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IDEALS AND RADICALS OF SOME ENDOMORPHISM RINGS

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Let R be the full ring of endomorphisms of a reduced abelian p-group G. A description is given, in terms of its action on G, of the Jacobson radical $J(R)$ of R for the case that G is sufficiently projective. Other ideals and radicals of R and their relation to $J(R)$ are discussed.

1. The results. In order to study the structure of a ring R, one of the most important tools is the investigation of radicals. In this context, the following ideals are of interest. Throughout, the word ideal will mean two-sided ideal.

$N(R)$: sum of all nilpotent ideals of R;
$B(R)$: intersection of all prime ideals of R (Baer radical);
$L(R)$: sum of all locally nilpotent ideals of R (Levitzki radical);
$K(R)$: sum of all nil ideals of R (Koethe radical);
$J(R)$: sum of all quasi-regular ideals of R (Jacobson radical).

One has

$$N(R) \subseteqq B(R) \subseteqq L(R) \subseteqq K(R) \subseteqq J(R) \,, \tag{1.1}$$

and all of these ideals, with the possible exception of $J(R)$, are nil [4, pp. 193-197].

In this note we consider the case where $R = \operatorname{End} G$ is the full ring of endomorphisms of a reduced p-primary abelian group G. Our interest will be focused primarily on the Jacobson radical.

In [3] and [9], respectively, lower and upper bounds for $J(\operatorname{End} G)$ where given, namely

$$I(G) \subseteqq J(\operatorname{End} G) \subseteqq H(G) \tag{1.2}$$

which are defined as follows. Note that we write mappings to the right; throughout, λ is a nonzero ordinal such that $p^\lambda G = 0$. Then let $I(G)$ be the set of all $\varepsilon \in \operatorname{End} G$ for which there exists a finite sequence of ordinals $\sigma_0, \sigma_1, \cdots, \sigma_{k+1}$, where $k = k(\varepsilon)$, such that

$$0 = \sigma_0 < \sigma_1 < \cdots < \sigma_k < \sigma_{k+1} = \lambda$$

and, for $i = 0, \cdots, k$, $p^{\sigma_i}G[p]\varepsilon \subseteqq p^{\sigma_{i+1}}G$. Let $H(G)$ be the set of all $\varepsilon \in \operatorname{End} G$ such that, for each nonnegative integer n, $p^n G[p]\varepsilon \subseteqq p^{n+1}G$. Clearly, both $I(G)$ and $H(G)$ are ideals of $\operatorname{End} G$. R. S. Pierce has shown that $J(\operatorname{End} G) = H(G)$ if G is torsion-complete, and $J(\operatorname{End} G) \subsetneqq H(G)$ if G contains an unbounded direct summand which is a direct sum of cyclic groups [9, pp. 287, 288].

We will prove the following theorem. A reduced abelian p-group G is called *sufficiently projective* if every countable subset of G is contained in a totally projective direct summand of G.

THEOREM 1.3. *If G is sufficiently projective then $J(\operatorname{End} G) = I(G)$.*

Clearly, totally projective groups are sufficiently projective; the totally projective p-groups without elements of infinite height are just the direct sums of cyclic p-groups [1, p. 89; 11, p. 251]. Thus, Theorem 1.3 generalizes Theorem 3.8 of [3] and the p-group version of (2.3) in [7].

As a consequence we obtain a characterization of the quasi-regular ideals of some endomorphism rings.

THEOREM 1.4. *Let G be sufficiently projective. Then the following properties of the ideal I of $\operatorname{End} G$ are equivalent.*

(i) *I is quasi-regular.*

(ii) *I induces in $G[p]$ a nil ring of endomorphisms.*

(iii) *The restriction of I to $G[p]$ is locally nilpotent.*

Radicals of endomorphism rings other than the Jacobson radical, have received little attention in the past. If G is unbounded then $R = \operatorname{End} G$ contains nil-ideals which are not nilpotent, for example $N(R)$ (see Corollary 2.5 below). This property characterizes the unbounded p-groups as the following result shows.

