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ON GENERALIZED POLARS OF THE PRODUCT OF ABSTRACT HOMOGENEOUS POLYNOMIALS

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Let E denote a vector space over an algebraically closed field K of characteristic zero. Our object is to investigate the location of null-sets of generalized polars of the product of certain given abstract homogeneous polynomials from E to K. Some special aspects of this general problem were studied in the complex plane by Bôcher and Walsh and, later, in vector spaces by Marden. Our present treatment furnishes further generalizations of the theorems of Marden, Bôcher, and Walsh and offers a systemmatic, abstract, and unified approach to their completely independent methods. One of our results, in special setting, relates to the polar of a product and reduces essentially to the author's earlier generalization [Trans. Amer. Math. Soc., 218 (1976), 115–131] of Hörmander's theorem on polars of abstract homogeneous polynomials. We show also that our theorems cannot be further generalized in certain natural directions.

1. Introduction. Let E be a vector space over a field K of characteristic zero. A mapping P from E to K is called [4, pp. 760–763], [7, p. 55], [8], [14] an *abstract homogeneous polynomial* ($a{\cdot}h{\cdot}p{\cdot}$) of degree n if for every $x, y \in E$,

$$p(sx + ty) = \sum_{k=0}^{n} A_k(x, y)s^k t^{n-k} \quad \forall s, t \in K ,$$

where the coefficients $A_k(x, y) \in K$ and are independent of s and t for any given x, y in E. We shall denote by $\boldsymbol{P}_n$ the class of all nth-degree a.h.p.'s from E to K. The nth-*polar of* P is the mapping (see [5, Lemma 1] for its existence and uniqueness) $P(x_1, x_2, \cdots, x_n)$ from E^n to K which is linear in each x_k and symmetric in the set $\{x_k\}$ such that $P(x, x, \cdots, x) = P(x)$ for every x in E. The kth-*polar* of P is then defined by

$$P(x_1, \cdots, x_k, x) = P(x_1, \cdots, x_k, x, \cdots, x) .$$

The *null-set* $Z_P(x, y)$ of P (relative to elements x, y in E) is defined [9, p. 28], [15] by

$$Z_P(x, y) = \{sx + ty \neq 0 \,|\, s, t \in K;\ P(sx + ty) = 0\} .$$

Now we shall assume throughout that K is an algebraically closed field of characteristic zero. It is known [5] (see also [2, pp. 38–40], [11, pp. 248-255]) that $K = K_0(i)$, where K_0 is a maximal ordered subfield of K and $-i^2$ is the unit element of K. If $z = a + ib \in K$ (with a, b in K_0), we define $\bar{z} = a - ib$, $\mathrm{Re}(z) = (z + \bar{z})/2$ and $|z| = (a^2 + b^2)^{1/2}$. If $A \subseteq K$, we call A to be *K_0-convex* if $\sum_{j=1}^{n} \mu_j a_j \in A$ for every $a_j \in A$ and $\mu_j \in K_{0+}$ (the set of all nonnegative elements of K_0) such that $\sum_{j=1}^{n} \mu_j = 1$. Adjoin to K an element ω (called infinity) and furnish $K \cup \{\omega\} \equiv K_\omega$ with the following structure: (1) the subset K of K_ω preserves its initial field structure; and (2) $a + \omega = \omega + a = \omega$ for every $a \in K$, $a \cdot \omega = \omega \cdot a = \omega$ for every $a \in K - \{0\}$, and $\omega^{-1} = 0$, $0^{-1} = \omega$. A subset A of K_ω is called [16, pp. 353, 373], [14, p. 116], [13, pp. 25-26] a *generalized circular region* ($g \cdot c \cdot r \cdot$) of K_ω if A is either one of the sets $\emptyset$, K, K_ω, or A satisfies the following two conditions:

(1) $\varphi_\zeta(A)$ is K_0-convex for every $\zeta \in K - A$, where $\varphi_\zeta(z) = (z - \zeta)^{-1}$ for every $z \in K_\omega$,

(2) $\omega \in A$ if A is not K_0-convex.

The empty set $\emptyset$, K, K_ω, and single-point sets (and their compliments in K_ω) are examples of trivial g.c.r.'s. We shall denote by $D(K_\omega)$ the class of all g.c.r.'s of K_ω. Zervos' characterization [16, pp. 372-387] of this class, when K is the field $\boldsymbol{C}$ of complex numbers, leads to the following result [16, p. 352], [14, p. 116], [15], namely: *The nontrivial g.c.r.'s of $\boldsymbol{C}_\omega$ are the open interior (or exterior) of circles or the open half-planes, adjoined with a connected subset (possibly empty) of their boundary.* The g.c.r.'s of $\boldsymbol{C}_\omega$, with all or no boundary points included, will be called *(classical) circular regions of $\boldsymbol{C}_\omega$*.

REMARK. Through we have defined the g.c.r.'s for an algebraically closed field of characteristic zero, but the definition remains the same for any maximal ordered field K_0 (see [16, pp, 353-373], [13, p. 26] for the definition of the class $D(K_\omega)$ when K is an arbitrary field).

We now give some concepts which were introduced earlier by the author [13, pp. 36-40], [14, p. 117-119], [15] to define circular cones in E and discuss some of their important properties which are found useful in later sections. Define an equivalence relation "$\sim$" among elements of E^2 by "$(x, y) \sim (x', y')$ if and only if $\mathscr{L}[x, y] = \mathscr{L}[x', y']$," where $\mathscr{L}[x, y]$ denotes the subspace of E generated by the elements $x, y \in E$. The equivalence class $[(x, y)]$, containing the element $(x, y) \in E^2$, is called *nontrivial* if x and y are linearly independent (it is called trivial, otherwise). The axiom of choice allows us to choose a unique element from each nontrivial equivalence class. The set $N(\subseteq E^2)$ ef elements thus chosen would be referred to as a *nucleus* of E^2. Obviously, $N \neq \emptyset$ if $\dim E \geqq 2$. Given a nucleus N of E^2 and a mapping $G: N \rightarrow D(K_\omega)$ (called *circular mapping* [14]), we define (cf. equations (2.1) and (2.2) in [14]) the *circular cone* $E_0(N, G)$, relative to N and G, by

$$E_0(N, G) = \bigcup T_G(x, y) , \tag{1.1}$$

where

$$T_G(x, y) = \{sx + ty \neq 0 \mid s, t \in K;\ s/t \in G(x, y)\} \tag{1.2}$$

and the union in (1.1) ranges over all elements $(x, y) \in N$.

REMARK 1.1. If $\dim E = 2$, then [14, Remark (2.1)] every circular cone $E_0(N, G)$ is of the form

$$E_0(N, G) = \{sx_0 + ty_0 \neq 0 \mid s, t \in K;\ s/t \in A\}$$

for some $A \in D(K_\omega)$, where x_0, y_0 are any two linearly independent elements of E and where $N = \{(x_0, y_0)\}$ and $G(x_0, y_0) = A$.

We define [13, p. 42], [14, p. 117], [15] *hermitian cones* to be subsets E_1 of E of the form $E_1 = \{x \in E \mid x \neq 0;\ H(x, x) \geqq 0\}$ (and the ones got by replacing in this expression the inequality "$\geqq$" by "$>$", "$\leqq$" or "$<$"), where $H(x, y)$ is a *hermitian symmetric* form [8, p. 270] from E^2 to K. For the first time, Hörmander [5] used hermitian cones in his attempt to generalize to vector spaces a theorem due to Laguerre [6], [7, Theorem (13, 2)] on polar derivatives and, later, Marden [8], [9] exploited these cones in generalizing to vector spaces certain classical results due to Bôcher [1], Grace [3], and to Szegö [10]. Recently, the author [13], [14], [15] succeeded in replacing the said role of hermitian cones by circular cones. The relationship between the class of hermitian cones and the class of circular cones is exhibited in the following propositions due to the author [14, pp. 117-119]. In the rest of our work, we assume that $\dim E \geqq 2$.

PROPOSITION 1.2. *Let E_1 be a hermitian cone in E. Given a nucleus $N \subseteqq E^2$, there exists a circular mapping $G: N \to D(K_\omega)$ such that $E_0(N, G) = E_1$ and $E_1 \cap \mathscr{L}[x, y] = T_G(x, y)$ for every $(x, y) \in N$, where T_G is as defined by* (1.2).

PROPOSITION 1.3. *The class of all circular cones in E contains properly the class of all hermitian cones.*

2. A generalization of Bôcher's theorem.

Before taking up our main result of this section, we shall give some definitions and useful properties. First, we establish the following proposition which expresses essentially the fact that any two circular cones can always be expressed relative to a common nucleus.

