PACIFIC JOURNAL OF MATHEMATICS
Vol. 74, No. 2, 1978

A REMARK ABOUT GROUPS OF CHARACTERISTIC 2-TYPE AND p-TYPE

ROBERT L. GRIESS, JR.

In the work of Klinger and Mason on groups of characteristic 2-type and p-type, a configuration for $p = 5$ emerges as a possibility in the conclusion to one of their theorems. In this note, we eliminate this possibility. Thus, the evidence that the only simple groups of characteristic 2-type and p-type are $G_2(3)$, $P\mathrm{Sp}(4, 3)$ and $U_4(3)$ is strengthened.

Introduction. We consider the following recent theorem of Klinger and Mason [2]:

THEOREM D. *Let G be a finite group, p an odd prime, $P \in \mathrm{Syl}_p(G)$. Assume $\mathsf{N}(P; 2) = \{1\}$, all 2-locals are 2-constrained with trivial core and all p-locals are p-constrained. Let A be an elementary abelian p-subgroup of G of maximal rank subject to $\mathsf{N}(A; 2) \neq \{1\}$. Assume $m(A) \geqq 2$. Then $m(A) = 2$ and one of the following holds.*

(a) *A contains every element of order p in $C_G(A)$.*

(b) *$p = 3$ and we can choose A and $T \in \mathsf{N}^*(A; 2)$ so that T is the central product of A-invariant quaternion subgroups $Q_1, \cdots, Q_w$, $w = 2, 3, 4$.*

(c) *$p = 5$ and for every $T \in \mathsf{N}^*(A; 2)$, T is the central product of five copies of Q_8 and five copies of D_8.*

It has been suspected that (c) never occurs. In fact, a conjecture of Gorenstein asserts that if G is a simple group, simultaneously of characteristic 2-type and p-type for some odd p (see [2] for definitions of these terms), then $p = 3$ and $G \cong G_2(3)$, $P\mathrm{Sp}(4, 3)$ or $U_4(3)$. It is the purpose of this paper to eliminate (c).

For terminology and notation used in this paper, see [2]. The reader is referred to [1] or [5] for basic material about groups or Lie type. See [3] for further results on groups of characteristic 2-type and 3-type.

In this paper, we consider the following hypothesis.

Hypothesis (KM). (1) G is a finite simple group, all of whose 2-local subgroups are 2-constrained.

(2) A is a noncyclic elementary abelian 5-group of maximal rank lying in a 2-local subgroup.

(3) $A \cong \mathbf{Z}_5 \times \mathbf{Z}_5$.

(4) $T \in \mathsf{N}^*(A; 2)$ and T is an extraspecial 2-group of order 2^{21},

type -, i.e., T is the central product of 5 quaternion groups and 5 dihedral groups. Also, $T = O_2(C_G(Z(T)))$ and $N_G(T)$ is corefree and 2-constrained.

(5) Let A_i, $0 \leqq i \leqq 5$, be all the subgroups of order 5 in A. Set $T_i = C_T(A_i)$ for $0 \leqq i \leqq 5$. Then $T_0 = \langle z \rangle = Z(T)$ and $T_i \cong D_8 \circ Q_8$, for $1 \leqq i \leqq 5$. Also, $C_G(A_i)$ is 5-constrained, $0 \leqq i \leqq 5$.

(6) $\mathit{И}(R; 2) = \{1\}$ for $R \in \mathrm{Syl}_5(G)$.

(7) A is contained in an elementary abelian subgroup of order 5^3 in G.

(8) $|O_{5'}(C_G(A_i))|$ is odd for $0 \leqq i \leqq 5$.

We observe that this hypothesis holds whenever the hypotheses and conclusion (c) but not conclusion (a) of Theorem D holds. This is clear, except possibly for (5), for which we refer to the proof of Lemma 3.8 of [2].

We now state our main result, which eliminates conclusion (c) of Theorem D.

THEOREM. *No finite group exists satisfying hypothesis (KM).*

The proof proceeds by contradiction. Let G be a counterexample to our theorem. First some notation.

