PACIFIC JOURNAL OF MATHEMATICS
Vol. 74, No. 1, 1978

THE DEGREE OF MONOTONE APPROXIMATION

R. K. BEATSON

Jackson type theorems are obtained for generalized monotone approximation. Let $E_{n,k}(f)$ be the degree of approximation of f by nth degree polynomials with kth derivative nonnegative on $[-1/4, 1/4]$. Then for each $k \geqq 2$ there exists an absolute constant D_k, such that for all $f \in C[-1/4, 1/4]$ with kth difference nonnegative on $[-1/4, 1/4]$; $E_{n,k}(f) \leqq D_k \omega(f, n^{-1})$. If in addition $f' \in C[-1/4, 1/4]$ then $E_{n,k}(f) \leqq D_k n^{-1} \omega(f', n^{-1})$.

Given a function f with nonnegative kth difference on $[-1/4, 1/4]$ (equivalently any finite real interval) it is natural to ask whether Jackson type estimates hold for

$$E_{n,k}(f) = \inf_{\{p \in \Pi_n : p^{(k)}(x) \geqq 0, x \in [-1/4, 1/4]\}} \|f - p\|$$

where the norm is the uniform norm, and Π_n is the space of algebraic polynomials of degree not exceeding n. In the case $k = 1$, Lorentz and Zeller [4] and Lorentz [5] have shown that there exists a constant D_1 such that if f is increasing on $[-1/4, 1/4]$

$$E_{n,1}(f) \leqq D_1 \omega(f, n^{-1}), \qquad n = 1, 2, \cdots, \tag{1}$$

where $\omega(f, \cdot)$ denotes the modulus of continuity of f. If, in addition, $f' \in C[1/4, 14]$ then

$$E_{n,1}(f) \leqq D_1 n^{-1} \omega(f', n^{-1}), \qquad n = 1, 2, \cdots. \tag{2}$$

DeVore [2, 3] has given a much simpler proof of the $k = 1$ results. The results of this paper are obtained with similar arguments.

NOTATION. Throughout $C_1, C_2, \cdots$ denote positive constants depending on k, but not depending on f, x or $n \geqq k$. Whenever it causes no confusion, $\|\cdot\|_\beta$ denotes $\|\cdot\|_{[-\beta,\beta]}$ and $\omega(e, \cdot)$ denotes $\omega_{[-1/4, 1/4]}(e, \cdot)$.

A function with nonnegative kth difference on $[a, b]$ cannot, in general, be extended to a function with nonnegative kth difference on a larger interval. For example the piecewise linear and convex function, $f \in C[0, \sum_{n=1}^\infty n^{-3}]$, with slope n on the interval

$$\left[\sum_{i=1}^{n-1} i^{-3}, \sum_{i=1}^{n} i^{-3}\right],$$

cannot be extended to the right and remain convex. This motivates the construction of a preapproximation (see Lemma 1) to f, to which we will apply appropriate polynomial convolution operators (see Lemma 2).

LEMMA 1. *Suppose $k \geqq 2$. Let*

$$L_n(h, x) = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} h(x + t_1 + \cdots + t_k) dt_1 \cdots dt_k \tag{3}$$

where $h \in C[-1/4, 1/4]$ and

$$\lambda = 1/8n \ , \qquad n = k, k+1, \cdots . \tag{4}$$

Extend the definition of $L_n(h)$ from

$$[-\alpha, \alpha] = \left[-\frac{1}{4} + \frac{k}{8n}, \frac{1}{4} - \frac{k}{8n}\right]$$

to $[-1/2, 1/2]$ by adjoining, to the right and left the Taylor polynomials of degree k, corresponding to $L_n(h)$ at the points $\alpha, -\alpha$. Then there exists constants E_k, F_k, G_k; $\bar{E}_k, \bar{F}_k, \bar{G}_k$; such that; for all $f \in C[-1/4, 1/4]$ with $f(-1/4) = f(1/4) = 0$ and nonnegative kth difference on $[-1/4, 1/4]$; for $n = k, k+1, \cdots$;

$$L_n(f, x)^{(k)} \geqq 0 \ , \qquad x \in R \ , \tag{5}$$

$$\| L_n(f)^{(j)} \|_{1/4} \leqq E_k n^j \omega(f, n^{-1}) \quad (j = 1, \cdots, k-1) \ , \tag{6}$$