PROPOSITION 2.1. *Given a circular cone $E_0(N, G)$ and an arbitrary nucleus $N' \subseteqq E^2$, there exists a circular mapping $G': N' \rightarrow D(K_\omega)$ such that $E_0(N, G) = E_0(N', G')$.*

Proof. From the definition of nucleus, we can define a mapping $\eta: N' \rightarrow N$ by assigning to every element $(x', y') \in N'$ a unique element $(x, y) \in N$ such that $(x, y) \sim (x', y')$. Then η is a $1-1$ and onto mapping. Consequently, every element $(x', y') \in N'$ determines uniquely an element $(x, y) \in N$, a set of scalars $\alpha, \beta, \gamma, \delta \in K$, and a homographic transformation [16, p. 353], [13, pp. 24-25] U of K_ω such that

$$\eta(x', y') = (x, y) , \tag{2.1}$$

$$x' = \alpha x + \beta y,\ y' = \gamma x + \delta y,\ \alpha\delta - \beta\gamma = \Delta(\text{say}) \neq 0 , \tag{2.2}$$

and

$$U(\rho) = (\delta\rho - \gamma)/(-\beta\rho + \alpha)\forall\rho \in K_\omega . \tag{2.3}$$

Let us now define $G'(x', y') = U(G(\eta(x', y'))) = U(G(x, y))$ for every $(x', y') \in N'$, where the element (x, y) and the corresponding homographic transformation U satisfies (2.1)-(2.3). Since $G(x, y) \in D(K_\omega)$ and since a homographic transformation permutes the class $D(K_\omega)$(cf. [16, p. 353], [13, p. 28]), we immediately infer that $G'(x', y') \in D(K_\omega)$ and, hence, G' is indeed a circular mapping from N' into $D(K_\omega)$. First, we claim that

$$E_0(N, G) \subseteqq E_0(N', G') . \tag{2.4}$$

If $z \in E_0(N, G)$, then there exists an element $(x, y) \in N$ and scalars $s, t \in K$ such that $z = sx + ty$ and $s/t = \rho(\text{say}) \in G(x, y)$. Since η is $1-1$ and onto, the above element (x, y) of N determines a unique element $(x', y') \in N'$ and the corresponding homographic transformation U satisfying the relations (2.1)-(2.3). This implies that $x = \Delta^{-1}(\delta x' - \beta y')$, $y = \Delta^{-1}(\alpha y' - \gamma x')$, and hence that

$$z = \Delta^{-1}[(\delta s - \gamma t)x' + (-\beta s + \alpha t)y'] = \Delta^{-1}(s'x' + t'y'),\ \text{say} .$$

Since $\rho = s/t \in G(x, y)$, the relations (2.1)-(2.3) and the definition of G' implies that $\rho' = s'/t' = (\delta\rho - \gamma)/(-\beta\rho + \alpha) = U(\rho) \in G'(x', y')$. That is, $z \in T_{G'}(x', y') \subseteqq E_0(N', G')$ and (2.4) holds.

Next, we claim that

$$E_0(N', G') \subseteqq E_0(N, G) . \tag{2.5}$$

For, if $z' \in E_0(N', G')$, then $z' = s'x' + t'y'$ for some $(x', y') \in N'$ and $s', t' \in K$ such that $s'/t' = \rho'(\text{say}) \in G'(x', y')$. Now, η determines uniquely an element $(x, y) \in N$, scalars $a, \beta, \gamma, \delta \in K$, and the corresponding U satisfying (2.1)-(2.3). Therefore,

$$z' = (s'\alpha + t'\gamma)x + (s'\beta + t'\delta)y = sx + ty \quad \text{(say)},$$

and since $s'/t' = \rho' \in U(G(x, y))$, it implies that

$$\rho = s/t = \frac{\alpha\rho' + \gamma}{\beta\rho' + \delta} = U^{-1}(\rho') .$$

Obviously, then $\rho' = U(\rho) \in U(G(x, y))$ and, hence, $\rho \in G(x, y)$. That is, $z' \in T_G(x, y)$ and (2.5) holds. The containments (2.4) and (2.5) finally establish the desired result.

In view of the above proposition, we shall assume (without loss of generality) that all the circular cones, whenever they appear in a particular theorem, have a common nucleus.

Conventionally speaking, the word "composite (a.h.) polynomial" has been used [7, pp. 65-106], [9], [15] to designate, in general, any (a.h.) polynomial which has been derived from given (a.h.) polynomials via certain kinds of composition. In what follows we define [8, p. 271], [13, pp. 118-119] a special kind of composite a.h.p.'s, derived from certain given a.h.p.'s and their first polars, and study the location of the null-sets of such polynomials.

DEFINITION 2.2. Given a.h.p.'s $P_k \in \boldsymbol{P}_{n_k}$ and scalars $m_k \in K$, $k = 1, 2, \cdots, q$, let us set

$$Q(x) = P_1(x) \cdot P_2(x) \cdots P_q(x) ,$$
$$Q_k(x) = P_1(x) \cdots P_{k-1}(x) \cdot P_{k+1}(x) \cdots P_q(x) ,$$

and define

$$\Phi(x_1, x) = \sum_{k=1}^{q} m_k Q_k(x) \cdot P_k(x_1, x) \forall x, x_1 \in E . \tag{2.6}$$

We shall call $\Phi(x_1, x)$ as a *generalized polar* of the product $Q(x)$. If $n = n_1 + n_2 + \cdots + n_q$, let us note that $Q \in \boldsymbol{P}_n$, $Q_k \in \boldsymbol{P}_{n-n_k}$ and $P_k(x_1, x)$ is an a.h.p. of degree $n_k - 1$ in x and of degree 1 in x_1, $1 \leqq k \leqq q$. Therefore, $\Phi(x_1, x)$ is an a.h.p. of degree $n - 1$ in x and an a.h.p. of degree 1 in x_1. The following proposition justifies the terminology for $\Phi(x_1, x)$ as "a generalized polar of $Q(x)$".

PROPOSITION 2.3. *In the notations of Definition 2.2, if $m_k = n_k$ for $k = 1, 2, \cdots, q$, then the generalized polar $\Phi(x_1, x)$ of the product $Q(x)$ is essentially the first polar $Q(x_1, x)$ of $Q(x)$, except for a nonzero constant factor. More precisely,*

$$\Phi(x_1, x) = n \cdot Q(x_1, x) \forall x, x_1 \in E ,$$

where $n = n_1 + n_2 + \cdots + n_q$ *(m_k being taken as n_k for all k).*

Proof. For each $k(1 \leqq k \leqq q)$, we use the properties of the n_kth polar of P_k and the fact that K is algebraically closed to obtain (for every $x, x_1 \in E$)

$$P_k(sx + tx_1) = P_k(sx + tx_1, sx + tx_1, \cdots, sx + tx_1)$$

$$(2.7) \qquad = \sum_{m=0}^{n_k} C(n_k, m) \cdot P_k(\overbrace{x, \cdots, x}^{m}, x_1, \cdots, x_1) \cdot s^m t^{n_k - m}$$

$$(2.8) \qquad = \prod_{j=1}^{n_k} (\delta_{jk} \cdot s - \gamma_{jk} \cdot t) , \quad \text{say}$$

$$(2.9) \qquad = \sum_{m=0}^{n_k} (-1)^{n_k - m} S(m, k) \cdot s^m t^{n_k - m} ,$$

where $\delta_{jk} \equiv \delta_{jk}(x, x_1)$, $\gamma_{jk} \equiv \gamma_{jk}(x, x_1)$ and where $S(m, k)$ denotes the sum of all possible products obtained from $[\delta_{1k}\delta_{2k} \cdots \delta_{mk} \cdot \gamma_{m+1k} \cdots \gamma_{n_k k}]$ by permuting the subscripts $1, 2, \cdots, n_k$ in all possible ways. The steps (2.7) and (2.9) imply that

$$(2.10) \qquad P_k(x) = S(n_k, k) = \prod_{j=1}^{n_k} \delta_{jk}$$

$$(2.11) \qquad P_k(x_1, x, x, \cdots, x) \equiv P_k(x_1, x) = -\frac{1}{n_k} \cdot S(n_k - 1, k)$$

for all $k = 1, 2, \cdots, q$. If we let $r_0 = 0$, $r_k = n_1 + n_2 + \cdots + n_k$ (with $r_q = n$) and define

$$(2.12) \qquad l = \psi(j, k) = r_{k-1} + j \forall j = 1, 2, \cdots, n_k, 1 \leqq k \leqq q ,$$

we easily notice that ψ determines a $1 - 1$ correspondence between the set $\{1, 2, \cdots, n\}$ and the set $\{(j, k) | 1 \leqq j \leqq n_k; 1 \leqq k \leqq q\}$. We may then write

$$(2.13) \qquad Q(sx + tx_1) = \prod_{k=1}^{q} \prod_{j=1}^{n_k} (\delta_{jk} s - \gamma_{jk} t) = \prod_{l=1}^{n} (\mu_l s - \nu_l t) , \quad \text{say} ,$$

where $\mu_l = \delta_{jk}$ and $\nu_l = \gamma_{jk}$ if and only if $l = \psi(j, k)$. Next, (2.10) and (2.11) gives, respectively,

$$(2.14) \qquad Q_k(x) = \prod_{i=1, i \neq k}^{q} \left(\prod_{j=1}^{n_i} \delta_{ji} \right) = \mu_1 \cdots \mu_{r_{k-1}} \cdot \mu_{r_k + 1} \cdots \mu_n ,$$

and

$$(2.15)\qquad \begin{aligned} n_k\cdot P_k(x_1, x) &= -\sum_{m=1}^{n_k}\left\{\prod_{j=1, j\neq m}^{n_k}\delta_{jk}\right\}\cdot\gamma_{mk} \\ &= -\sum_{m=1}^{n_k}\left\{\sum_{j=1, j\neq m}^{n_k}\mu_{r_{k-1}+j}\right\}\cdot\nu_{r_{k-1}+m}\,. \end{aligned}$$

Consequently, (2.14) and (2.15) imply that

$$\begin{aligned} n_k\cdot Q_k(x)\cdot P_k(x_1, x) &= -\sum_{m=1}^{n_k}(\mu_1\mu_2\cdots\mu_{r_{k-1}+m-1})\cdot(\mu_{r_{k-1}+m+1}\cdots\mu_n)\nu_{r_{k-1}+m} \\ &= -\sum_{l=r_{k-1}+1}^{r_k}(\mu_1\mu_2\cdots\mu_{l-1}\cdot\nu_l\cdot\mu_{l+1}\cdots\mu_n)\,. \end{aligned}$$

Finally, if $m_k = n_k$ for all k, we obtain

$$\begin{aligned} \Phi(x_1, x) &= -\sum_{k=1}^{q}\left[\sum_{l=r_{k-1}+1}^{r_k}(\mu_1\cdots\mu_{l-1}\cdot\nu_l\cdot\mu_{l+1}\cdots\mu_n)\right] \\ &= -\sum_{l=1}^{n}(\mu_1\cdots\mu_{l-1}\cdot\nu_l\cdot\mu_{l+1}\cdots\mu_n) \\ &= n\cdot Q(x_1, x), \end{aligned}$$

due to the corresponding formula (2.11) for the polynomial Q.