Notation. $C = C_G(A_1)$, $N = N_G(A_1)$, $\bar{N} = N/0_{5'}(N)$, P a Sylow 5-group of $O_{5',5}(N)$ normalized by T_1.

For simplicity, we have singled out A_1 and T_1. It will be obvious that the following results apply to A_i, T_i, $C_G(A_i)$, etc. for $1 \leqq i \leqq 5$.

(1) For every $x \in T_1^\#$, $C_{\bar{P}}(x)$ has rank at most 2.

Proof. (*KM*.2)

(2) z inverts $\bar{P}/\bar{A}_1$ and $\bar{P}/\bar{A}_1 \neq 1$.

Proof. $C_C(z)$ contains A. Let $R = C_{\bar{P}}(z)$ and assume $R > \bar{A}_1$. Then there is $U \leqq PA_0 \, |U| = p^3$, $A \leqq U$, $A_1 \leqq Z(U)$ and $[U, z] = 1$. By (*KM*.4), $T = O_2(C_G(z))$ and U acts faithfully on T. Since $A \lhd U$ and A_0 is the unique A_i with $C_T(A_i) = \langle z \rangle$, we get $A \leqq Z(U)$ and so U is abelian. Since U acts faithfully on T and leaves invariant each T_i, U is elementary of rank 3, against (*KM*.2). Therefore $R = \bar{A}_1$, which proves that z inverts $\bar{P}/\bar{A}_1$. If $\bar{P} = \bar{A}_1$, then $T_1 \leqq 0_{5'}(C)$. A Frattini argument and (*KM*.7) then contradicts (*KM*.2).

(3) $\bar{P}$ is extraspecial of order 5^{1+4m}, some $m \geqq 1$.

Proof. Let R be a characteristic elementary abelian subgroup of P and suppose $|R| > 5$. Since z inverts R/A_1, T_1 acts faithfully on R/A_1, whence $|R/A_1| = 5^m$, $m \equiv 0 \pmod 4$. But then, if $x \in T_1$, $|x| = 2$, $|C_R(x)| \geqq 5^3$, against (KM.2). So, $|R| = 5$. Thus $\bar{P}$ is of symplectic type, $P = P_0P_1$, P_0 cyclic, P_1 trivial or extraspecial. Since $P_0 = Z(P)$, z normalizes P_0. Since z centralizes A_1, (2) implies that $P_0 = A_1$. Since $P > A_1$, $P = P_1$ is extraspecial as required. Since T_1 operates faithfully on $\bar{P}$, $|\bar{P}| = 5^{1+2k}$ where k is even, because $T_1 \cong D_8 \circ Q_8$.

(4) $|\bar{P}| = 5^5$, $\exp(P) = 5$.

Proof. By (3) and the fact that z inverts $\bar{P}/\bar{A}_1$, $\exp(P) = 5$. Let $|\bar{P}| = 5^{1+4m}$, m an integer. By (KM.5), $m \geqq 1$. Suppose $m \geqq 2$. Let $x \in T_1 - \langle z \rangle$, $|x| = 2$. Then $|C_{\bar{P}}(x)| = 5^{1+2m}$ and $C_{\bar{P}}(x)$ is extraspecial.

Since $C_{\bar{P}}(x)$ has exponent 5, $C_{\bar{P}}(x)$ contains an elementary group of rank $m + 1 \geqq 3$, against (KM.2). Thus $m = 1$.

(5) $\bar{P}$ is an indecomposable module for A_0. Thus P is the unique group of its isomorphism type in PA_0. Also, $\mathrm{Scn}_3(PA_0)$ has one member.

Proof. The first statement follows from the fact that T_1A_0 acts faithfully on $\bar{P}$. The second follows from $A_1 = Z(PA_0)$ and the fact that P/A_1 is the unique maximal abelian subgroup of PA_0/A_1. The third statement is obvious.

