$ with its Seifert matrix B satisfying the condition in Theorem 1, we can construct a simple knot (Σ_1, M) with an $(n-1)$-connected Seifert manifold V and associ ated Seifert matrix A, [5]. Then we construct the 2-fold branched covering of (Σ_1, M) to obtain a simple knot (Σ_2, M) as in [4], [9], [12]. If we are in the p.l. category, then both Σ and Σ_2 are the standard sphere S^{2n+1}. Both (Σ, M) and (Σ_2, M) have the same Seifert matrix, hence they are actually equivalent, [8]. The involution T is given by the covering translation for the branched covering.

4. **Free involutions.** Since the study of knots invariant under free involutions on spheres is very similar to that of knots fixed under involutions, [9], [10], the following theorem can be proved in a similar fashion.

THEOREM 1'. *A simple knot $(\Sigma^{2n+1}, M^{2n-1})$, $n \geqq 3$, admits a free p.l. involution T leaving M invariant if and only if it has an associated Seifert matrix B of the form $B = A(A-\varepsilon A')^{-1}A$ for some matrix A with both $A+\varepsilon A'$ and $A-\varepsilon A'$ being unimodular.*

5. **The differentiable case.** Let T denote a differentiable involution on Σ^{2n+1} fixing M^{2n-1}, $n \geqq 3$. We want to study the relation between the differentiable structure of Σ and $\Sigma_1 = \Sigma/T$. If $\Sigma_1 \neq S^{2n+1}$, then we may view (Σ_1, M) as the connected sum of (S^{2n+1}, M) and Σ_1 along a disk disjoint from the Seifert manifold V and M. We then construct the 2-fold branched covering (Σ_3, M) of (S^{2n+1}, M) with branched point set M. By the uniqueness of differentiable structure of the cyclic branched covering ([2] or [4]), it is easy to see that $\Sigma = 2\Sigma_1 + \Sigma_3$, where the sum denotes the connected sum in the group of homotopy spheres Γ_{2n+1}, [6].

In the case n is odd, we let Σ_0 denote the generator of $bP_{4k} = \{y \in \Gamma_{4k-1} \mid y$ bounds parallelizable manifolds$\}$. Then we have the following proposition.

PROPOSITION 3. $\Sigma = 1/8(\text{index}\,(A+A'))\Sigma_0 + 2\Sigma_1$.

Proof. We first note that $A + A'$ is a unimodular, symmetric, even matrix, hence its index is divided by 8, [6]. According to the remark in the preceeding paragraphs, we only have to determine the differentiable structure of Σ_3, the 2-fold branched covering of (S^{4k-1}, M).

We push the Seifert manifold V into D^{4k}, a disk having S^{4k-1} as its boundary; then use V as the branched point set to construct a 2-fold branched covering N of D^{4k} with $\partial N = \Sigma_3$, [4, §4]. Proposition 5.6 in [4] tells us that the intersection form on $H_{2k}(N)$ is given by $A + A'$. All we have to do now is to show that N is parallelizable. The Seifert manifold V^{4k-2} has the homotopy type of a wedge of r copies of S^{2k-1}, hence we may represent each of the basis element of $H_{2k-1}(V) = r$ copies of Z by an embedded $(2k-1)$-sphere S_i. Each S_i bounds a $2k$-disk D_i in D^{4k}. Let x denote the covering translation in the 2-fold branched covering N over D^{4k}. Then $Q_i = xD_i \cup (-D_i)$ represent a basis for $H_{2k}(N)$, [4, p. 155]. N has the homotopy type of a wedge of r copies of S^{2k}, represented by the Q_i's. Then the argument used in Lemma 4 (i) of [12] shows that the normal bundle of each Q_i in N is stably trivial. Thus N is parallelizable, and it follows that $\Sigma_3 = 1/8(\text{index}\,(A + A'))\Sigma_0$.

In particular, we see that Σ_0 does not admit an involution T fixing a codimension 2 simple knot M with $1/8(\text{index}\,(A + A')) =$ even integer. In contrast, if G is a free differentiable involution acting on Σ^{4k-1} leaving M invariant, and A a Seifert matrix for the equivariant knot complement $(\Sigma - M \times D^2)/G$; then we proved in [10] that $1/8(\text{index}\,(A + A')) = \sigma(G, \Sigma) =$ the Browder-Livesay index desuspension invariant, [11]. But we know that Σ_0^7, the generator of bP_8, admits a free involution G with $\sigma(G, \Sigma_0^7) = 0$, [3], [11, p. 63]. Thus (G, Σ_0^7) admits an unknotted invariant S^5, [11], which implies $1/8(\text{index}\,(A + A')) = 0$.

In the case n is odd, we know that $bP_{4k+2} = Z_2$ or 0, [6]. Recall that $\Sigma_1 = \Sigma/T$, where the involution T fixes a simple knot M in Σ^{4k+1}. Then we have the following proposition.

PROPOSITION 4. $\Sigma = 2\Sigma_1$.

Proof. As in Proposition 3, we only have to determine the differentiable structure of Σ_3, the 2-fold branched cover of (S^{4k+1}, M). The proof of Proposition 3 shows that Σ_3 bounds a $2k$-connected parallelizable manifold N^{4k+2} with intersection form $A - A'$. Then the argument in [5, p. 256-257] enables us to embed N in S^{4k+3} in such a way that (S^{4k+3}, Σ_3) is a simple knot with Seifert manifold N and Seifert matrix A (see [4, p. 153] and [5, p. 256]). We know from [7, p. 544] that the Kervaire invariant of N is the Arf invariant of A. Since $A + A'$ is a symmetric, even, unimodular matrix, Lemma 2 in [11, p. 36] shows that the Arf invariant of A is zero. Hence Σ_3 is the standard sphere.

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