REMARK 2.4. If $q = 1$, $m_1 = n_1$, the above proposition tells us that $\Phi(x_1, x) = n_1\cdot P_1(x_1, x)$.

Now we prove the following main result [13, Theorem (18.1)] of this section which generalizes a theorem due to Marden [8, Theorem (3.1)], concerning the generalized polar $\Phi(x_1, x)$ of the product $Q(x)$ as defined by (2.6). The complex plane version leads to certain improvements in Bôcher's theorem [1], [13, Corollary (19.3)] and in Walsh's theorem [7, Theorem (20.1)]. We prove

THEOREM 2.5. *Let $E_0^{(i)} = E_0(N, G_i)$, $i = 1, 2$, be two disjoint circular cones in E and let $P_k \in \boldsymbol{P}_{n_k}(k = 1, 2, \cdots, q)$ such that*

$$(2.16)\qquad Z_{P_k}(x, y) \subseteqq \begin{cases} T_{G_1}(x, y)\,, & k = 1, 2, \cdots, p(<q) \\ T_{G_2}(x, y)\,, & k = p+1, \cdots, q \end{cases}$$

for all $(x, y) \in N$. If $\Phi(x_1, x)$ is the generalized polar of the product $Q(x)$ (cf. Definition 2.2) with $m_k > 0$ for $k = 1, 2, \cdots, p$ and $m_k < 0$ for $k = p+1, \cdots, q$ such that

$$(2.17)\qquad \sum_{k=1}^{q} m_k = 0$$

then $\Phi(x_1, x) \neq 0$ for all linearly independent elements x, x_1 of E such that $x \in E - E_0^{(1)} \cup E_0^{(2)}$.

Proof. Let $P_k(sx + tx_1)$ be as given by the Equation (2.8) in the proof of Proposition 2.3. Let x, x_1 be linearly independent elements of E such that $x \in E - E_0^{(1)} \cup E_0^{(2)}$. Then x, x_1 are nonzero elements such that $x \notin E_0^{(1)} \cup E_0^{(2)}$, so that $P_k(x) \neq 0$ for $1 \leqq k \leqq q$ and (due to (2.10)) $\delta_{jk} \neq 0$ for $1 \leqq j \leqq n_k, 1 \leqq k \leqq q$. Let $\rho_{jk} = \gamma_{jk}/\delta_{jk}$. Now, x and x_1 determine uniquely an element $(x_0, y_0) \in N$ and a set of scalars $\alpha, \beta, \gamma, \delta \in K$ (with $\alpha\delta - \beta\gamma \neq 0$) such that $(x_0, y_0) \sim (x, x_1)$ and such that $x = \alpha x_0 + \beta y_0$, $x_1 = \gamma x_0 + \delta y_0$. Since (for each $k = 1, 2, \cdots, q$),

$$P_k(\rho_{jk}x + x_1) = P_k[(\alpha\rho_{jk} + \gamma)x_0 + (\beta\rho_{jk} + \delta)y_0] = 0 \quad \forall 1 \leqq j \leqq n_k ,$$

we see (due to (2.16)) that

$$(\rho_{jk}x + x_1) \in \begin{cases} T_{G_1}(x_0, y_0) & \forall 1 \leqq j \leqq n_k , \quad 1 \leqq k \leqq p \\ T_{G_2}(x_0, y_0) & \forall 1 \leqq j \leqq n_k , \quad p + 1 \leqq k \leqq q \end{cases}$$

and, hence, that

$$\left(\frac{\alpha\rho_{jk} + \gamma}{\beta\rho_{jk} + \delta}\right) \in \begin{cases} G_1(x_0, y_0) & \forall 1 \leqq j \leqq n_k , \quad 1 \leqq k \leqq p \\ G_2(x_0, y_0) & \forall 1 \leqq j \leqq n_k , \quad p + 1 \leqq k \leqq q . \end{cases}$$

Let us put $\rho'_{jk} = (\alpha\rho_{jk} + \gamma)/(\beta\rho_{jk} + \delta)$, so that

$$\rho_{jk} = (\delta\rho'_{jk} - \gamma)/(-\beta\rho'_{jk} + \alpha) = U(\rho'_{jk})$$

for all j, k, where U is the homographic transformation given by $U(\rho) = (\delta\rho - \gamma)/(-\beta\rho + \alpha)$ for $\rho \in K_\omega$. That is,

$$\rho_{jk} \in \begin{cases} U(G_1(x_0, y_0)) & \forall 1 \leqq j \leqq n_k , \quad 1 \leqq k \leqq p \\ U(G_2(x_0, y_0)) & \forall 1 \leqq j \leqq n_k , \quad p + 1 \leqq k \leqq q \end{cases} \tag{2.18}$$

where $U(G_i(x_0, y_0)) \in D(K_\omega)$ for $i = 1, 2$, because $G_i(x_0, y_0) \in D(K_\omega)$ and U preserves the class $D(K_\omega)$ (cf. [16, p. 353], [13, p. 28]). But clearly $\omega \notin U(G_i(x_0, y_0))$ for $i = 1, 2$. For, otherwise, $\alpha/\beta \in G_1(x_0, y_0) \cup G_2(x_0, y_0)$ (since $U(\rho) = \omega$ if and only if $\rho = \alpha/\beta$) and, hence, $x = \alpha x_0 + \beta y_0 \in T_{G_1}(x_0, y_0) \cup T_{G_2}(x_0, y_0) \subseteqq E_0^{(1)} \cup E_0^{(2)}$, contradicting the choice of x already made. Now, the definition of g.c.r. implies that the sets $U(G_i(x_0, y_0))$, $i = 1, 2$, are K_0-convex g.c.r.'s of K_ω and, hence, (2.18) implies that

$$\sum_{j=1}^{n_k} \frac{1}{n_k}\rho_{jk} \in \begin{cases} U(G_1(x_0, y_0)) & \forall 1 \leqq k \leqq p \\ U(G_2(x_0, y_0)) & \forall p + 1 \leqq k \leqq q . \end{cases} \tag{2.19}$$

If we let $A_1 = m_1 + m_2 + \cdots + m_p$ and $A_2 = m_{p+1} + \cdots + m_q$, we infer from the hypotheses on the m_k that the scalars m_k/A_1 (resp. m_k/A_2) are positive elements of K_0 for $k = 1, 2, \cdots, p$ (resp. $k = p + 1, \cdots, q$) with sum as 1. This fact, together with the statements (2.19) and the K_0-convexity of the sets $U(G_i(x_0, y_0))$ for $i = 1, 2$, implies that $\mu_i/A_i \in U(G_i(x_0, y_0))$ for $i = 1, 2$, where

$$\mu_1 = \sum_{k=1}^{p}\sum_{j=1}^{n_k} \frac{m_k}{n_k}\rho_{jk},\ \mu_2 = \sum_{k=p+1}^{q}\sum_{j=1}^{n_k} \frac{m_k}{n_k}\rho_{jk}\ .$$

Therefore, there exist elements $\rho_i \in G_i(x_0, y_0)$, $i = 1, 2$, such that $\mu_i/A_i = U(\rho_i)$ for $i = 1, 2$, and we have

$$\rho_i = [(\mu_i/A_i)\alpha + \gamma]/[(\mu_i/A_i)\beta + \delta] \in G_i(x_0, y_0)\ .$$

That is,

$$[(\mu_i/A_i)\alpha + \gamma]x_0 + [(\mu_i/A_i)\beta + \delta]y_0 \in T_{G_i}(x_0, y_0)$$

and, hence, $(\mu_i/A_i)x + x_1 \in T_{G_i}(x_0, y_0)$ for $i = 1, 2$. We claim that $\mu_1 + \mu_2 \neq 0$. For, otherwise, since $A_1 + A_2 = m_1 + m_2 + \cdots + m_q = 0$, we observe that $\mu_1/A_1 = \mu_2/A_2$ and that $(\mu_1/A_1)x + x_1$ belongs to $T_{G_1}(x_0, y_0) \cap T_{G_2}(x_0, y_0)$. That is, $(\mu_1/A_1)x + x_1 \in E_0^{(1)} \cap E_0^{(2)}$, contradicting the hypothesis that $E_0^{(1)}$ and $E_0^{(2)}$ are disjoint. Hence

$$\mu_1 + \mu_2 = \sum_{k=1}^{q}\sum_{j=1}^{n_k} \frac{m_k}{n_k}\cdot\rho_{jk} \neq 0\ . \tag{2.20}$$

Since $P_k(x) \neq 0$ for all k, we obtain (cf. (2.10) and (2.11))

$$P_k(x_1, x) = -\frac{1}{n_k}\Big(\sum_{j=1}^{n_k}\rho_{jk}\Big)\cdot P_k(x) \quad \text{for} \quad 1 \leqq k \leqq q\ . \tag{2.21}$$

Finally, since $Q_k(x)\cdot P_k(x) = Q(x) \neq 0$ for all k (cf. Definition 2.2), we get (due to (2.20) and (2.21))

$$\Phi(x_1, x) = -\Big[\sum_{k=1}^{q}\sum_{j=1}^{n_k} \frac{m_k}{n_k}\cdot\rho_{jk}\Big]\cdot\Big(\prod_{k=1}^{q} P_k(x)\Big) \neq 0 \tag{2.22}$$

as was to be proved.

The above theorem deduces as corollary the following result due to Marden in terms of hermitian cones (a proper subclass of circular cones).

COROLLARY 2.6 (Marden [8, Theorem (3.1]]). *Let*

$$E_i = \{x \in E \,|\, x \neq 0;\ H_i(x, x) > 0\}\ ,\quad i = 1, 2\ ,$$

be two hermitian cones corresponding to the hermitian symmetric forms $H_i(x, y)$ *from* E^2 *to* K *such that* $(E - E_1 \cup \{0\}) \cap (E - E_2 \cup \{0\}) = \varnothing$ *and let* $P_k \in \boldsymbol{P}_{n_k}(k = 1, 2, \cdots, q)$ *such that* $P_k(x) \neq 0$ *for* $x \in E_1$ *when* $k = 1, 2, \cdots, p$ *and such that* $P_k(x) \neq 0$ *for* $x \in E_2$ *when* $k = p + 1, p + 2, \cdots, q$. *If the scalars* m_k *sotisfy the hypotheses of Theorem*

2.5, *then* $\Phi(x_1, x) \neq 0$ *for all linearly independent elements* x, x_1 *of* E *such that* $x \in E_1 \cap E_2$.