(6) Let $B > A_1$ satisfy $B/A = C_{P/A_1}(A_0)$. Then BA_0 is abelian of rank 3 and $|[BA_0, z]| = 5$.

Proof. Suppose BA_0 were nonabelian. A Lie ring analysis of the action of z on BA_0 gives a contradiction. Thus, BA_0 is abelian and, furthermore, must be elementary because of the way z acts.

(7) Let p be an odd prime and let $S = \mathrm{Sp}(4, p)$. Let $L \leqq S$ and $R \in \mathrm{Syl}_p(L)$. Assume $|R| \geqq p^2$. Let M be the standard 4-dimensional module for F_pS. Then one of the following holds: (i) L contains a transvection on M; (ii) $|R| = p^2$ and every element of $R^\#$ has quadratic minimal polynomial on M.

Proof. Let $\{a, b\}$ be a set of fundamental roots for a root, system Σ of type C_2 with a short, b long (see [1] or [5] for the basic machinery about groups of Lie type). For $r \in \Sigma$, let X_r be the usual one-parameter subgroup. On the standard 4-dimensional module M, elements of X_r, for r long, act as transvections.

We assume that L contains no transvection, then obtain (ii). Let $R \in \mathrm{Syl}_p(L)$ and embed R in U, the standard Sylow p-group, $U = \langle X_r \mid r \in \Sigma,\ r \text{ positive}\rangle$. Let $s = 2a + b$ be the root of maximal height and let $K = N_G(X_s)$ be the parabolic subgroup associated with the set $\{a\}$ of fundamental roots. Then $O_p(K)$ is extraspecial of order p^3, exponent p, and $K'/O_p(K) \cong \mathrm{SL}(2, p)$. Now, $R_1 = R \cap O_p(K) \neq 1$, as $|R| \geqq p^2$, so that $R_1 \cap X_s = 1$, $|R_1| = p$ and $|R| = p^2$. Since K is transitive on the nonidentity elements of $O_p(K)/X_s \cong \boldsymbol{Z}_p \times \mathbf{Z}_p$, we may assume $R_1 = X_{a+b}$, ($a + b$ is a short root). Then $N_S(R_1) = (X_b X_{a+b} X_{2a+b})H$, where H is the standard Cartan subgroup, so that $R \leqq U_1 = X_b X_{a+b} X_{2a+b}$. But, $U_1 = O_p(K_1)$, where K_1 is the parabolic subgroup of S which stabilizes a maximal totally isotropic subspace, say N (i.e., K_1 is associated with the set $\{b\}$ of fundamental roots). Since $K_1'/U_1 \cong \mathrm{SL}(2, p)$ is faithful on N and on M/N, U_1 stabilizes the chain: $M > N > 0$. Thus, (ii) holds, as required.

(8) $|G|_5 = 5^6$, $PA_0 \in \mathrm{Syl}_5(G)$, A_i is G-conjugate to A_1 and lies in the center of a Sylow 5-group, $1 \leqq i \leqq 5$.

Proof. By (KM.5) and (3), A_1 is a Sylow 5-center. Since all of the preceeding analysis could be applied to $N_G(A_i)$, $1 \leqq i \leqq 5$, we are done once we show $|N|_5 = 5^6$, i.e., $|\bar{N}/\bar{P}|_5 = 5$. Firstly, $|\bar{N}/\bar{C}|$ divides 4 and $\bar{C}/\bar{P}$ is isomorphic to a subgroup of $\mathrm{Sp}(4, 5)$. If $|\bar{C}/\bar{P}|_5 = 5$, we have $|N|_5 = 5^6$, as required. So, we may assume $|\bar{C}/\bar{P}|_5 \geqq 5^2$. Let $C^* = C/O_{5',5}(C)$. Suppose first that C^* contains a transvection in its action on $M = \bar{P}/\bar{A}_1$. Let L be the subgroup of C^* generated by elements inducing transvections on M. Then T_1^* normalizes L and the structure of $S = \mathrm{Sp}(4, 5)$ implies that L is not a 5-group. Therefore, $L \cong \mathrm{SL}(2, 5)$ or $\mathrm{Sp}(4, 5)$, by [4] and the structure of S. If $L \cong \mathrm{SL}(2, 5)$, the fact that $(T_1 A_0)^*$ normalizes L forces $[T_1^*, L] = 1$, which is absurd since T_1^* is irreducible on M. So, $L \cong \mathrm{Sp}(4, 5)$. Therefore, $C_C(Z)$ contains T_1 as a nonnormal subgroup, whereas $T_1 \leqq O_2(C_G(Z)) = T$, by ($KM$.4) and ($KM$.5), contradiction.