THEOREM 1.5. *Let R be the endomorphism ring of a reduced abelian p-group G. Then the following conditions are equivalent.*

(i) *$N(R)$ is nilpotent.*

(ii) *$J(R)$ is nilpotent.*

(iii) *$J(R)$ is nil.*

(iv) *$J(R) = K(R)$.*

(v) *$N(R) = B(R) = L(R) = K(R) = J(R)$.*

(vi) *$J(R) = N(R)$.*

(vii) *G is bounded.*

As pointed out in [5, p. 312], not all radicals that have been defined in the past are of equal importance. The Jacobson radical has been so successful since it seems to have a strong bearing on the structure of a ring. This is in coincidence with our investigations of endomorphism rings. The characterization of $J(\operatorname{End} G)$ in terms of its action on G seems to be more tangible than the characterization of the other ideals of $R = \operatorname{End} G$ in (1.1). Moreover, in all cases such a characterization of $J(R)$ has been obtained [2, p. 27; 7, p. 170; 9, p. 287; Theorem 1.3 above], it was possible to completely describe $J(R)$ in terms of its action on $G[p]$. That this is not possible for the other ideals of (1.1), can be seen from the following result. Throughout, for subgroups S and T of G, $S \subseteq T$, $\mathrm{Ann}\,(T/S)$ denotes the set of all $\varepsilon \in \mathrm{End}\, G$ such that $T\varepsilon \subseteq S$, and $\mathrm{Ann}\, T = \mathrm{Ann}\,(T/0)$. Clearly, if S and T are fully invariant, then $\mathrm{Ann}\,(T/S)$ is an ideal of $\mathrm{End}\, G$.

THEOREM 1.6. *Let R be the endomorphism ring of a sufficiently projective abelian p-group G without elements of infinite height and let $J = \mathrm{Ann}\, G[p]$. Then*

$$J(R) = K(R) + J = L(R) + J = B(R) + J = N(R) + J\,.$$

We conclude with a remark on various classes of p-groups.

The significance of the class of totally projective p-groups lies in the fact that it is the largest natural class of p-groups distinguishable by certain cardinal invariants [1, p. 100]. Recently, a number of new classes of p-groups have been introduced which properly contain the totally projectives and, for some of them, a classification by invariants has been obtained. Among those we mention Warfield's S-groups [13, 14] and the C_λ-groups of Megibben and Wallace [8, 12] where λ is an uncountable limit ordinal. However, for p-groups without elements of infinite height all of these classes coincide and reduce to the class of direct sums of cyclic groups which are the totally projective groups of length at most ω. This is not the case for the class of sufficiently projective p-groups introduced in this article: there exist sufficiently projective p-groups of length ω which are not direct sums of cyclic groups and, hence, not totally projective [1; p. 50].

2. The proofs.

Throughout the following, $R = \mathrm{End}\, G$. We postpone the proof of Theorem 1.3 and start with a few observations on endomorphism rings of arbitrary reduced p-groups. As above, λ is a nonzero ordinal such that $p^\lambda G = 0$.

LEMMA 2.1. *The restriction of $I(G)$ to $G[p]$ is a locally nilpotent ring of endomorphisms of $G[p]$.*

Proof. Note that $I(G)$ is the sum (and set union) of ideals whose restriction to $G[p]$ is nilpotent, namely

$$I(G) = \bigcup_{\substack{\Phi \subseteq \lambda \\ \Phi \text{ finite}}} I(\Phi) = \sum_{\substack{\Phi \subseteq \lambda \\ \Phi \text{ finite}}} I(\Phi)\,,$$

where, for Φ a finite subset of $\lambda = \{\mu \mid \mu < \lambda\}$, $I(\Phi)$ is defined as follows. If $\Phi = \{\sigma_1, \cdots, \sigma_m\}$ such that $\sigma_1 < \sigma_2 < \cdots < \sigma_m$, put $\sigma_0 = 0$ and $\sigma_{m+1} = \lambda$, and let

$$I(\Phi) = \bigcap_{i=0}^{m} \mathrm{Ann}\,(p^{\sigma_i}G[p]/p^{\sigma_{i+1}}G[p]) \, .$$

Theorem 1.4 will be an immediate consequence of Theorem 1.3 and the following result.