Proof. Starting with the hermitian cones $E - E_i \cup \{0\} = E_i'$ (say), $i = 1, 2$, and taking an arbitrary nucleus N of E^2, we can always get (due to Proposition 1.2) two circular mappings $G_i: N \to D(K_\omega)$ for $i = 1, 2$, such that $E_i' = E_0(N, G_i) \equiv E_0^{(i)}$(say) and such that $E_i' \cap \mathscr{L}[x, y] = T_{G_i}(x, y)$ for every $(x, y) \in N$ and $i = 1, 2$. We easily notice that

$$Z_{P_k}(x, y) \subseteqq \begin{cases} T_{G_1}(x, y) & \forall k = 1, 2, \cdots, p \\ T_{G_2}(x, y) & \forall k = p + 1, \cdots, q \end{cases}$$

for all $(x, y) \in N$. Since $E_0^{(1)}$ and $E_0^{(2)}$ are disjoint circular cones, all the hypotheses of Theorem 2.5 are satisfied and we conclude that $\Phi(x_1, x) \neq 0$ for all linearly independent elements x, x_1 of E such that $x \notin E_0^{(1)} \cup E_0^{(2)}$. Since $x \neq 0$ and since $E_0^{(i)} = E - E_i \cup \{0\}$, we see that $\Phi(x_1, x) \neq 0$ for all linearly independent elements x_1, x such that $x \in E_1 \cap E_2$. This completes the proof.

Our second application of Theorem 2.5 gives the following corollary, which is an improved version of a theorem due to Bôcher [1], [7, Theorem (20.2)], [13, Corollary (19.3)] on the vanishing of the Jacobian of two binary forms in complex variables. The improvement is in the sense that we use g.c.r.'s, whereas Bôcher used the (classical) c.r.'s in his theorem. Our result runs as follows:

COROLLARY 2.7. *Let* C_1, C_2 *be two disjoint g.c.r.'s of* C_ω *and let* $C_i^* = \{(s, t) \in \boldsymbol{C}^2 \mid (s, t) \neq 0; s/t \in C_i\}$, $i = 1, 2$. *If*

$$P_i(s, t) = \sum_{k=0}^{n} a_{ki} s^k t^{n-k}, \quad i = 1, 2,$$

are two binary forms in the complex variables s, t *such that all the nontrivial zeros of* P_i *lie in* C_i^* *for* $i = 1, 2$, *then all the nontrivial zeros of the Jacobian of* P_1 *and* P_2 *lie in* $C_1^* \cup C_2^*$. *(Note that the origin* $(0, 0) \in \boldsymbol{C}^2$ *is the trivial zero of every binary form).*

Proof. Letting $x_0 = (1, 0)$, $y_0 = (0, 1)$, $N = \{(x_0, y_0)\}$, and $G_i(x_0, y_0) = C_i$, we observe (cf. Remark 1.1) that the sets C_i^* are precisely the disjoint circular cones $E_0^{(i)} \equiv E_0(N, G_i) = T_{G_i}(x_0, y_0)$ for $i = 1, 2$, and that the P_i are basically the a.h.p.'s of degree n (from $\boldsymbol{C}^2$ to $\boldsymbol{C}$), given by

$$P_i(x) \equiv P_i(sx_0 + ty_0) = \sum_{k=0}^{n} a_{ki}\, s^k t^{n-k} \forall x = (s, t) \in \boldsymbol{C}^2, \quad i = 1, 2,$$

such that $Z_{P_i}(x_0 y_0) \subseteqq T_{G_i}(x_0, y_0)$ for $i = 1, 2$. For all element $x = (s, t)$

and $x_1 = (s_1, t_1)$ of C^2, we know [8, Equation (2.4)] that

$$P_i(x_1, x) = \frac{1}{n}\left(s_1\frac{\partial P_i}{\partial s} + t_1\frac{\partial P_i}{\partial t}\right), \quad i = 1, 2,$$

and, for $x_1 = x$, it gives

$$P_i(x) = \frac{1}{n}\left(s\frac{\partial P_i}{\partial s} + t\frac{\partial P_i}{\partial t}\right), \quad i = 1, 2.$$

If we take $q = 2$, $p = 1$, and $m_1 = -m_2 = 1$, then $\Phi(x_1, x)$ in Theorem 2.5 is given by

$$\begin{aligned}\Phi(x_1, x) &= P_1(x_1, x)\cdot P_2(x) - P_1(x)\cdot P_2(x_1, x)\\ &= \frac{1}{n^2}(s_1t - st_1)\cdot\left[\frac{\partial P_1}{\partial s}\cdot\frac{\partial P_2}{\partial t} - \frac{\partial P_1}{\partial t}\cdot\frac{\partial P_2}{\partial s}\right] \qquad (2.23)\\ &= \frac{1}{n^2}(s_1t - st_1)\cdot J(s, t), \quad \text{say},\end{aligned}$$

where $J(s, t)$ denotes the Jacobian of P_1 and P_2. Since $\Phi(x_1, x)$, the a.h.p.'s P_i, and the circular cones $E_0^{(i)} = C_i^*$ satisfy the hypotheses of Theorem 2.5, we conclude that $\Phi(x_1, x) \neq 0$ whenever x, x_1 are linearly independent and $x \notin C_1^* \cup C_2^*$, i.e., given any nonzero element $x = (s, t) \notin C_1^* U C_2^*$, we can always choose an element $x_1 = (s_1, t_1) \in C^2$ which is linearly independent to x (so that $s_1t - st_1 \neq 0$) and for which $\Phi(x_1, x) \neq 0$. The equality (2.23) then says that $J(s, t) \neq 0$. Therefore, all the nontrivial zeros of the Jacobian $J(s, t)$ lie in $C_1^* U C_2^*$, as was be to proved.

If Corollary 2.7 is restated in terms of ordinary polynomials (from C to C), it reduces essentially to an improved version of the second part of the two-circle theorem due to walsh [12], [7, Theorem (20, 1)] on the derivative of the quotient of two polynomials. The improvement is in the sense in which Corollary 2.7 improves upon Bôcher's theorem.

COROLLARY 2.8. *If all the zeros of the complex valued polynomial* $f_i(z)$ *of degree* n *lie in the g.c.r.* C_i *of* $C_\omega (i = 1, 2)$ *and if* $C_1 \cap C_2 = \varnothing$, *then all the finite zeros of the derivative of the quotient* $f(z) = f_1(z)/f_2(z)$ *lie in* $C_1 \cup C_2$.

Proof. Let us take the sets C_i^* in the manner of Corollary 2.7 and, writing $f_i(z) = \sum_{k=0}^n a_{ki}z^k$ for $i = 1, 2$, let us define

$$P_i(s, t) = t^n\cdot f_i(s/t) = \sum_{k=0}^{n} a_{ki}s^kt^{n-k} \forall s, t \in C, \quad i = 1, 2. \qquad (2.24)$$

Then the Jacobian of the binary forms P_1 and P_2 is given by (cf. [7, pp. 93-94]).

(2.25) $$J(s, t) = nt^{2(n-1)} \cdot f'(s/t)[f_2(s/t)]^2 \quad \forall s, t \in \boldsymbol{C} .$$

Next, we notice from (2.24) that s/t is a zero of f_i if and only if (s, t) is a nontrivial zero of P_i and, from (2.25), that s/t is a finite zero of f' if and only if (s, t) is a nontrivial zero of $J(s, t)$. The proof is now self-evident in view of Corollary 2.7.

REMARK. Since there do exist [14, pp. 123–125] circular cones (both hermitian and otherwise) and a.h.p.'s satisfying the hypotheses of Theorem 2.5, it follows from Proposition 1.3 that our Theorem 2.5 is a strengthened generalization of Marden's theorem expressed in Corollary 2.6.

The following example shows that Theorem 2.5 cannot be generalized for vector spaces over nonalgebraically closed fields of characteristic zero.

EXAMPLE 2.9. Let K_0 be a maximal ordered field (so that K_0 is a nonalgebraically closed field of characteristic zero [11, pp. 233, 250]) and let $C_1 = \{-1\}$ and $C_2 = \{1\}$ be two generalized circular regions of K_0 (see Remark in § 1 concerning the definition of g.c.r.'s in K_0). With $x_0 = (1, 0)$, $y_0 = (0, 1)$ as basis elements of the vector space $E = K_0^2$, if we define $N = \{(x_0, y_0)\}$ and $G_i(x_0, y_0) = C_i$ for $i = 1, 2$, then the corresponding circular cones $E_0^{(i)} \equiv E_0(N, G_i)$, for $i = 1, 2$, are disjoint. If we take two a.h.p.'s $P_1, P_2 \in \boldsymbol{P}_3$, defined by

$$\begin{aligned} P_1(x) &\equiv P_1(sx_0 + ty_0) = s^3 + 3s^2t + 3st^2 + t^3 = (s + t)^3 \\ P_2(x) &\equiv P_2(sx_0 + ty_0) = s^3 + 5s^2t + 4st^2 - 10t^3 \\ &= (s - t)[(s + 3t)^2 + t^2] \end{aligned}$$

for all $x = (s, t) \in E$, then $Z_{P_i}(x_0, y_0) \subseteqq T_{G_i}(x_0, y_0)$ for $i = 1, 2$ (since $[(s + 3t)^2 + t^2]$ cannot vanish unless $s = t = 0$ (cf. [1, p. 36])). Also, we know [8, Equation 2.4] that

$$P_1(x_1, x) = \frac{1}{3} \cdot [s_1(3s^2 + 6st + 3t^2) + t_1(3s^2 + 6st + 3t^2)]$$

$$P_2(x_1, x) = \frac{1}{3} \cdot [s_1(3s^2 + 10st + 4t^2) + t_1(5s^2 + 8st - 30t^2)]$$

for all elements $x = (s, t)$ and $x_1 = (s_1, t_1)$ in E. Let us set

$$\Phi(x_1, x) = P_1(x_1, x) \cdot P_2(x) - P_1(x) \cdot P_2(x_1, x) .$$

Now, $\Phi(x_1, x)$, the polynomials P_i, and the circular cones $E_0^{(i)}$ satisfy the hypotheses of Theorem 2.5, whereas it can be easily verified that $\Phi(x_1, x) = 0$ for the linearly independent elements $x_1 = (1, 1)$ and $x = (1 + \sqrt{69}, 2)$ in E, violating the conclusion in Theorem 2.5.