$$\| L_n(f)^{(k)} \|_{1/2} \leqq E_k n^k \omega(f, n^{-1}) \ , \tag{7}$$

$$\| f - L_n(f) \|_{1/4} \leqq F_k \omega(f, n^{-1}) \ , \tag{8}$$

and

$$\| L_n(f) \|_{1/4} \leqq G_k n \omega(f, n^{-1}) \ . \tag{9}$$

If in addition $f' \in C[-1/4, 1/4]$ then

$$\| L_n(f)^{(j)} \|_{1/4} \leqq \bar{E}_k n^{j-1} \omega(f', n^{-1}) \quad (j = 2, \cdots, k-1) \ , \tag{6'}$$

$$\| L_n(f)^{(k)} \|_{1/2} \leqq \bar{E}_k n^{k-1} \omega(f', n^{-1}) \ , \tag{7'}$$

$$\| f - L_n(f) \|_{1/4} \leqq \bar{F}_k n^{-1} \omega(f', n^{-1}) \ , \tag{8'}$$

and

$$\| L_n(f)^{(2-j)} \|_{1/4} \leqq \bar{G}_k n \omega(f', n^{-1}) \ . \qquad (j = 1, 2) \ . \tag{9'}$$

Proof. For $x \in [-\alpha, \alpha]$

$$L_n(f, x) = (2\lambda)^{-k} \int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} \int_{x+t_2+\cdots+t_k-\lambda}^{x+t_2+\cdots+t_k+\lambda} f(\gamma) d\gamma dt_2 \cdots dt_k$$

implying

$$L_n(f, x)' = (2\lambda)^{-k}\int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} \Delta_{2\lambda} f(x + t_2 + \cdots + t_k - \lambda)dt_2 \cdots dt_k ;$$

repeating the argument, j times, $j = 1, \cdots k$,

$$L_n(f, x)^{(j)} = (2\lambda)^{-k}\int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} \Delta_{2\lambda}^{j} f(x + t_{j+1} + \cdots + t_k - j\lambda)dt_{j+1} \cdots dt_k . \tag{10}$$

(5) follows immediately. (10) and the definition of λ imply

$$\| L_n(f)^{(j)} \|_\alpha \leqq C_1 n^j \omega(f, n^{-1}) \qquad (j = 1, \cdots, k) . \tag{11}$$

(6), (7) follow from (11) on estimating the derivatives of the Taylor polynomials extending $L_n(f)$ to the larger interval.

To prove (8). The definition of $L_n(f, x)$ clearly implies

$$\| f - L_n(f) \|_\alpha \leqq C_2 \omega(f, n^{-1}) . \tag{12}$$

Also

$$\| f - L_n(f) \|_{[\alpha,1/4]} \leqq \| f - f(\alpha) \|_{[\alpha,1/4]} + | f(\alpha) - L_n(f, \alpha) | + \| L_n(f) - L_n(f, \alpha) \|_{[\alpha,1/4]} ;$$

so by (4); (12); (6), (7); and the manner in which $L_n(f)$ was extended

$$\| f - L_n(f) \|_{[\alpha,1/4]} \leqq C_3 \omega(f, n^{-1}) .$$

A similar result holds on $[-1/4, -\alpha]$; (8) follows.

To prove (9). Note that (8) implies both

$$\omega(L_n(f), n^{-1}) \leqq C_4\, \omega(f, n^{-1})$$

and

$$L_n(f, -1/4) \leqq F_k \omega(f, n^{-1}) ;$$

the second since $f(-1/4) = 0$; (9) follows.

We proceed to prove the results for $f' \in C[-1/4, 1/4]$. Arguments analogous to those leading from (10) to (6), (7); lead from

$$L_n(f, x)^{(j)} = (2\lambda)^{-k}\int_{-\lambda}^{\lambda} \cdots \int_{-\lambda}^{\lambda} \Delta_{2\lambda}^{j-1} f'(x + t_j + \cdots + t_k - (j-1)\lambda)dt_j \cdots dt_k ,$$

$(j = 1, \cdots, k)$ to (6′), (7′).