COROLLARY 2.2. *Suppose that $J(R) = I(G)$ and let I be an ideal of R. Then the following conditions are equivalent.*

(i) *I is quasi-regular.*

(ii) *The restriction of I to $G[p]$ is nil.*

(iii) *The restriction of I to $G[p]$ is locally nilpotent.*

Proof. Obviously, (iii) implies (ii). By (2.1) of [3], (ii) implies (i). Assume (i). Then $I \subseteqq J(R) = I(G)$, and Lemma 2.1 completes the proof.

For the next result, G need not be reduced. For $X \subseteqq G$ and $I \subseteqq R$, XI denotes the set of all $x\varepsilon$ where $x \in X$ and $\varepsilon \in I$.

LEMMA 2.3. *For each integer $n \geqq 0$, $H(G) \cap \mathrm{Ann}\, p^nG$ is a nilpotent ideal of R.*

Proof. Let $I = H(G) \cap \mathrm{Ann}\, p^nG$. Clearly, I is an ideal of R. Moreover,

$$G[p]I^{n+1} \subseteqq p^nGI = 0 \, ,$$

which implies

$$G[p^k]I^{n+1} \subseteqq G[p^{k-1}] \quad \text{for each} \quad k \geqq 1 \, . \tag{2.4}$$

Furthermore,

$$GI \subseteqq G[p^n] \, .$$

Hence, using (2.4),

$$GI^{1+(n+1)n} \subseteqq G[p^n]^{(n+1)n} \subseteqq G[p^{n-n}] = 0 \, ,$$

as desired.

The set $T(R)$ of all torsion elements of R is an ideal of R [1; p. 278] which, as an abelian group, is a p-group. Hence

$$\begin{aligned} T(R) &= \{\varepsilon \in R \mid p^n\varepsilon = 0 \text{ for some } n < \omega\} \\ &= \bigcup_{n<\omega} \mathrm{Ann}\, p^nG = \sum_{n<\omega} \mathrm{Ann}\, p^nG \, . \end{aligned}$$

COROLLARY 2.5. *$H(G)\cap T(R)=\sum_{n<\omega}[H(G)\cap \operatorname{Ann} p^nG]$ is a locally nilpotent ideal contained in $N(R)$.*

Proof of Theorem 1.5. We first show the equivalence of conditions (ii)-(vii). By definition of the various ideals, (ii) implies (vi), which, in turn, using (1.1), implies (v). That (v) implies (iv) and (iv) implies (iii) is obvious. Assume (iii). By (1.2), $\operatorname{Ann} G[p]\subseteqq I(G)\subseteqq J(G)$. Thus, (iii) implies that the multiplication $p\cdot 1_G$ is nilpotent. Consequently, $(p\cdot 1_G)^n=p^n\cdot 1_G=0$ for some integer $n\geqq 0$ and $p^nG=0$. Thus (vii) follows from (iii). Assume (vii). Then $p^nG=0$ for some integer $n\geqq 0$, and $\operatorname{Ann} p^nG=R$. Thus

$$H(G)=H(G)\cap R=H(G)\cap \operatorname{Ann} p^nG\ .$$

Using Lemma 2.3, it follows that $H(G)$ is nilpotent and, observing (1.2), so is $J(R)$. Hence (vii) implies (ii), and the last six conditions are equivalent. Clearly, (ii) implies (i). The proof will be completed once we show that $N(R)$ is not nilpotent if G is unbounded. Suppose the latter and pick an integer $n\geqq 1$. We construct $\varepsilon\in N(R)$ such that $\varepsilon^n\neq 0$. Since G is unbounded and reduced, G has a decomposition

$$G=\bigoplus_{i=1}^{n+1}\langle a_i\rangle\oplus C$$

where, for $i=1,\cdots,n$, the order of a_i, call it m_i, is strictly less than the order of a_{i+1}. Define $\varepsilon\in R$ by

$$\begin{aligned}&a_i\varepsilon=p^{m_{i+1}-m_i}a_{i+1}\ ,\quad i=1,\cdots,n\\&a_{n+1}\varepsilon=0\\&C\varepsilon=0\ .\end{aligned}$$