We have that C^* contains no transvections. Therefore, (7) tells us that a Sylow 5-group of C^* consists of elements operating trivially or quadratically on M. This contradicts (5), since A_0 has minimal polynomial of degree 4. Therefore, we have $|N|_5 = 5^6$, as reguired.

(9) Let $W = BA_0$, as in (6). Then $W \in \mathrm{Syl}_5(C_G(W))$. For each $i = 1, \cdots, 5$, $A_G(W)$ contains a transvection t_i centralizing A_i. Also,

$1 \neq [A, t_1] \leqq B$ but $A_1 \nleqq [A, t_1]$. If V is a conjugate of W for which $|N_{PA_0}(V)| \geqq 5^5$, then $V \leqq P$. Moreover, if $V \lhd P$, then $\{V\} = \mathrm{Scn}_3(PA_0)$.

Proof. Choose $g \in G$ so that $V = W^g \leqq PA_0$. If $V \leqq P$, then $[V, P] = A_1$ and $V = C_{PA}(V)$ by (5). Say $|V \cap P| = 5^2$. Then $V_0 = V \cap P$ is centralized by $b \in PA_0 - P$. By (5), $V_1 = C_P(V_0)$ has index 5 in P and $V = C_{PA_0}(V)$.

Recall that an extremal conjugate of a subgroup S_1 of a p-Sylow group S of G is a G-conjugate S_2 of S_1 for which $N_S(S_2) \in \mathrm{Syl}_p(N_G(S_2))$. If we now take V to be an extremal conjugate of W in PA_0, we see from the previous paragraph that $V = C_{PA_0}(V)$, whence $W \in \mathrm{Syl}_5(C_G(W))$. The existence of t_i follows from embedding W in a Sylow 5-group of $C_G(A_i)$. The further properties $1 \neq [A_1, t_1] \leqq B$ and $A_1 \nleqq [A, t_1]$ are forced if we take $t_1 \in N_{PA_0}(W)$. The final statement follows from $V = C_{PA_0}(V)$ and (5).

(10) Let X be the subgroup of $A_G(W)$ generated by transvections. Then one of the following holds

(i) X is a 5-group and $|X| \geqq 5^2$.

(ii) $X/O_5(X) \cong \mathrm{SL}(2, 5)$.

(iii) $X \cong \mathrm{SL}(3, 5)$.

Proof. This follows from the classification of groups in odd characteristic generated by transvections [4]. To get $|X| \geqq 5^2$ in Case (i), use (9).

(11) X is not isomorphic to $\mathrm{SL}(2, 5)$.

Proof. Suppose false. Then $X \lhd A_G(W)$ implies that $A_G(W)$ is isomorphic to a subgroup of $\mathbf{Z}_4 \times \mathrm{GL}(2, 5)$ and that W is extremal in PA_0. Write $W = W_0 \oplus W_1$ where $W_0 = C_W(X)$, $W_1 = [W, X]$. Now, $A \cap W_1 \neq 1$. Suppose $A_i \leqq W_i$ for some i. Then, for a Sylow 5-group R of $N_G(W)$, $A_i = [W, R]$. If $i \neq 0$, this contradicts (9) and the action of $N_P(W)$ on W. Therefore, $[W, R] = A \cap W_1 = A_0$ and every group of order 5 in W_1 is fused in $N_G(W)$ to A_0. The orbits of X on the 25 subgroups of order 5 in W which meet W_1 trivially have lengths 1 and 24. By (8), they lie in the G-conjugacy class of A_1. On the other hand, the action of P on B implies that every group of order 5 in B distinct from A_1 is fused to the one in $B \cap W_1 \cong \mathbf{Z}_5$. Therefore all subgroups of order 5 in W are fused in G to A_1. Now embed R in a Sylow 5-group of $C_G(A_0)$. Since $R' = A_0$, the proof of (9) shows that $W \leqq O_{5',5}(C_G(A_0))$. Since P is extraspecial, we get $|N_G(W)|_5 \geqq 5^5$, a contradiction to our first remark.