Next, we ask ourselves a natural question as to whether or not the g.c.r.'s $G_i(x, y)$ or C_i (employed in the hypotheses of Theorem 2.5 and Corollaries 2.7 and 2.8) can be replaced, in general, by g.c.r.'s adjoined with arbitrary subsets of their boundary, without effecting the conclusion therein. The answer is in the negative in view of the following

EXAMPLE 2.10. With $E = \boldsymbol{C}^2$, $K = \boldsymbol{C}$, $x_0 = (1, 0)$, $y_0 = (0, 1)$, and $N = \{(x_0, y_0)\}$, let us define the g.c.r.'s of $\boldsymbol{C}_\omega$ by

$$G_1(x_0, y_0) = \{z \in \boldsymbol{C}_\omega \mid \mathrm{Im}\,(z) > 0\} \quad \text{and} \quad G_2(x_0, y_0) = \{z \in \boldsymbol{C}_\omega \mid \mathrm{Im}\,(z) < 0\}$$

so that the corresponding circular cones $E_0^{(i)} \equiv E_0(N, G_i) = T_{G_i}(x_0, y_0)$, $i = 1, 2$, are disjoint. If we put $A_1 = G_1(x_0, y_0) \cup \{1, 2\}$, $A_2 = G_2(x_0, y_0) \cup \{-1, -2\}$, and

$$S_i = \{sx_0 + ty_0 \neq 0 \mid s, t \in \boldsymbol{C};\ s/t \in A_i\}\ , \quad i = 1, 2\ ,$$

then

$$S_1 = E_0^{(1)} \cup \{sx_0 + ty_0 \neq 0 \mid s/t = 1, 2\}\ ,$$
$$S_2 = E_0^{(2)} \cup \{sx_0 + ty_0 \neq 0 \mid s/t = -1, -2\}\ ,$$

so that S_1, S_2(resp. A_1, A_2) are disjoint subsets of E(resp. $\boldsymbol{C}_\omega$) none of which are circular cones (resp. g.c.r.'s). Next, we define

$$P_1(x) \equiv P_1(sx_0 + ty_0) = s^2 - 3st + 2t^2 = (s - t)(s - 2t)\ ,$$
$$P_2(x) \equiv P_2(sx_0 + ty_0) = s^2 + 3st + 2t^2 = (s + t)(s + 2t)\ ,$$

for all $x = (s, t) \in \boldsymbol{C}^2$. Then $P_1, P_2 \in \boldsymbol{P}_2$ such that $Z_{P_i}(x_0, y_0) \subseteqq S_i$ for $i = 1, 2$. Now, the generalized polar $\Phi(x_1, x)$ of P_1 and P_2, with $q = 2$, $p = 1$, $m_1 = -m_2 = +1$, is given by (cf. (2.23))

$$\Phi(x_1, x) = 3(s_1 t - s t_1)\cdot(s^2 - 2t^2) \tag{2.26}$$

for all elements $x = (s, t)$ and $x_1 = (s_1, t_1)$ of E. But, we see that $\Phi(x_1, x) = 0$ for the linearly independent elements $x = (\sqrt{2}, 1)$ and $x_1 = (1, 1)$, where $x \notin S_1 U S_2$. I.e., *Theorem* 2.5 *no longer holds when the g.c.r.'s* $G_i(x_0, y_0)$ *are replaced, in general, by the above sets* A_i.

In the language of Corollary 2.7, the above example says the following: The nontrivial zeros of the binary forms P_i (defined above) lie in A_i^* (cf. definition of C_i^* in Corollary 2.7) for $i = 1, 2$, but the Jacobian $J(s, t) = 12(s^2 - 2t^2) = 0$ for the element $(\sqrt{2}, 1) \notin A_1^* U A_2^*$. I.e., Corollary 2.7 *does not hold, in general, when the sets* C_i *are replaced by the above sets* A_i. Similarly, as in passing from Corollary 2.7 to Corollary 2.8, we may express the above result in terms of ordinary polynomials and infer that Corollary 2.8 does not hold, in general, when the sets C_i are replaced by the type of sets A_i chosen above.

3. A generalization of Marden's theorem. In the previous section, we have studied the generalized polars $\Phi(x_1, x)$ subject to the condition that the scalar multipliers m_k are nonzero elements of K_0 with a vanishing sum. This section primarily deals with a similar study in the case when all the m_k's are taken as positive. Our main theorem generalizes a result of Marden [8, Theorem (4.1)] and it involves essentially the generalization of a theorem each due to the author [14, Theorem (3.1)] and to Hörmander [5, Theorem 1]. We prove

THEOREM 3.1. *Let $E_0 \equiv E_0(N, G)$ be a circular cone in E and let $P_k \in \boldsymbol{P}_{n_k}(k = 1, 2, \cdots, q)$ such that $Z_{P_k}(x, y) \subseteq T_G(x, y)$ for all $(x, y) \in N$ and $k = 1, 2, \cdots, q$. If $\Phi(x_1, x)$ is a generalized polar of the product $Q(x)$ (cf. Definition 2.2) with $m_k > 0$ for $k = 1, 2, \cdots, q$, then $\Phi(x_1, x) \neq 0$ for all nonzero elements $x, x_1 \in E - E_0$.*

Proof. Take any two nonzero elements $x, x_1 \in E - E_0$. If x, x_1 are linearly dependent (i.e., if $x_1 = \alpha x$ for some nonzero scalar α), then $P_k(x_1, x) = P_k(\alpha x, x) = \alpha P_k(x)$ for all k and hence

$$\Phi(x_1, x) = \alpha\Big(\sum_{k=0}^{q} m_k\Big) \cdot \prod_{k=1}^{q} P_k(x) \neq 0 \tag{3.1}$$

due to the fact that $P_k(x) \neq 0$ for all k.

Now, we prove the theorem for the case when x, x_1 are linearly independent. Let

$$P_k(sx + tx_1) = \prod_{j=1}^{n_k} (\delta_{jk}s - \gamma_{jk}\cdot t)\ , \quad k = 1, 2, \cdots, q\ .$$

Since $P_k(x) = \delta_{1k}\cdot\delta_{2k}\cdots\delta_{n_kk} \neq 0$ and $P_k(x_1) = (-1)^{n_k}\cdot\gamma_{1k}\cdot\gamma_{2k}\cdots\gamma_{n_kk} \neq 0$ for all k, we see that $\delta_{jk}, \gamma_{jk} \neq 0$ for all j and k. Consequently, the elements $\gamma_{jk}/\delta_{jk} = \rho_{jk}$ (say) $\neq 0$ for $1 \leqq j \leqq n_k$ and $1 \leqq k \leqq q$. Now, proceeding exactly as in the proof of Theorem 2.5, we easily conclude (cf. (2.18)) that $\rho_{jk} \in U(G(x_0, y_0))$ for all j and k, where $(x_0, y_0) \in N$ such that $x = \alpha x_0 + \beta y_0$, $x_1 = \gamma x_0 + \delta y_0$, and where $U(\rho) = (\delta\rho - \gamma)/(-\rho\beta + \alpha)$ for all $\rho \in K_\omega$. As before, $U(G(x_0, y_0)) \in D(K_\omega)$. Since $x, x_1 \notin T_G(x_0, y_0)$, we notice that $\alpha/\beta, \gamma/\delta \notin G(x_0, y_0)$ and (hence) that $0, \omega \notin U(G(x_0, y_0))$. That is, $U(G(x_0, y_0))$ is a K_0-convex g.c.r. of K_ω which does not contain the origin. Hence, (2.19) and the succeeding arguments in the proof of Theorem 2.5 imply that $\mu/n \in U(G(x_0, y_0))$, where $n = n_1 + n_2 + \cdots + n_q$ and where

$$\mu = \sum_{k=1}^{q} \sum_{j=1}^{n_k} (m_k/n_k)\cdot\rho_{jk} \neq 0 \text{ (since } 0 \notin U(G(x_0, y_0)))\ .$$

Since $P_k(x) \neq 0$ for all k, we obtain (cf. (2.22))

$$\Phi(x_1, x) = -\mu \cdot \prod_{k=1}^{q} P_k(x) \neq 0$$

and the proof is complete.

The above theorem deduces as corollary the following result due to Marden and may thus be regarded (cf. Remark following Corollary 2.8) as a strengthened generalization of his theorem.

COROLLARY 3.2. (Marden [8, Theorem (4.1)]). *Let*

$$E_1 = \{x \in E \mid x \neq 0;\ H(x, x) > 0\}$$

be a hermitian cone in E, *when* $H(x, y)$ *is a hermitian symmetric form from* E^2 *to* K, *and let* $P_k \in \boldsymbol{P}_{n_k}(k = 1, 2, \cdots, q)$ *such that* $P_k(x) \neq 0$ *for all* $x \in E_1$ *and* $k = 1, 2, \cdots, q$. *If* $\Phi(x_1, x)$ *is the generalized polar of the product* $Q(x)$ (*cf. Definition* 2.2) *with* $m_k > 0$ *for* $k = 1, 2, \cdots, q$, *then* $\Phi(x_1, x) \neq 0$ *for all nonzero elements* $x, x_1 \in E_1$.

