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GENERATING $O(n)$ WITH REFLECTIONS

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For $r \in C_n \equiv \{x | x \in R^n, ||x|| = 1\}$, let $S_r = I_n - 2rr'$ where r is a column vector. $O(n)$ denotes the orthogonal group on R^n. If $R \subseteqq C_n$, let $\mathscr{R} = \{S_r | r \in R\}$ and let G be the smallest closed subgroup of $O(n)$ which contains $\mathscr{R}$. G is *reducible* if there is a nontrivial subspace $M \subseteqq R^n$ such that $gM \subseteqq M$ for all $g \in G$. Otherwise, G is *irreducible*.

THEOREM. **If G is infinite and irreducible, then $G = O(n)$.**

In what follows, R^n denotes Euclidean n-space with the standard inner product, $O(n)$ is the orthogonal group of R^n, and $C_n = \{x | x \in R^n, ||x|| = 1\}$. If U is a subset of $O(n)$, $\langle U \rangle$ denotes the group generated algebraically by U and $\langle \bar{U} \rangle$ denotes the closure of $\langle U \rangle$. Thus, $\langle \bar{U} \rangle$ is the smallest closed subgroup of $O(n)$ containing U. For an integer k, $1 \leqq k < n$, M_k denotes a k-dimensional linear subspace of R^n. If $r \in C_n$, let $S_r = I - 2rr'$ where r is a column vector. Thus S_r is a reflection through r-henceforth called a reflection.

Suppose $R \subseteqq C_n$ and let $\mathscr{R} = \{S_r | r \in R\}$. Set $G = \langle \bar{\mathscr{R}} \rangle$. The group G is *reducible* if there is an M_k such that $gM_k \subseteqq M_k$ for all $g \in G$; otherwise, G is *irreducible*. The main result of this note is the following.

THEOREM 1. *If G is infinite and irreducible, then $G = O(n)$.*

Proof of Theorem 1. First note that if $S_r \in \mathscr{R}$ and $g \in G$, then $gS_rg^{-1} = S_{gr} \in G$. Let $\Delta = \{gr | g \in G, r \in R\}$. Thus, $t \in \Delta$ implies that $S_t \in G$. Since G is infinite, Δ must be infinite (see Benson and Grove (1971), Proposition 4.1.3). Since every Γ in $O(n)$ is a product of a finite number of reflections, to show that $G = O(n)$, it suffices to show that G is transitive on C_n (if G is transitive on C_n, then $\Delta = C_n$ so every reflection is an element of G and hence $G = O(n)$).

The proof that G is transitive on C_n follows. By Lemma 1 (below), there is a subgroup $K_2 \subseteqq G$ and a subspace $M_2 \subseteqq R^n$ such that $kx = x$ if $x \in M_2^\perp$ and $k \in K_2$ and K_2 is transitive on $D_2 \equiv M_2 \cap C_n$. Since G is irreducible, there is an $r_2 \in R$ such that $r_2 \notin M_2$ and $r_2 \notin M_2^\perp$. Let $M_3 = \text{span}\{r_2, M_2\}$ and let $K_3 = \langle\{K_2, S_{r_2}\}\rangle > \subseteqq G$. With $D_3 \equiv M_3 \cap C_n$, Lemma 3 (below) implies that $kx = x$ for all $x \in M_3^\perp$ and $k \in K_3$, and K_3 is transitive on D_3. Again, since G is irreducible, there is an $r_3 \in R$ such that $r_3 \notin M_3$ and $r_3 \notin M_3^\perp$. With $M_4 = \text{span}\{r_3, M_3\}$, let $K_4 = \langle\{K_3, S_{r_3}\}\rangle > \subseteqq G$ and let $D_4 \equiv M_4 \cap C_n$. By Lemma 3 (below)

$kx = x$ for $x \in M_4^\perp$ and $k \in K_4$ and K_4 is transitive on D_4. Applying this argument $(n-2)$ times, we obtain $K_n \subseteqq G$ and K_n is transitive on $D_n = C_n$. Thus, G is transitive on C_n and the proof is complete.