Since, for each $i\leqq n$, $m_{i+1}-m_i\geqq 1$, $\varepsilon\in H(G)$. Clearly,

$$p^{m_{n+1}}G\varepsilon\subseteqq C\varepsilon=0\ .$$

Hence $\varepsilon\in H(G)\cap \operatorname{Ann} p^{m_{n+1}}G$ and $\varepsilon\in N(R)$, by Corollary 2.5. One easily verifies that

$$a_1\varepsilon^n=p^{m_{n+1}-m_1}a_{n+1}\neq 0\ .$$

Hence $\varepsilon^n\neq 0$ as desired.

LEMMA 2.6. *If $p^\omega G=0$ then*

$$I(G)=[H(G)\cap T(R)]+\operatorname{Ann} G[p]\ .$$

Proof. Since $\operatorname{Ann} G[p]\subseteqq I(G)$, it suffices to show that, for each

$\varepsilon \in I(G)$, there exists $\eta \in H(G) \cap T(R)$ so that ε and η coincide on $G[p]$. Let $\varepsilon \in I(G)$. By definition of $I(G)$, $p^\omega G = 0$ implies the existence of $m < \omega$ such that $p^m G[p]\varepsilon = 0$. It is will known that G has a decomposition

$$G = A \oplus B, \quad p^m G = p^m B \supseteq B[p].$$

Define $\eta \in R$ by

$$\begin{aligned} B\eta &= 0 \\ a\eta &= a\varepsilon \quad \text{for all} \quad a \in A. \end{aligned}$$

Since $A(\eta - \varepsilon) = 0$ and

$$B[p]\varepsilon \subseteq p^m G[p]\varepsilon = 0 = B\eta,$$

ε and η agree on $G[p]$. Thus, $\varepsilon \in I(G)$ implies $\eta \in I(G) \subseteq H(G)$. By construction, $p^m G\eta \subseteq B\eta = 0$, hence $p^m\eta = 0$ and $\eta \in T(R)$, completing the proof.

Observing (1.1), the following corollary together with Theorem 1.3 immediately yields Theorem 1.6.

COROLLARY 2.7. *Suppose that* $p^\omega G = 0$ *and* $J(R) = I(G)$. *Then*

$$J(R) = N(R) + \operatorname{Ann} G[p].$$

Proof. Lemma 2.6, Corollary 2.5 and (1.1).

Only Theorem 1.3 remains to be proven. For this we require the following observation.

LEMMA 2.8. *Let* $G = P \oplus C$ *and let* $\iota: P \to G$ *and* $\pi: G \to P$ *be the corresponding canonical injection and projection respectively. Then, if* J *is a quasi-regular ideal of* End G, *the set*

$$\iota J\pi = \{\iota\varepsilon\pi \mid \varepsilon \in J\}$$

is a quasi-regular ideal of End P.

Proof. Define f: End $G \to$ End P by $f(\varepsilon) = \iota\varepsilon\pi$ for all $\varepsilon \in \operatorname{End} G$. Then f is a surjective ring homomorphism [1; p. 217]. Such functions map quasi-regular ideals to quasi-regular ideals [4; p. 8].

From now on we assume that G is sufficiently projective. As above, $p^\lambda G = 0$.

PROPOSITION 2.9. *Let* J be *a quasi-regular ideal* End G. *Then, for each ordinal* $\sigma < \lambda$, $p^\sigma G[p]J \subseteq p^{\sigma+1}G$.