Proof. The proof is exactly similar to that of Corollary 2.6.

The following corollary is an immediate consequence of Theorem 3.1. If $q = 1$, this corollary reduces essentially to the author's generalization [14, Theorem 3.1] of Laguerre's theorem, and if, in addition, E_0 is taken as a hermitian cone, it is essentially (due to Remark 2.4) a result due to Hörmander [5, Lemma 2].

COROLLARY 3.3. *Let* $E_0 \equiv E_0(N, G)$ *be a circular cone in* E *and let* $P_k \in \boldsymbol{P}_{n_k}(k = 1, 2, \cdots, q)$ *such that* $Z_{P_k}(x, y) \subseteq T_G(x, y)$ *for all* $(x, y) \in N$ *and* $k = 1, 2, \cdots, q$. *If* $Q(x) = P_1(x)P_2(x) \cdots P_q(x)$ *then the first polar* $Q(x_1, x) \neq 0$ *for all nonzero elements* $x, x_1 \in E - E_0$.

Proof. The proof is obvious in view of Proposition 2.3 and Theorem 3.1.

REMARK. In view of the examples given earlier by the author [14, p. 122], Corollary 3.3 and hence Theorem 3.1 cannot be further generalized in the two directions already discussed in case of Theorem 2.5.

4. On two-circle theorems of Walsh. In Theorem 2.5, the circular cones $E_0^{(i)}(i = 1, 2)$ were assumed to be disjoint and the constants m_k were taken as nonzero elements of K_0 such that $\sum_{k=1}^{q} m_k = 0$, whereas Theorem 3.1 uses only positive elements $m_k \in K_0$ (so that $\sum_{k=1}^{q} m_k \neq 0$) and utilizes only one circular cone. In this section, we study the same problem for the case when the constants m_k are nonzero elements of K_0 such that $\sum_{k=1}^{q} m_k \neq 0$ and the two cones $E_0^{(1)}$ and $E_0^{(2)}$ are not necessarily disjoint. In fact we establish two main results in this section. The first one, which is somewhat like a theorem due to Marden [8, Theorem (4.2)], deduces as corollary the first part of Walsh's two-circle theorem [12], [7, Theorem (20.1)] on the critical points of rational functions. (The second part of Walsh theorem has already been considered as a corollary of Theorem 2.5.) Our second result is essentially a generalization of Walsh's two-circle theorem [7, Theorem (19, 1)] on the critical points of the product of two polynomials. Before we take up these results, we give the following definition and some relevant explanations.

DEFINITION 4.1. Given distinct elements $\rho_1, \rho_2, \rho_3 \in K$, we define the *cross-ratio mapping* (with respect to ρ_1, ρ_2, ρ_3) to be the homographic transformation [16, p. 353], [13, pp. 24–25] $h: K_\omega \to K_\omega$ given by

$$(4.1) \qquad h(\rho) = \frac{\rho - \rho_2}{\rho - \rho_3} \cdot \frac{\rho_1 - \rho_3}{\rho_1 - \rho_2} = (\rho, \rho_1, \rho_2, \rho_3) \quad \forall \rho \in K_\omega .$$

We call $(\rho, \rho_1, \rho_2, \rho_3)$ as the *cross-ratio* of ρ with ρ_1, ρ_2, ρ_3. In the case when any one of the ρ_i's is taken as ω, we define the corresponding cross-ratio to be the expression got by deleting in (4.1) the factors which thereby involve ω. E.g., $(p, \omega, \rho_2, \rho_3) = (\rho - \rho_2)/(\rho - \rho_3)$, etc.

It is trivial to verify that the homographic transformation in (4.1) maps ρ_1, ρ_2, ρ_3 to $1, 0, \omega$, respectively, and that there is no other homographic transformation with this property. Consequently, identity mapping is the only homographic transformation which can map $1, 0, \omega$ to $1, 0, \omega$, respectively. Furthermore, cross-ratios are invariant under every homographic transformation T, i.e., $(\rho, \rho_1, \rho_2, \rho_3) = (T\rho, T\rho_1, T\rho_2, T\rho_3)$. This follows from the fact that T^{-1} is also a homographic transformation and that hT^{-1} is a homographic transformation which maps $T\rho_1, T\rho_2, T\rho_3$ to $1, 0, \omega$, respectively. Now we prove

THEOREM 4.2. *If all the hypotheses of Theorem* 2.5 *are assumed, except that the circular cones* $E_0^{(i)} (i = 1, 2)$ *are not necessarily disjoint and that* (2.17) *is replaced by the condition* $\sum_{k=1}^{q} m_k \neq 0$, *and if* $A_1 = \sum_{k=1}^{p} m_k$ *and* $A_2 = \sum_{k=p+1}^{q} m_k$, *then* $\Phi(x_1, x) \neq 0$ *for all linearly independent elements* x, x_1 *of* E *such that* $x_1 \in E - E_0^{(1)} \cap E_0^{(2)}$ *and* $x \in E - E_0^{(1)} \cup E_0^{(2)} \cup T_S(x_0, y_0)$, *where* $(x_0, y_0) \in N \cap \mathscr{L}[x, x_1]$, $x_1 = \gamma x_0 + \delta y_0$, *and where*

(4.2) $S(x_0, y_0) = \left\{\rho \in K_\omega \mid (\rho, \gamma/\delta, \rho_1, \rho_2) = \frac{-A_1}{A_2};\ \rho_i \in G_i(x_0, y_0),\ i = 1, 2\right\}$.

(Of course, $\Phi(x_1, x) \neq 0$ for any two nonzero and linearly dependent elements x, x_1 such that $x \in E - E_0^{(1)} \cap E_0^{(2)}$.)

Proof. The statement within parenthesis is self-evident in view of (3.1). In order to prove the other case, we first observe that every linearly independent pair (x, x_1) of elements $x, x_1 \in E$ determines a unique element $(x_0, y_0) \in N \cap \mathscr{L}[x, x_1]$, a unique set of scalars $\alpha, \beta, \gamma, \delta$ (with $\alpha\delta - \beta\gamma \neq 0$) such that $x = \alpha x_0 + \beta y_0$ and $x_1 = \gamma x_0 + \delta y_0$, and, thereby, a unique set $S(x_0, y_0)$ defined by (4.2). Let us take two linearly independent elements x, x_1 of E such that $x_1 \in E - E_0^{(1)} \cap E_0^{(2)}$ and $x \in E - E_0^{(1)} \cup E_0^{(2)} \cup T_S(x_0, y_0)$, where $S(x_0, y_0)$ is the unique subset of K_ω determined in the above manner by the pair (x, x_1). If $P_k(sx + tx_1)$ is given by (2.8), then proceeding as in the proof of Theorem 2.5 we see that $\mu_i/A_i \in U(G_i(x_0, y_0))$ for $i = 1, 2$, where $U(\rho) = (\delta\rho - \gamma)/(-\beta\rho + \alpha)$ for $\rho \in K_\omega$ and where

$$\mu_1 = \sum_{k=1}^{p} \sum_{j=1}^{n_k} \frac{m_k}{n_k} \rho_{jk} \quad \text{and} \quad \mu_2 = \sum_{k=p+1}^{q} \sum_{j=1}^{n_k} \frac{m_k}{n_k} \rho_{jk} .$$

At this point, we note that μ_1 and μ_2 cannot vanish simultaneously. For, otherwise, $0 \in U(G_1(x_0, y_0)) \cap U(G_2(x_0, y_0))$ and, therefore, γ/δ would lie in $G_1(x_0, y_0) \cap G_2(x_0, y_0)$. This would imply that $x_1 = \gamma x_0 + \delta y_0 \in T_{G_1}(x_0, y_0) \cap T_{G_2}(x_0, y_0)$, contradicting the fact that $x_1 \notin E_0^{(1)} \cap E_0^{(2)}$. Next, we observe that $\mu_1 + \mu_2 \neq 0$ whenever $\mu_1 = 0 \neq \mu_2$ or $\mu_1 \neq 0 = \mu_2$. In case, however, $\mu_1, \mu_2 \neq 0$, we again show that $\mu_1 + \mu_2 \neq 0$ as follows: Since μ_i/A_i belongs to $U(G_i(x_0, y_0))$ for $i = 1, 2$, there exist elements $\rho_i \in G_i(x_0, y_0)$ such that $\mu_i/A_i = U(\rho_i) = (\delta\rho_i - \gamma)/(-\beta\rho_i + \alpha)$ for $i = 1, 2$. If (on the contrary) $\mu_1 + \mu_2 = 0$, then $\mu_1/\mu_2 = -1$ and $\gamma/\delta, \rho_1, \rho_2$ are distinct elements (since $\mu_1, \mu_2 \neq 0$ and $A_1/A_2 \neq -1$) and hence

$$\frac{\delta\rho_2 - \gamma}{-\beta\rho_2 + \alpha} \cdot \frac{-\beta\rho_1 + \alpha}{\delta\rho_1 - \gamma} = -A_1/A_2 .$$

That is (cf. Definition 4.1),

$$(\alpha/\beta, \gamma/\delta, \rho_1, \rho_2) = \frac{\alpha/\beta - \rho_1}{\alpha/\beta - \rho_2} \cdot \frac{\gamma/\delta - \rho_2}{\gamma/\delta - \rho_1} = -A_1/A_2$$

and, hence, $\alpha/\beta \in S(x_0, y_0)$. This implies at once that $x = \alpha x_0 + \beta y_0 \in T_S(x_0, y_0)$, contradicting the choice of x already made above. (In the above arguments, let us note that β and δ cannot vanish simultaneously (since $A_1/A_2 \neq -1$).) We have, therefore, shown that in all cases $\mu_1 + \mu_2 \neq 0$. Finally, the proof follows from (2.20) – (2.22).

The above theorem leads to the following corollary, which is the first part of the (so-called) two-circle theorem due to Walsh on the critical points of rational functions. In the following result we shall write $D(c, r) = \{z \in C \mid |z - c| \leqq r\}$ and call it a *disc* with center c and radius r.