To fill in the gaps in the above argument, it remains to prove Lemmas 1, 2, and 3. Lemma 1 provides the starting point for the stepwise argument used in the proof of Theorem 1.

LEMMA 1. *If G is irreducible and infinite, there is a subspace M_2 and a subgroup $K_2 \subseteqq G$ such that $kx = x$ for $x \in M_2^\perp$, $k \in K_2$ and K_2 acts transitively on $D_2 \equiv M_2 \cap C_n$.*

Proof. As noted in the proof of Theorem 1, the set $\Delta = \{gr \mid r \in R, g \in G\}$ is infinite. Thus, there is a point $\delta_0 \in C_n$ such that every neighborhood of δ_0 contains infinitely many points in Δ. Thus we can select a sequence of pairs (r_i, t_i), $r_i, t_i \in \Delta$, such that r_i and t_i are linearly independent and $1 - 1/i < r_i't_i < r_{i+1}'t_{i+1} < 1$ for $i = 1, 2, \cdots$.

For $0 \leqq \eta < 2\pi$, set

$$\Psi(\eta) = \begin{pmatrix} \cos\eta & \sin\eta \\ -\sin\eta & \cos\eta \end{pmatrix} \in O(2)\,. \tag{1}$$

Define θ_i by $\cos\theta_i = r_i't_i$, $0 \leqq \theta_i < \pi$ so $\theta_i \to 0$ as $i \to \infty$. Let $\Gamma_i \in O(n)$ have first row t_i' and second row

$$(r_i - t_i'r_it_i)'/\|r_i - t_i'r_it_i\|\,.$$

Then an easy calculation shows that

$$S_{t_i}S_{r_i} = \Gamma_i'\begin{pmatrix} \Psi(2\theta_i) & 0 \\ 0 & I_{n-2} \end{pmatrix}\Gamma_i\,, \qquad i = 1, 2, \cdots \tag{2}$$

where I_{n-2} is an $(n-2) \times (n-2)$ identity matrix. Setting $H_i = \langle\Psi(2\theta_i)\rangle \subseteqq O(2)$, it is clear that

$$\left\{\Gamma_i'\begin{pmatrix} h & 0 \\ 0 & I_{n-2} \end{pmatrix}\Gamma_i \,\middle|\, h \in H_i\right\} \subseteqq G\,, \qquad i = 1, 2, \cdots\,. \tag{3}$$

By selecting an appropriate subsequence, we can assume without loss of generality that $\Gamma_i \to \Gamma_0 \in O(n)$, as $i \to \infty$.

If $\Psi(\eta)$ is given by (1), we now claim that

$$\Gamma_0'\begin{pmatrix} \Psi(\eta) & 0 \\ 0 & I_{n-2} \end{pmatrix}\Gamma_0 \in G\,. \tag{4}$$

Since G is closed and (3) holds, to establish (4), it suffices to show there is a subsequence i_j and $h_{i_j} \in H_{i_j}$ such that $h_{i_j} \rightarrow \Psi(\eta)$ as $i_j \rightarrow \infty$. However, the existence of such a sequence is assured since $\theta_i \rightarrow 0$ as $i \rightarrow \infty$. Thus (4) holds. Hence we see that

$$(5) \qquad K_2 \equiv \left\{ \Gamma_0' \begin{pmatrix} h & 0 \\ 0 & I_{n-2} \end{pmatrix} \Gamma_0 \middle| h \in H^* \right\} \subseteqq G$$

where H^* is the full rotation group of R^2.

To complete the proof of Lemma 1, let M_2 be the span of the first two columns of Γ_0'. With $D_2 \equiv M_2 \cap C_n$, it is easy to check that $kx = x$ for all $x \in M_2^{\perp}$, $k \in K_2$ and that K_2 acts transitively on D_2. This completes the proof.