Proof. Let $\varepsilon \in J$ and let $x \in p^{\sigma}G[p]$. By hypothesis, G has a decomposition

$$G = P \oplus C \quad \text{where } x,\ x\varepsilon \in P,\ P \text{ totally projective.}$$

Let ι and π be as in Lemma 2.8. Then

$$x\varepsilon = x\iota\varepsilon\pi \in (p^{\sigma}G[p] \cap P)(\iota\varepsilon\pi) = p^{\sigma}P[p](\iota\varepsilon\pi) \subseteqq (p^{\sigma}P[p])(\iota J\pi) .$$

By Lemma 2.8, $\iota J\pi$ is a quasi-regular ideal of End P and, since P is totally projective, (2.2) of [3] implies $(p^{\sigma}P[p])(\iota J\pi) \subseteqq p^{\sigma+1}P$. Hence,

$$x\varepsilon \in p^{\sigma+1}P \subseteqq p^{\sigma+1}G ,$$

completing the proof.

For $x \in G$, let $h(x)$ denote the (possibly transfinite) height of x.

PROPOSITION 2.10. *Let J be a quasi-regular ideal of* End G, *let τ be an ordinal such that $0 < \tau \leqq \lambda$, and let $\varepsilon \in J$, then there exists an ordinal $\sigma < \tau$ such that $p^{\sigma}G[p]\varepsilon \subseteqq p^{\tau}G$.*

Proof. Assume, by way of contradiction, that $p^{\sigma}G[p]\varepsilon \nsubseteq p^{\tau}G$ for all $\sigma < \tau$. Then, for each $\sigma < \tau$, there exists $x_{\sigma} \in p^{\sigma}G[p]$ such that $x_{\sigma}\varepsilon \notin p^{\tau}G$ and thus, by Proposition 2.9, $\sigma \leqq h(x_{\sigma}) < h(x_{\sigma}\varepsilon) < \tau$. The same result implies that τ is a limit ordinal. Consequently, there exist $y_i \in G[p]$, $i = 0, 1, 2, \cdots$, such that

$$h(y_i) < h(y_i\varepsilon) < h(y_{i+1}) < \tau \quad \text{for} \quad i = 0, 1, 2, \cdots . \tag{2.11}$$

Let

$$\rho = \sup\{h(y_i) \mid i = 0, 1, 2, \cdots\} . \tag{2.12}$$

Since G is sufficiently projective, it has a decomposition

$$\begin{cases} G = P \oplus C,\ P \text{ totally projective,} \\ y_i,\ y_i\varepsilon \in P \quad \text{for} \quad i = 0, 1, 2, \cdots . \end{cases} \tag{2.13}$$

By (2.12), P has length at least ρ. Let ι and π be the maps of Lemma 2.8. Then $\iota\varepsilon\pi \in \iota J\pi$ which, by Lemma 2.8, is a quasi-regular ideal of End P. Since P is totally projective, Theorem 3.5 of [3] implies the existence of an ordinal $\sigma < \rho$ such that

$$p^{\sigma}P[p](\iota\varepsilon\pi) \subseteqq p^{\rho}P . \tag{2.14}$$

By (2.11) and (2.12), there exists an i such that

$$\sigma \leqq h(y_i) < \rho ,$$

and (2.14), together with (2.13), implies

$$y_i\varepsilon = y_i\iota\varepsilon\pi \in p^\sigma P[p](\iota\varepsilon\pi) \subseteqq p_\rho P .$$

Hence, using (2.11), $\rho \leqq h(y_i\varepsilon) < h(y_{i+1})$. This contradiction to (2.14) completes the proof.

Proof of Theorem 1.3. By (1.2), it suffices to show that $J(\text{End } G) \subseteqq I(G)$. Let $\varepsilon \in J(\text{End } G)$. Proposition 2.10 implies the existence of $\gamma < \lambda$ such that $p^\gamma G[p]\varepsilon \subseteqq p^\lambda G = 0$. Apply Proposition 2.10 repeatedly and use the fact that every properly descending sequence of ordinal numbers terminates after finitely many steps [10; p. 270].

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