COROLLARY 4.3 (Walsh [7, Theorem (20, 1)]). *If f_1 (resp. f_2) is a polynomial from C to C of degree n_1 (resp. n_2) such that all the zeros of f_1 (resp. f_2) lie in the disc $D(c_1, r_1) \equiv D_1$ (resp. $D(c_2, r_2) \equiv D_2$) and if $n_1 \neq n_2$, then all the finite zeros of the derivative of the quotient $f(z) = f_1(z)/f_2(z)$ lie in $\bigcup_{i=1}^{3} D(c_i, r_i)$, where*

$$c_3 = \frac{n_2 c_1 - n_1 c_2}{n_2 - n_1}, \quad r_3 = \frac{n_2 r_1 + n_1 r_2}{|n_2 - n_1|} . \tag{4.3}$$

Proof. Letting $x_0 = (1, 0)$, $y_0 = (0, 1)$, $N = \{(x_0, y_0)\}$, $G_i(x_0, y_0) = D_i$, $f_i(z) = \sum_{k=0}^{n_i} a_{ki} z^k$ for $i = 1, 2$, we notice (cf. Remark 1.1) that the sets

$$E_0^{(i)} \equiv E_0(N, G_i) = \{sx_0 + ty_0 \neq 0 \mid (s, t) \in C^2, s/t \in D_i\} (i = 1, 2)$$

are circular cones in C^2 and that the mappings $P_i \colon C^2 \to C$, defined by

$$P_i(x) \equiv P_i(sx_0 + ty_0) = \sum_{k=0}^{n_i} a_{ki} s^k t^{n_i - k} \quad \forall x = (s, t) \in C^2$$

for $i = 1, 2$, are a.h.p.'s of degree n_i such that $Z_{P_i}(x_0, y_0) \subseteqq T_{G_i}(x_0, y_0)$ for $i = 1, 2$. Now the generalized polar $\Phi(x_1, x)$ of the product $P_1(x)P_2(x)$, given by

$$\Phi(x_1, x) = n_1 P_1(x_1, x) P_2(x) - n_2 P_1(x) P_2(x_1, x) \tag{4.4}$$

for all elements $x = (s, t)$ and $x_1 = (s_1, t_1)$ of C^2, satisfies all the hypotheses of Theorem 4.2 with $m_1 = A_1 = n_1$ and $m_2 = A_2 = -n_2$. For the special choice of x_1 as x_0 (so that $s_1 = 1$ and $t_1 = 0$), we proceed as in the proof of Corollary 2.7 and observe that (for nonzero elements x and for $i = 1, 2$) $P_i(x) = t^{n_i} f_i(s/t) \partial P_i/\partial s = t^{n_i - 1} f_i'(s/t)$, $\partial P_i/\partial t = n_i t^{n_i - 1} f_i(s/t) - s t^{n_i - 2} f_i'(s/t)$ and (hence) that

$$\begin{aligned} \Phi(x_0, x) &= \frac{1}{n_1 n_2}\left[n_1 \frac{\partial P_1}{\partial s}\left(s\frac{\partial P_2}{\partial s} + t\frac{\partial P_2}{\partial t}\right) - n_2 \frac{\partial P_2}{\partial s}\left(s\frac{\partial P_1}{\partial s} + t\frac{\partial P_1}{\partial t}\right)\right] \\ &= t^{n_1 + n_2 - 1} \cdot [f_1'(s/t) f_2(s/t) - f_1(s/t) f_2'(s/t)] \\ &= t^{n_1 + n_2 - 1} \cdot f'(s/t) \cdot \{f_2(s/t)\}^2 . \end{aligned} \tag{4.5}$$

Since $x_0 \notin E_0^{(1)} \cap E_0^{(2)}$, Theorem 4.2 implies that $\Phi(x_0, x) \neq 0$ whenever x is linearly independent to x_0 such that $x \notin E_0^{(1)} \cup E_0^{(2)} \cup T_S(x_0, y_0)$. That is $\Phi(x_0, x) \neq 0$ for all elements $x = (s, t)$ for which $t \neq 0$ and

$s/t \notin D_1 \cup D_2 \cup S(x_0, y_0)$, where $S(x_0, y_0)$ is given by (since $\gamma = 1$ and $\delta = 0$ in the notations of Theorem 4.2).

$$\begin{aligned} S(x_0, y_0) &= \left\{\rho \in C_\omega \,|\, (\rho, \omega, \rho_1, \rho_2) = \frac{n_1}{n_2};\ \rho_i \in G_i(x_0, y_0),\ i = 1, 2\right\} \\ &= \left\{\rho \in C_\omega \,|\, (\rho - \rho_1)/(\rho - \rho_2) = \frac{n_1}{n_2};\ \rho_i \in D_i;\ i = 1, 2\right\} \\ &= \{(n_2\rho_1 - n_1\rho_2)/(n_2 - n_1) \,|\, \rho_1 \in D_1,\ \rho_2 \in D_2\} \\ &= D(c_3, r_3) \quad \text{(due to (4.3))}. \end{aligned}$$

From (4.5) it follows that $f'(s/t) \neq 0$ for all $s, t \in C$ such that $t \neq 0$ and $s/t \notin D_1 \cup D_2 \cup D(c_3, r_3)$ and, hence, the corollary follows.

In the above theorem, the constants $m_k \in K_0$ have been assumed to have a nonvanishing sum, with at least one $m_k > 0$ and at least one $m_k < 0$. Next, we deal with a case when all the m_k's in Theorem 4.2 are taken as positive elements of K_0 and obtain the following corresponding result.

THEOREM 4.4. *If the a.h.p.'s $P_k (k = 1, 2, \cdots, q)$ and the circular cones $E_0^{(1)}$ and $E_0^{(2)}$ (not necessarily disjoint) satisfy the conditions* 2.16 *of Theorem* 2.5 *for some $1 \leqq p < q$ and if $m_k > 0$ for $k = 1, 2, \cdots, q$, then $\Phi(x_1, x) \neq 0$ for all linearly independent elements x, x_1 of E such that $x_1 \in E - E_0^{(1)} \cap E_0^{(2)}$ and $x \in E - E_0^{(1)} \cup E_0^{(2)} \cup T_S(x_0, y_0)$, where $S(x_0, y_0)$ is as defined in Theorem* 4.2. *(Of course, $\Phi(x_1, x) \neq 0$ whenever x, x_1 are nonzero and linearly dependent such that $x \in E - E_0^{(1)} \cup E_0^{(2)}$.)*

Proof. The proof is exactly the same as in Theorem 4.2.

An application of this theorem furnishes the following result on the zeros of the *formal derivative* of the product of two polynomials (from K to K). For $K = C$, this result reduces essentially to the two-circle theorem due to Walsh [7, Theorem (19, 1)]. By the *formal derivative* [16, p. 360], [14, p. 121] f' of a polynomial $f(z) = \sum_{k=0}^{n} a_k z^k$ (from K to K), we mean the polynomial $f'(z) = \sum_{k=1}^{n} k a_k z^{k-1}$. If, however, $P(s, t)$ is a polynomial (from K^2 to K) in s and t, we define the *formal partial derivative* $\partial P/\partial s$ of P with respect to s (say) as the formal derivative of P when P is regarded as a polynomial in s(t being held fixed). In the following corollary, we shall write $D(c, r) = \{z \in K \,||\, z - c| \leqq r\}$ and call it a *ball* with center c (c being in K) and radius r (r being in K_{0+}). The balls are usually called discs when $K = C$.

COROLLARY 4.5. *If f_1 (resp. f_2) is a polynomial from K to K of degree n_1 (resp. n_2) such that all the zeros of f_1 (resp. f_2) lie in the ball $D(c_1, r_1) \equiv D_1$(resp. $D(c_2, r_2) \equiv D_2$), then all the zeros of the formal derivative of the product $f(z) = f_1(z) \cdot f_2(z)$ lie in $\bigcup_{i=1}^{3} D(c_i, r_i)$, where*

$$c_3 = \frac{n_1 c_2 + n_2 c_1}{n_1 + n_2}, \quad r_3 = \frac{n_1 r_2 + n_2 r_1}{n_1 + n_2}. \tag{4.6}$$

Proof. Proceeding as in the proof of Corollary 4.3, with C replaced by K and $\Phi(x_1, x)$ in (4.4) replaced by

$$\Phi(x_1, x) = n_1 P_1(x_1, x) \cdot P_2(x) + n_2 P_1(x) \cdot P_2(x_1, x),$$

we notice that $\Phi(x_1, x)$ satisfies the hypotheses of Theorem 4.4 with $m_1 = A_1 = n_1$, $m_2 = A_2 = n_2$. Following the computation used for obtaining (4.5), we can easily verify that (for all nonzero elements $x = (s, t) \in K^2$)

$$\begin{aligned} \Phi(x_0, x) &= t^{n_1+n_2-1} \cdot [f_1'(s/t) f_2(s/t) + f_1(s/t) \cdot f_2'(s/t)] \\ &= t^{n_1+n_2-1} \cdot f'(s/t), \end{aligned}$$

where $x_1 = x_0 \notin E_0^{(1)} \cap E_0^{(2)}$, (since $\omega \notin G_i(x_0, y_0) \equiv D_i$ for $i = 1, 2$) and f_1', f_2', f' denote the formal derivatives of f_1, f_2, f respectively. By Theorem 4.4, $\Phi(x_0, x) \neq 0$ whenever the element $x = (s, t)$ is linearly independent to x_0 and is such that $x \notin E_0^{(1)} \cup E_0^{(2)} \cup T_S(x_0, y_0)$. That is, $f'(s/t) \neq 0$ for all $s, t \in K$ such that $t \neq 0$ and $s/t \notin D_1 \cup D_2 \cup S(x_0, y_0)$, where

$$\begin{aligned} S(x_0, y_0) &= \{\rho \in K_\omega \mid (\rho, \omega, \rho_1, \rho_2) = -n_1/n_2;\ \rho_i \in G_i(x_0, y_0),\ i = 1, 2\} \\ &= \{\rho \in K_\omega \mid (\rho - \rho_1)/(\rho - \rho_2) = -n_1/n_2;\ \rho_i \in D_i,\ i = 1, 2\} \\ &= \{(n_1\rho_2 + n_2\rho_1)/(n_1 + n_2) \mid \rho_1 \in D_1,\ \rho_2 \in D_2\} \\ &= D(c_3, r_3) \quad \text{(due to (4.6))}. \end{aligned}$$

Hence, all the zeros of f lie in $\bigcup_{i=1}^{3} D(c_i, r_i)$, as was to be proved.