The following result is used in the proof of Lemma 3.

LEMMA 2. *For $u_0 \in (0, 1]$, define a function $f: [0, 1] \rightarrow [0, 1]$ by*

$$(6) \qquad f(u) = \begin{cases} 0 & \text{if } 0 \leqq u \leqq u_0 \\ 1 - [\sqrt{uu_0} + \sqrt{(1-u)(1-u_0)}]^2 & \text{if } u_0 \leqq u \leqq 1 \,. \end{cases}$$

Let $v_1 = f(1)$ and define $v_i = f(v_{i-1})$ for $i = 2, 3, \cdots$. Then, there exists an index i_0 such that $v_i = 0$ for $i \geqq i_0$.

Proof. It is not hard to verify that f is a continuous convex function. Since $0 \leqq v_1 < 1$, $v_2 = f(v_1) = f((1 - v_1)0 + v_1 1) \leqq v_1 f(1) = v_1^2$. Proceeding by induction, $v_i \leqq v_1^i$ so $\lim_{i\to\infty} v_i = 0$. Since f is 0 in the interval $[0, u_0]$, there is an index i_0 such that $v_i = 0$ for $i \geqq i_0$. This completes the proof.

After establishing Lemma 1, the key to Theorem 1 is Lemma 3. Although the proof of Lemma 3 is quite long, the geometric idea behind the proof is fairly simple. Consider R^3 and let $D_2 = \{x \mid x \in R^3, x_3 = 0, x_1^2 + x_2^2 = 1\}$. Also, let $H = \left\{ \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \middle| k \text{ is any rotation of } R^2 \right\}$. Thus H acts transitively on D_2. Consider a fixed vector $t \in R^3$ with $\|t\| = 1$ such that t is not in the (x_1, x_2) plane and t is not in the x_3-line. Let $S_t = I\text{-}2tt'$ be the reflection across the plane $\{t\}^{\perp}$ and let $\tilde{H}$ be the group generated by S_t and H. The claim is that $\tilde{H}$ is transitive on $D_3 = \{x \mid x \in R^3, \|x\| = 1\}$. For example, suppose the angle between t and the (x_1, x_2) plane is 45°. Geometrically, it is clear that the set $H(S_t(D_2)) \equiv \{x \mid x = hS_t u$ for some $h \in H$, and some $u \in D_2\}$ is just D_3—that is, $S_t(D_2)$ is a circle passing through $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and the transitivity of H implies that H moves the set $S_t(D_2)$ everywhere onto D_3 (picture this on the surface of a basketball). Thus, given

$v_1, v_2 \in D_3$, $v_i = h_i S_t u_i$, for $h_i \in H$ and $u_i \in D_2$ for $i = 1, 2$. Since $u_1 = h_0 u_2$ for some $h_0 \in H$, it follows that $v_1 = h_1 S_t h_0 S_t h_2^{-1} v_2$ so $\tilde{H}$ is transitive on D_3. For other t-vectors, D_3 does not get covered by one application of HS_t to D_2, but D_3 is covered by a finite number of applications of HS_t to D_2—that is, $D_3 = (H(S_t(\cdots)H)S_t)(D_2)$ for some finite string $HS_tHS_t \cdots HS_t$. Again, this implies the transitivity of $\tilde{H}$ on D_3. Lemma 3 and its proof make all of the above precise.