(12) $|X|_5 = 5^3$ and the unique extremal conjugate V of W in PA_0 lies in P.

Proof. By (11), $|X|_5 \geqq 5^2$, so by (9) any extremal conjugate V of W in PA_0 lies in P. Suppose $|X|_5 \leqq 5^2$. Then (10.i) holds and we have $|X|_5 = 5^2$. Thus W is not extremal in PA_0. The action of P on V and t_1 on W show that $|X|_5 = 5^3$, a contradiction.

(13) X is not a 5-group and $C_W(X) = 1$. Thus $X \cong \mathrm{SL}(3, 5)$ or X stabilizes a unique hyperplane W_1 of W, in which case $X/O_5(X) \cong \mathrm{SL}(2, 5)$ acts faithfully on W_1.

Proof. By embedding W in a Sylow 5-group of $N_G(D)$, where $D \leqq W$, $D \sim_G A_1$, we see that some Sylow 5-group of X centralizes D. Since $|C_W(R)| = 5$ for $R \in \mathrm{Syl}_5(N_G(W))$, we take $D = A_i$, $i = 1, 2, 3, 4, 5$ to get the first statement. The second statement now follows, using (10) and (12).

(14) $A = W_1$ in case $X \not\cong \mathrm{SL}(3, 5)$. Thus each group of order 5 in A is central in some Sylow 5-group of $N_G(W)$.

Proof. Suppose X normalizes the hyperplane W_1. Then W_1 contains the center of every Sylow 5-group of $N_G(W)$. Therefore, using (9), $W_1 \geqq A_1A_2 = A$, as required.

(15) $X \cong \mathrm{SL}(3, 5)$.

Proof. Suppose not. By (14), W is normal in a Sylow 5-group of C, so we may assume $W \leqq P$. However, again by (14), $W \geqq A_0 \not\leqq P$, contradiction.

(16) $X \not\cong \mathrm{SL}(3, 5)$.

Proof. Suppose $X \cong \mathrm{SL}(3, 5)$. The action of z on W has determinant -1 (see (6)). So, there is $u \in N_G(W)$ inducing -1 on W. In fact $|u| = 2$, by (*KM*.6). Thus, $N = O_{5',5}(N) \cdot C_N(u)$, by the Frattini argument. It follows that V is complemented in PA_0 by $C_{PA_0}(u')$, for an appropriate involution u'. But then there are two Jordan blocks of degree 2 for the action of an element of order 5 in $C_{PA_0}(u) - P$ on P/A_1. This contradicts (5).

Since evidently (15) and (16) are in conflict, the proof of our theorem is complete.

REFERENCES

1. R. Carter, *Simple Groups of Lie Type*, Wiley-Interscience, London, New York, 1972.
2. K. Klinger and G. Mason, *Centralizers of p-groups of characteristic* 2, *p-type*, J. Algebra, **37** (1975), 362–375.
3. G. Mason, *Two theorems on groups of characteristic 2-type*, Pacific J. Math., **57** (1975), 233–253.
4. J. McLaughlin, *Some groups generated by transvection*, Arch. Math. (Basel), **18** (1967), 364–368.
5. R. Steinberg, *Lectures on Chevalley Groups*, Yale University Notes, New Haven, Conn., 1967.

Received April 12, 1976 and in revised form July 20, 1977.

UNIVERSITY OF MICHIGAN
ANN ARBOR, MI 48109