If Theorem 4.4 is specialized for the case when $G_1(x, y) = G_2(x, y) = G(x, y)$(say) for all $(x, y) \in N$ (so that $E_0^{(1)} = E_0^{(2)} = E_0(N, G) = E_0$, say) we easily conclude that $\Phi(x_1, x) \neq 0$ for all linearly independent elements x, x_1 such that $x_1 \in E - E_0$ and $x \in E - E_0 \cup T_S(x_0, y_0)$, where $(x_0, y_0) \in N \cap \mathscr{L}[x, x_1]$, $x_1 = \gamma x_0 + \delta y_0$ and (cf. (4.2))

$$S(x_0, y_0) = \{\rho \in K_\omega \mid (\rho, \gamma/\delta, \rho_1, \rho_2) = -A_1/A_2;\ \rho_1, \rho_2 \in G(x_0, y_0)\}.$$

Since $\sigma = \gamma/\delta \notin G(x_0, y_0)$ and since ρ_1, ρ_2 can vary over only distinct elements of $G(x_0, y_0)$, we see that every element ρ of $S(x_0, y_0)$ is given by

$$\rho = (A_1\rho_2 + A_2\rho_1)/(A_1 + A_2) \qquad \text{if} \quad \sigma = \omega$$

or

$$\varphi_\sigma(\rho) = [A_1\varphi_\sigma(\rho_2) + A_2\varphi_\sigma(\rho_1)]/(A_1 + A_2) \qquad \text{if} \quad \sigma \neq \omega$$

for some distinct elements ρ_1, ρ_2 in $G(x_0, y_0)$, where $\varphi_\sigma(z) = 1/(z - \sigma) \forall z \in K_\omega$. In the first case, $G(x_0, y_0)$ happens to be K_0-convex (since $\omega \notin G(x_0, y_0)$) and $\rho \in G(x_0, y_0)$. While in the second case, $\varphi_\sigma(G(x_0, y_0))$ is K_0-convex (cf. definition of g.c.r.) and so $\varphi_\sigma(\rho) \in \varphi_\sigma(G(x_0, y_0))$ and $\rho \in G(x_0, y_0)$. Consequently, in either case, we discover that $S(x_0, y_0) \subseteq G(x_0, y_0)$ and so $T_S(x_0, y_0) \subseteq T_G(x_0, y_0) \subseteq E_0$. Therefore, we conclude that $\Phi(x_1, x) \neq 0$ for all linearly independent elements $x, x_1 \in E - E_0$. This fact together with the statement of Theorem 4.4 within parenthesis, suggests that in the present set up Theorem 4.4 reduces essentially to Theorem 3.1. In view of this and the remark following Corollary 3.3 we again notice that Theorem 4.4 and Corollary 4.5 cannot be further generalized in the two directions in which Theorem 2.5 could not be extended.

Next, we give an example to show that Theorem 4.2 cannot be generalized to vector spaces over nonalgebraically closed fields of characteristic zero.

EXAMPLE 4.6. In the notations of Example 2.9, take $E = K_0^2$, $G_i(x_0, y_0) = C_i = \{0\}$ for $i = 1, 2$ (so that the circular cones $E_0^{(1)}$ and $E_0^{(2)}$ are identical), and define

$$P_1(x) \equiv P_1(sx_0 + ty_0) = s^3 + st^2 = s(s^2 + t^2)$$
$$P_2(x) \equiv P_2(sx_0 + ty_0) = s^3$$
$$\Phi(x_1, x) = 2P_1(x_1, x)\cdot P_2(x) - P_1(x)\cdot P_2(x_1, x)$$

for all elements $x = (s, t)$ and $x_1 = (s_1, t_1)$ of E. Proceeding as in Example 2.9, we can easily verify that $Z_{P_i}(x_0, y_0) \subseteq T_{G_i}(x_0, y_0)$ for $i = 1, 2$ and that $\Phi(x_1, x) = (1/3)s^3(3s^2 - t^2)$ if we take $x_1 = x_0 = (1, 0)$. In the notations of Theorem 4.2, let us note (since $\gamma = 1$, $\delta = 0$, $A_1 = 2$, $A_2 = -1$) that $S(x_0, y_0)$ consists of all elements $\rho \in K_\omega$ such that $(\rho, \omega, \rho_1, \rho_2) = (\rho - \rho_1)/(\rho - \rho_2) = 2$. That is, $S(x_0, y_0) = \emptyset$. Now, the polynomials P_i and the generalized polar $\Phi(x_1, x)$ satisfy all the hypotheses of Theorem 4.2, but $\Phi(x_1, x) = 0$ for the linearly independent elements $x_1 = (1, 0)$ and $x = (1, \sqrt{3})$, where $x_1 \notin E_0^{(1)} \cap E_0^{(2)}$ and $x \notin E_0^{(1)} \cap E_0^{(2)} \cup T_S(x_0, y_0)$, contrary to the conclusion in Theorem 4.2.

FINAL REMARK. At the end, let us recall that the condition "$E_0^{(1)} \cap E_0^{(2)} = \emptyset$" has been used as hypothesis only in case of Theorem 2.5. In what follows, we show that this hypothesis is necessary in order for the conclusion in Theorem 2.5 to hold. To this effect, we reconsider Example 2.9 with necessary modifications: In fact, we replace the maximal ordered field K_0 by an algebraically closed field K of characteristic zero and take the same polynomials $P_1(x)$, $P_2(x)$, $\Phi(x_1, x)$ and the same g.c.r. $C_1 = G_1(x_0, y_0)$ as in Example 2.9, but this time we define

$$C_2 = G_2(x_0, y_0) = \{z \in K_\omega \mid |8z + 9| \leqq 17\}$$
$$= \{z \in K_\omega \mid 64z\bar{z} + 72(z + \bar{z}) - 208 \leqq 0\} .$$

Indeed, $C_2 \in D(K_\omega)$ [14, p. 116] and the elements $z = \pm 1$ and $z = -3 \pm i$ belong to C_2, so that $C_1 \cap C_2 = \{-1\}$ and (hence) $E_0^{(1)} \cap E_0^{(2)} \neq \varnothing$. Also $Z_{P_i}(x_0, y_0) \subseteqq T_{G_i}(x_0, y_0)$ for $i = 1, 2$. Therefore, all the hypotheses of Theorem 2.5 are satisfied by the polynomials $P_1(x)$, $P_2(x)$, $\Phi(x_1, x)$ and the circular cones $E_0^{(1)}$, $E_0^{(2)}$ (except that they are disjoint), whereas $\Phi(x_1, x) = 0$ (see Example 2.9) for the linearly independent elements $x_1 = (1, 1)$ and $x = (1 + \sqrt{69}, 2) \notin E_0^{(1)} \cup E_0^{(2)}$ (since $(1 + \sqrt{69})/2 \notin C_2$). This is contrary to the conclusion in Theorem 2.5.

REFERENCES

1. M. Bôcher, *A problem in Statics and its relation to certain algebraic invariants*, Proc. Amer. Acad. Arts Sci., **40** (1904), 469–484.
2. N. Bourbaki, *Éléments de mathématique.* XIV. Part 1, *Let structures fondamentales de l'analyse.* Livre II: *Algèbre.* Chap. VI: *Groupes et corps ordonnés*, Actualitiés Sci. Indust., no. 1179, Hermann, Paris, 1952.
3. J. H. Grace, *On the zeros of a polynomial*, Proc. Cambridge Philos. Soc., **11** (1900–1902), 352–357.
4. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, rev. ed., Amer. Math. Soc. Colloq. Publ., Vol. 31, Amer. Math. Soc., Providence, R. I., 1957.
5. L. Hörmander, *On a theorem of Grace*, Math. Scand., **2** (1954), 55–64.
6. E. Laguerre, *Oeuvres* 1, Paris, 1898.
7. M. Marden, *Geometry of polynomials*, rev. ed., Math. Surveys, no. 3, Amer. Math. Soc., Providence, R. I., 1966.
8. ———, *A generalization of a theorem of Bôcher*, SIAM J. Numer. Anal., **3** (1966), 269–275.
9. ———, *On composite abstract homogeneous polynomials*, Proc. Amer. Math. Soc., **22** (1969), 28–33.
10. G. Szegö, *Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen*, Math. Z., **13** (1922), 28–55.
11. B. L. Van der Waerden, *Algebra*, Vol. I, 4th ed., Die Grundlehren der math. Wissenschaften, Band 33, Springer-Verlag, Berlin, 1955; English transl., Ungar, New York, 1970.
12. J. L. Walsh, *On the location of the roots of the Jacobian of two binary forms, and of the derivative of a rational function*, Trans. Amer. Math. Soc., **22** (1921), 101–116.
13. N. Zaheer, *Null-sets of abstract homogeneous polynomials in vector spaces*, Doctoral Thesis, Univ. of Wisconsin, Milwaukee, Wis., 1971.
14. ———, *On polar relations of abstract homogeneous polynomials*, Trans. Amer. Math. Soc., **218** (1976), 115–131.
15. ———, *On composite abstract homogeneous polynomials*, Trans. Amer. Math. Soc., **228** (1977), 345–358.

16. S. P. Zervos, *Aspects modernes de la localisation des zéros des polynômes d'une variable*, Ann. Sci. Ecole Norm Sup. (3) **77** (1960), 303–410.

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