LEMMA 3. *Consider a subspace* $M_m \subseteq R^n$, $2 \leqq m < n$, *and suppose that* K *is a subgroup of* $O(n)$ *such that*

$$\begin{cases} kx = x & \text{for all} \quad x \in M_m^\perp,\ k \in K \\ K \text{ is transitive on } D_m \equiv M_m \cap C_n\,. \end{cases} \tag{7}$$

Let $t \in C_n$ *be such that* $t \notin M_m$ *and* $t \notin M_m^\perp$. *With* $M_{m+1} = \operatorname{span}\{t, M_m\}$, *let* $D_{m+1} \equiv M_{m+1} \cap C_n$. *Then the group* $K^* \subseteq O(n)$ *generated by* K *and* $S_t = I - 2tt'$ *satisfies*

$$\begin{cases} kx = x & \text{for all} \quad x \in M_{m+1}^\perp,\ k \in K^* \\ K^* & \text{is transitive on } D_{m+1}\,. \end{cases} \tag{8}$$

Proof. That $kx = x$ for all $x \in M_{m+1}^\perp$ and $k \in K^*$ is not hard to verify. To establish the transitivity of K^* on D_{m+1}, define a set B_1 by

$$B_1 = K(S_t(D_m)) = \{x \mid x = kS_t u \text{ for some } u \in D_m, \text{ some } k \in K\} \tag{9}$$

and then define B_i inductively by

$$B_i = K(S_t(B_{i-1})) = \{x \mid x = kS_t u \text{ for some } u \in B_{i-1}, \text{ some } k \in K\} \tag{10}$$

$i = 2, 3, \cdots$. Since $K(S_t(D_{m+1})) \subseteq D_{m+1}$, it follows that $B_i \subseteq D_{m+1}$ for all i. The remainder of the proof is devoted to showing that there is an index i_0 such that $B_{i_0} = D_{m+1}$, because this implies the transitivity of K^* on D_{m+1}.

Claim 1. If $B_{i_0} = D_{m+1}$, then K^* is transitive on D_{m+1}.

Proof of Claim 1. Consider $z_1, z_2 \in D_{m+1}$. If $B_{i_0} = D_{m+1}$, then

$$\underbrace{K(S_t(K(S_t \cdots (D_m))))}_{i_0\text{-terms}} = D_{m+1}\,.$$

Thus, there exists $k_1, \cdots, k_{i_0} \in K$ and $g_1, \cdots, g_{i_0} \in K$ such that

$$z_1 = \left[\prod_{j=1}^{i_0} (k_j S_t)\right] u_1 \equiv h_1 u_1$$

and

$$z_2 = \left[\prod_{j=1}^{i_0} (g_j s_t)\right] u_2 \equiv h_2 u_2$$

for some $u_1, u_2 \in D_m$. Since K is transitive on D_m, there exists a $k_0 \in K$ such that $k_0 u_1 = u_2$. Thus, $z_2 = h_2 k_0 h_1^{-1} z_1$ which shows that K^* is transitive on D_{m+1} as $h_2 k_0 h_1^{-1} \in K^*$. This completes the proof of Claim 1.

We now continue with the proof. Let P denote the orthogonal projection onto M_m and define Z_c, $0 \leqq c \leqq 1$ by

$$Z_c = \{x \mid x \in D_{m+1}, \|Px\|^2 \geqq c\} . \tag{11}$$

Note that $Z_1 = D_m$ and $Z_0 = D_{m+1}$.

REMARK. Geometrically, Z_c is an equatorial zone (with equator D_m) which partially covers D_{m+1}. Smaller values of c correspond to more of D_{m+1} being covered.

Define φ on $[0, 1]$ by

$$\varphi(c) = \inf_{x \in Z_c} \|PS_t x\|^2 , \quad 0 \leqq c \leqq 1 , \tag{12}$$

and let

$$b_1 = \inf_{x \in B_1} \|Px\|^2 . \tag{13}$$

Since each $k \in K$ commutes with P, we have

$$b_1 = \inf_{k \in K} \inf_{x \in D_m} \|PkS_t x\|^2 = \inf_{x \in D_m} \|PS_t x\|^2 = \inf_{x \in Z_1} \|PS_t x\|^2 = \varphi(1) . \tag{14}$$

Claim 2. $B_1 = Z_{b_1}$.