To show (8′) we use the quantitative Korovkin type estimate (see e.g., DeVore [2, p. 28-32])

(13) $|L_n(f, x) - f(x)| \leqq |f(x)||1 - L_n(1, x)| + |f'(x)||L_n((t-x). x)| + (1 + \sqrt{L_n(1, x)})\alpha_n(x)\omega(f', \alpha_n(x))$

where

(14) $$\alpha_n^2(x) = L_n((t-x)^2, x) .$$

Now $||1 - L_n(1)|| = ||L_n((t-x), x)|| = 0$, while

$$L_n((t-x)^2, x) = (2\lambda)^{-k}\int_{-\lambda}^{\lambda}\cdots\int_{-\lambda}^{\lambda}(t_1 + t_2 + \cdots + t_k)^2 dt_1\cdots dt_k$$
$$= k(2\lambda)^{-1}\int_{-\lambda}^{\lambda} t^2 dt \leqq C_5 n^{-2} .$$

Substituting into (13), (14) we find

(12′) $$||L_n(f) - f||_\alpha \leqq C_6 n^{-1}\omega(f', n^{-1}) .$$

Since for this particular operator

$$L_n(f, x)' = L_n(f', x) , \quad x \in [-\alpha, \alpha]$$

and $L_n(f, x)'$ is continued outside $[-\alpha, \alpha]$ by adjoining the Taylor polynomials of degree $k-1$, corresponding to f', at either end point; reasoning, similar to that yielding (8), implies

(15) $$||f' - L_n(f)'||_{1/4} \leqq C_7\,\omega(f', n^{-1}) .$$

Writing

$$||f - L_n(f)||_{[\alpha,1/4]} \leqq |f(\alpha) - L_n(f, \alpha)| + \int_\alpha^{1/4}|f'(t) - L_n(f, t)'|dt ;$$

(12′); (4) and (15) imply

$$||f - L_n(f)||_{[\alpha,1/4]} \leqq C_8 n^{-1}\omega(f', n^{-1}) .$$

Combining the above, the similar result on $[-1/4, -\alpha]$, and (12′) proves (8′).

To show (9′). Note (15) implies

$$\omega(L_n(f)', n^{-1}) \leqq C_9\,\omega(f', n^{-1})$$

and also

$$|L_n(f, \xi)'| \leqq C_7\,\omega(f', n^{-1}) \quad \text{where} \quad f'(\xi) = 0,\ -\frac{1}{4} < \xi < \frac{1}{4} ;$$

the existence of such an ξ following from $f(-1/4) = f(1/4) = 0$. Hence

$$||L_n(f)'||_{1/4} \leqq C_{10}\, n\omega(f', n^{-1}) .$$

(9′) follows since (8′) implies

$$\left|L_n\left(f, -\frac{1}{4}\right)\right| \leqq \bar{F}_k\, n^{-1}\omega(f', n^{-1}) .$$

We now know how well $L_n(f)$ approximates f, and concern ourselves with how well $L_n(f)$ may be approximated by convolutions with positive polynomials.

LEMMA 2. *Suppose $k \geqq 2$. Then there exist constants H_k, I_k and a sequence of even positive algebraic polynomials $\{\lambda_n\}_{n=k}^{\infty}$ satisfying*

$$\int_{-1}^{1} \lambda_n(t)dt = 1 , \tag{16}$$

and

$$||\lambda_n^{(j)}||_{[-1,1]/[-1/4,1/4]} \leqq H_k n^{2-4k+2j}(\leqq H_k n^{-2k}) , \quad (j = 0, \cdots, k-1) . \tag{17}$$

Further if f satisfies the conditions of Lemma 1, $g = L_n(f)$ and

$$L_n^*(g) = \int_{-1/2}^{1/2} g(t)\lambda_n(t - x)dt ; \tag{18}$$

then if $f \in C[-1/4, 1/4]$

$$||g - L_n^*(g)||_{1/4} \leqq I_k\omega(f, n^{-1}) ; \tag{19}$$

and if $f' \in C[-1/4, 1/4]$

$$||g - L_n^*(g)||_{1/4} \leqq I_k n^{-1}\omega(f', n^{-1}) . \tag{20}$$

Proof. Let $\lambda_k = \lambda_{k+1} = \cdots = \lambda_{4k-1} \equiv 1/2$. For $n \geqq 2k$, let

$$\lambda_{4n-4k}(t) = c_n[P_{2n}(t)/((t^2 - x_{1,2n}^2) \cdots (t^2 - x_{k,2n}^2))]^2 , \tag{21}$$

where P_{2n} is the Legendre polynomial of degree $2n$ and $x_{1,2n}, \cdots, x_{n,2n}$ are its positive zeros in increasing order. c_n is a normalizing constant for (16). Define the remaining λ_n's with the relation

$$\lambda_{4n+1} = \lambda_{4n+2} = \lambda_{4n+3} = \lambda_{4n} , \qquad n \geqq k .$$

Observe firstly that a theorem of Bruns (see e.g., DeVore [2, p. 20]) implies

$$C_{11}n^{-1} \leqq x_{1,2n} < \cdots < x_{k,2n} \leqq C_{12}n^{-1} , \qquad n > k . \tag{22}$$

Using the normalization $||P_n||_{[-1,1]} = 1$ and the corresponding Taylor expansion of P_n (see e.g., Davis [1, p. 365]),

$$(23) \qquad |P_{2n}(0)| = 2^{-2n}\begin{bmatrix} 2n \\ n \end{bmatrix} = (1 + o(1))/\sqrt{\pi n}\,,$$

the last equality being a consequence of Stirlings formula. (21), (22), and (23) together imply

$$(24) \qquad \lambda_{4n-4k}(0) \geqq C_{13} c_n n^{4k-1}\,, \qquad n \geqq 2k\,.$$

Let $n \geqq 2k$. Write

$$1 = \int_{-1}^{1} \lambda_{4n-4k}(t)dt = \sum_{k=-n}^{n} A_k(2n+1)\lambda_{4n-4k}(x_{k,2n+1})\,;$$

where the $A_k(2n+1)$ are the weights of the Gaussian quadrature formula, exact for polynomials of degree $4n+1$, with nodes at the zeros of the Legendre polynomial of degree $2n+1$. Therefore

$$1 \geqq A_0(2n+1)\lambda_{4n-4k}(0)$$

and since (Szego [6, p. 350]), $A_0(2n+1) = \pi(1+o(1))/(2n+1)$

$$(25) \qquad \lambda_{4n-4k}(0) \leqq C_{14}\, n\,.$$

(24) and (25) imply

$$c_n \leqq C_{15}\, n^{2-4k}\,;$$

which together with the normalization of the P_n, the definition of the λ_n, and (22) implies

$$\|\lambda_n\|_{[-1,1]/[-1/4,1/4]} \leqq C_{16}\, n^{2-4k}\,.$$

(17) follows by means of Markov's inequality.

It remains to show the order of approximation results. We cannot use the standard quantitative Korovkin theorem as

$$\omega_{[-1/2,1/2]}(g, n^{-1}) \neq 0(\omega_{[-1/4,1/4]}(f, n^{-1}))\,;$$

at least not in general. However a related method is applicable.

Again let $n \geqq 2k$. $t^{2k}\lambda_{4n-4k}(t)$ is a polynomial of degree $4n-2k$. Therefore for $j = 1, \cdots, k$

$$M_j = \int_{-1}^{1} t^{2j}\lambda_{4n-4k}(t)dt = 2\sum_{i=1}^{n} x_{i,2n}^{2j} A_i(2n)\lambda_{4n-4k}(x_{i,2n})\,;$$

where the $A_i(2n)$ are the weights of the Gaussian quadrature formula, exact for polynomials of degree $4n-1$, with nodes at the zeros of the Legendre polynomial of degree $2n$. Since λ_{4n-4k} has zeros at $x_{k+1,2n}, \cdots, x_{n,2n}$,

$$M_j = 2 \sum_{i=1}^{k} x_{i,2n}^{2j} A_i(2n) \lambda_{4n-4k}(x_{i,2n}) .$$

Since also λ_{4n-4k} has a local maximum on $[-x_{k+1,2n}, x_{k+1,2n}]$ at zero, and Szego [6, p. 350]

$$A_i(2n) \leqq \frac{\pi}{2n}(1 + o(1)) \qquad (i = 1, \cdots, k) ,$$

(22), (25) and the definition of the λ_n imply

$$\int_{-1}^{1} t^{2j}\lambda_n(t)dt \leqq C_{17}\, n^{-2j} , \qquad j = 1, \cdots, k; n \geqq k . \tag{26}$$

(26) and (17) may be used to estimate certain quantities involving L_n^*. All the estimates are uniform in $|x| \leqq 1/4$.