Proof of Claim 2. If $x \in B_1$, $\|Px\|^2 \geqq b_1$ which implies that $x \in Z_{b_1}$. Conversely, consider $x \in Z_{b_1}$ and let Q denote the orthogonal projection onto the one-dimensional subspace $M_m^\perp \cap M_{m+1}$ which is spanned by the vector $t^* \equiv (I - P)t/\|(I - P)t\|$. Since Z_c is compact and arcwise connected, the continuous function $u \rightarrow \|PS_t u\|^2 (u \in Z_c)$ takes on all values between 1 and $\varphi(c)$. As $x \in Z_{b_1}$,

$$\|Px\|^2 \geqq b_1 = \varphi(1) = \inf_{u \in D_m} \|PS_t u\|^2 .$$

Hence, there exists a $u \in D_m$ such that $\|PS_t u\|^2 = \|Px\|^2$. Thus, $1 = \|Px\|^2 + \|Qx\|^2 = \|PS_t u\|^2 + \|QS_t u\|^2$, so $\|QS_t u\|^2 = \|Qx\|^2$. Since Q is a projection onto a one-dimensional subspace, u can be chosen (by changing to $-u$ if necessary) such that $Qx = QS_t u$. The transitivity of K on D_m implies there is a $k \in K$ such that $kPS_t u = Px$. Thus,

$kS_t u = kPS_t u + kQS_t u = Px + kQS_t u = Px + QS_t u = Px + Qx = x$, so $x = kS_t u \in B_1$. This completes the proof of Claim 2.

Using Claim 2, $B_2 = K(S_t(B_1)) = K(S_t(Z_{b_1}))$. Consider

$$b_2 \equiv \inf_{x \in B_2} ||Px||^2 . \tag{15}$$

Using (15) and the fact that each $k \in K$ commutes with P, we have

$$b_2 = \inf_{x \in B_2} ||Px||^2 = \inf_{x \in Z_{b_1}} \inf_{k \in K} ||PkS_t x||^2 = \inf_{x \in Z_{b_1}} ||PS_t x||^2 = \varphi(b_1) . \tag{16}$$

Claim 3. $B_2 = Z_{b_2}$.

Proof of Claim 3. If $x \in B_2$, then $x \in D_{m+1}$ and $||Px||^2 \geqq b_2$, so $x \in Z_{b_2}$. Conversely, consider $x \in Z_{b_2}$. As u varies over Z_{b_1}, the function $u \to ||PS_t u||^2$ takes on all values between 1 and b_2. Since $||Px||^2 \geqq b_2$, there is a $u \in Z_{b_1}$ such that $||PS_t u||^2 = ||Px||^2$. As in the proof of Claim 2, $1 = ||Px||^2 + ||Qx||^2 = ||PS_t u||^2 + ||QS_t u||^2$ so $||Qx||^2 = ||QS_t u||^2$, and we can choose u such that $Qx = QS_t u$. The transitivity of K implies that there is a $k \in K$ such that $kPS_t u = Px$. Thus, $x = Px + Qx = kPS_t u + QS_t u = kPS_t u + kQS_t u = kS_t u \in B_2$ since $u \in Z_{b_1} = B_1$. The proof of Claim 3 is complete.

Arguing as in the proof of Claim 3, it is an easy matter to show that $B_i = Z_{b_i}$ and $b_i = \varphi(b_{i-1})$ where

$$b_i = \inf_{x \in B_i} ||Px||^2, \; i = 3, 4, \cdots . \tag{17}$$

As noted earlier, the proof of Lemma 3 will be complete if we can show there is an index i_0 such that $B_{i_0} = Z_0 = D_{m+1}$. To establish the existence of an i_0, we will explicitly calculate the function φ defined in (12) and then apply Lemma 2. Define $z_0 \in D_{m+1}$ by

$$z_0 = S_t t^* \tag{18}$$

where $t^* = (I - P)t/||(I - P)t||$. Then,

$$\begin{aligned} a \equiv ||Pz_0||^2 &= \frac{||PS_t(I - P)t||^2}{||(I - P)t||^2} = \frac{||P(I - 2tt')(I - P)t||^2}{||(I - P)t||^2} \\ &= \frac{4||Pt||^2(t'(I - P)t)^2}{||(I - P)t||^2} = = 4||Pt||^2(1 - ||Pt||^2) . \end{aligned} \tag{19}$$

Since $t \notin M_m$ and $t \notin M_m^\perp$, $0 < ||Pt||^2 < 1$ so $0 < a \leqq 1$.