$$\begin{aligned} 1 - L_n^*(1, x) &= \int_{-1}^{1} \lambda_n(t)dt - \int_{-1/2-x}^{1/2-x} \lambda_n(t)dt \\ &\leqq 2\int_{1/4}^{1} \lambda_n(t)dt \leqq C_{18} n^{2-4k} . \end{aligned}$$

$$\begin{aligned} L_n^*((t-x)^{2j}, x) &= \int_{-1/2}^{1/2} (t-x)^{2j}\lambda_n(t-x)dt \\ &= \int_{-1/2-x}^{1/2-x} t^{2j}\lambda_n(t)dt \\ &\leqq \int_{-1}^{1} t^{2j}\lambda_n(t)dt \end{aligned} \tag{27}$$

and applying (26)

$$L_n^*((t-x)^{2j}, x) \leqq C_{19}\, n^{-2j} , \qquad j = 1, \cdots, k . \tag{28}$$

$$\begin{aligned} L_n^*(|t-x|^k, x) &\leqq \int_{-1}^{1} |t|^k\lambda_n(t)dt \\ &\leqq \left[\int_{-1}^{1} t^{2k}\lambda_n(t)dt\right]^{1/2} \\ &\leqq C_{20} n^{-k} , \end{aligned} \tag{29}$$

where we have used the Schwartz inequality, (16) and (28).

For j odd,

$$\begin{aligned} |L_n^*((t-x)^j, x)| &= \left|\int_{-1/2-x}^{1/2-x} t^j\lambda_n(t)\,dt\right| \\ &\leqq 2\int_{1/4}^{1} t^j\lambda_n(t)dt \end{aligned}$$

since λ_n is even. Applying (17)

$$|L_n^*((t-x)^j, x)| \leqq C_{21}\, n^{2-4k} , \qquad j = 1, 3, 5, \cdots . \tag{30}$$

If $t \in [-1/2, 1/2]$ and $x \in [-1/4, 1/4]$, Taylor's theorem gives

$$(31) \quad g(t) = \left[\sum_{j=0}^{k-1} \frac{g^{(j)}(x)(t-x)^j}{j!}\right] + \frac{1}{(k-1)!}\int_x^t g^{(k)}(u)(t-u)^{k-1}du \;.$$

Since the last term on the right hand side is bounded in modulus by $(1/k!)|t-x|^k \|g^{(k)}\|_{[-1/2,1/2]}$,

$$|L_n^*(g, x) - g(x)| \leqq |g(x)|\,|1 - L_n^*(1)| + \sum_{j=1}^{k-1} \frac{|g^{(j)}(x)|}{j!}|L_n^*((t-x)^j, x)| + \frac{1}{k!}\|g^{(k)}\|_{[-1/2,1/2]} L_n^*(|t-x|^k, x) \;.$$

Thus

$$\|L_n^*(g, x) - g(x)\|_{[-1/4,1/4]} \leqq \|g\|_{[-1/4,1/4]}\|1 - L_n^*(1)\|_{[-1/4,1/4]} + \sum_{j=1}^{k-1} \frac{\|g^{(j)}\|}{j!}{}_{[-1/4,1/4]}\|L_n^*((t-x)^j, x)\|_{[-1/4,1/4]} + \frac{1}{k!}\|g^{(k)}\|_{[-1/2,1,2]}\|L_n^*(|t-x|^k, x)\|_{[-1/4,1/4]} \;.$$

Combining the above, the estimates of all the terms involving g from Lemma 1 ($g = L_n(f)$), and the estimates (27), (28), (29), (30) of all the $L_n^*(\cdot,\cdot)$ yields (19), (20).

Given Lemmas 1 and 2 it remains to discuss how close $L_n^*(g)$ is to a polynomial with nonnegative kth derivative on $[-1/4, 1/4]$.

THEOREM. *For each $k \geqq 2$ there exists a constant D_k, such that for all $h \in C[-1/4, 1/4]$ with* kth *difference nonnegative on* $[-1/4, 1/4]$

$$E_{n,k}(h) \leqq D_k \omega_{[-1/4,1/4]}(h, n^{-1}) \;, \qquad n = k, k+1, \cdots \;.$$

If in addition $h' \in C[-1/4, 1/4]$ then

$$E_{n,k}(h) \leqq D_k n^{-1}\omega_{[-1/4,1/4]}(h', n^{-1}) \;, \quad n = k, k+1, \cdots \;.$$

Proof. Fix $k \geqq 2$. Let $f = h - \rho$ where

$$\rho(x) = h\left(-\frac{1}{4}\right) + 2\left(h\left(\frac{1}{4}\right) - h\left(-\frac{1}{4}\right)\right)\left(x + \frac{1}{4}\right) \;.$$

Clearly $\omega(f, n^{-1}) \leqq 2\omega(h, n^{-1})$ and when h' exists $\omega(f', n^{-1}) = \omega(h', n^{-1})$. Lemmas 1 and 2 apply to f. Writing

$$\bar{L}_n(h) = \rho(x) + L_n^*(L_n(f))$$

Lemmas 1 and 2 imply

$$(32)\qquad \begin{aligned} \|h - \bar{L}_n(h)\|_{1/4} &= \|f - L_n^*(L_n(f))\| \\ &\leqq \|f - L_n(f)\|_{1/4} + \|L_n(f) - L_n^*(L_n(f))\|_{1/4} \\ &\leqq \left| \begin{array}{ll} C_{22}\omega(h, n^{-1}) & h \in C\left[-\frac{1}{4}, \frac{1}{4}\right], \\ C_{22}n^{-1}\omega(h', n^{-1}), & h' \in C\left[-\frac{1}{4}, \frac{1}{4}\right]. \end{array} \right. \end{aligned}$$

Let $g = L_n(f)$. Then

$$\bar{L}_n(h) = \rho(x) + L_n^*(g) = \rho(x) + \int_{-1/2}^{1/2} g(t)\lambda_n(t - x)dt ,$$

$$\begin{aligned} \bar{L}_n(h, x)' &= \rho'(x) + \int_{-1/2}^{1/2} g(t)\cdot -\lambda_n'(t - x)dt \\ &= \rho'(x) + [-g(t)\lambda_n(t - x)]_{-1/2}^{1/2} + \int_{-1/2}^{1/2} g'(t)\lambda_n(t - x)dt . \end{aligned}$$

$k \geqq 2$ alternate differentiations and integrations by parts yield;

$$\begin{aligned} \bar{L}_n(h, x)^{(k)} &= (-1)^k\left[\sum_{j=0}^{k-1}(-1)^j\left[g^{(j)}(t)\lambda_n^{(k-1-j)}(t - x)\right]_{t=-1/2}^{t=1/2}\right] \\ &\quad + \int_{-1/2}^{1/2} g^{(k)}(t)\lambda_n(t - x)dt \\ &= r(x) + \int_{-1/2}^{1/2} g^{(k)}(t)\lambda_n(t - x)dt . \end{aligned}$$

(5) and the positivity of the kernels imply the second term on the right hand side is nonnegative. Lemma 1 implies

$$\|g^{(j)}\|_{1/2} \leqq C_{23}n^k\omega(h, n^{-1}) , \qquad j = 0, \cdots, k - 1, h \in C, \left[-\frac{1}{4}, \frac{1}{4}\right].$$

Hence using (17)

$$\|r\|_{1/4} \leqq C_{24}n^{-k}\omega(h, n^{-1}) .$$

Let

$$p_n(x) = \bar{L}_n(h, x) + \frac{x^k}{k!}C_{29}n^{-k}\omega(h, n^{-1}) ;$$

$p_n^{(k)}(x)$ is nonnegative on $[-1/4, 1/4]$, and by (32) p_n provides the first estimate of the theorem. Similarly, when $h' \in C[-1/4, 1/4]$

$$p_n(x) = \bar{L}_n(h, x) + \frac{x^k}{k!}C_{25}n^{-k-1}\omega(h', n^{-1})$$

provides the second estimate of the theorem.

References

1. P. J. Davis, *Interpolation and Approximation*, Blaisdell, N.Y., 1963.
2. R. A. DeVore, *The Approximation of Continuous Functions by Positive Linear Operators*, Lecture Notes in Mathematics #293, Springer, 1972.
3. ———, *Degree of Monotone Approximation*, Linear Operators and Approximation 11, editors P. L. Butzer and B. Sz.-Nagy, Birkhauser Verlag, Stuttgart, 1974, 337-351.
4. G. G. Lorentz and K. L. Zeller, *Degree of approximation by monotone polynomials* 1, J. Approximation Theory, **1** (1968), 501-504.
5. G. G. Lorentz, *Monotone Approximation*, Inequalities 111, O. Shisha editor, Academic Press, New York, 1972, 201-215.
6. G. Szego, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ. Vol. XXIII, N. Y., 1959.

Received April 28, 1977.

University of Canterbury
Christchurch, New Zealand