Claim 4. The function φ is given by

$$\varphi(c) = \begin{cases} 0 & \text{if } 0 \leqq c \leqq a \\ 1 - [\sqrt{ac} + \sqrt{(1-a)(1-c)}]^2 & \text{if } a \leqq c \leqq 1 . \end{cases} \tag{20}$$

Proof of Claim 4. Since $Q = t^* t^{*\prime}$ (see the proof of Claim 2), for each $x \in R^n$, $\|QS_t x\|^2 = x'S_t Q S_t x = x' S_t t^* t^{*\prime} S_t x = (z_0' x)^2$. Thus,

$$\varphi(c) = \inf_{x \in Z_c} \|PS_t x\|^2 = \inf_{x \in Z_c} (1 - \|QS_t x\|^2) = 1 - \sup_{x \in Z_c} (z_0' x)^2 . \tag{21}$$

If $a = 1$, then $z_0 \in D_m \subseteqq Z_c$, so $\sup_{x \in Z_c} (z_0'x)^2 = 1$ and $\varphi(c) = 0$ for all $c \in [0, 1]$.

Now, consider $a \in (0, 1)$. For $x \in Z_c$, let $\gamma = \|Px\|^2 \geqq c$. Then, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} z_0'x &= z_0'Px + z_0'Qx = (Pz_0)'Px + (Qz_0)'Qx \\ &\leqq \|Pz_0\| \, \|Px\| + \|Qz_0\| \, \|Qx\| = \sqrt{a}\sqrt{\gamma} + \sqrt{1-a}\sqrt{1-\gamma} . \end{aligned} \tag{22}$$

Further, there is equality in the above inequality for $x = x_0$ where

$$x_0 = \sqrt{\gamma/a} P z_0 + \sqrt{(1-\gamma)/(1-a)} Q z_0 . \tag{23}$$

Clearly, $\|Px_0\|^2 = \gamma \geqq c$ so $x_0 \in Z_c$. Thus,

$$\varphi(c) = 1 - \sup_{\gamma \in [c,1]} [\sqrt{a\gamma} + \sqrt{(1-a)(1-\gamma)}]^2 . \tag{24}$$

If $c \leqq a$, then $\gamma = a \in [c, 1]$ and $\varphi(c) = 0$. If $c > a$, then the sup in (24) is achieved at $\gamma = c$. Thus φ is given by (20) and the proof of Claim 4 is complete.

Now, by Lemma 2, there is an index i_0 such that $b_{i_0} = 0$ since $b_1 = \varphi(1)$ and $b_i = \varphi(b_{i-1})$. Thus, $B_{i_0} = Z_0 = D_{m+1}$ and by Claim 1, K^* is transitive on D_{m+1}. This completes the proof of Lemma 3.

The following is an immediate consequence of Theorem 1.

COROLLARY 1. *Let* $G_1 = \langle \mathscr{R} \rangle$ *where* $\mathscr{R} = \{S_r \mid r \in R\}$. *If* G_1 *is infinite and irreducible, then the closure of* G_1 *is* $O(n)$. *Also, for each* $x \in C_n$, $\{gx \mid g \in G_1\}$ *is dense in* C_n.

REMARK. The assumption that G is generated by reflections cannot be removed since $O^+(n)$, $n \geqq 2$ is infinite, closed and irreducible but $O^+(n) \neq O(n)$. Our interest in Theorem 1 arose in connection with results for G-monotone functions when G is generated by reflections (see Eaton and Perlman (1976)).

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