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SUPER TRIANGULATIONS

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This paper concerns itself with continuous families of linear embeddings of triangulated complexes into E^2. In [2] Cairns showed that if f and g are two linear embeddings of a triangulated complex (C, T) into E^2 so that there is an orientation preserving homeomorphism k of E^2 with $k \circ f = g$, then there is a continuous family of linear embeddings h_t: $(C, T) \to E^2(t \in [[0, 1])$ so that $h_0 = f$ and $h_1 = g$. In this paper we prove various relative versions of this result when C is an arc, a θ-curve, or a disk.

Introduction. To appreciate where the results in this paper fit into the literature, it is useful to be aware of the following examples which were described in [1, Example 4.1].

EXAMPLE 1. This example is a triangulated 1-complex (C, S) linearly embedded in E^2 consisting of a simple closed curve J with two disjoint spanning arcs in its interior. The complex C is homeomorphic to ⦶. There is a homeomorphism $g: E^2 \to E^2$ fixed on J such that $f = g|C$ is linear with respect to S but there is no linear isotopy $h_t: (C, S) \to E^2(t \in [0, 1])$ with $h_0 = id$ and $h_1 = f$ which keeps J fixed.

EXAMPLE 2. Example 1 can be modified by incorporating (C, S) into the 1-skeleton of a triangulated disk (P, T) with boundary J to produce an example of a disk with properties similar to those of (C, S). Namely, the triangulated disk (P, T) is linearly embedded in E^2 and admits a linear homoemorphism k fixed on $\operatorname{Bd} P$ for which there is no linear isotopy $h_t: (P, T) \to E^2(t \in [0, 1])$ with $h_0 = \mathrm{id}$ and $h_1 = k$ which leaves the boundary fixed throughout.

It is known that no such example can be found where P is convex [1, Corollary 4.4] nor could P be star-like if T has no spanning edge [1, Theorem 4.1].

In this paper it is shown (Theorem 2.4) that no 1-complex homeomorphic to a θ-curve can have the properties of Example 1. Then in Theorem 4.2 it is proved that Example 2 can not retain its properties under all subdivisions. In fact each triangulation T of a disk P has a subdivision T' which is a super triangulation of P. A super triangulation T' of a disk P is one which is as flexible as possible. Namely, any linear embedding of $\operatorname{Bd} P$ into E^2 extends to a linear embedding of (P, T') and for any two linear homeomorphisms f, g of (P, T') into E^2 with $f | \operatorname{Bd} P = g | \operatorname{Bd} P$, there is a linear isotopy

$h_t: (P, T') \to E^2 (t \in [0, 1])$ with $h_0 = f$ and $h_1 = g$ which agrees with f and g on Bd P throughout. (The formal definition of super triangulation appears in §3.)

DEFINITION. Let (C, T) be a finite complex C with triangulation T. A *linear embedding* of C (or (C, T)) into E^n is an embedding which is linear on each simplex of T. A *linear isotopy* $h_t: (C, T) \to E^n (t \in [0, 1])$ is a continuous family of linear embeddings of (C, T) into E^n. A *simple push* is a linear isotopy which is fixed except on the open star of one vertex. A *push* is a sequence of simple pushes each one performed after the preceding one.

2. Linear isotopies of arcs and θ-curves.

This section begins with a theorem which states that two triangulated arcs linearly embedded in E^2 with the same endpoints can be linearly isotoped to a common arc in E^2 keeping the endpoints fixed where the movement takes place only in the closure of the unbounded component of the complement of the union of the two arcs.

THEOREM 2.1. *Let (A, T_A) and (B, T_B) be two triangulated arcs linearly embedded in E^2 which share common endpoints v and w. Then there are linear isotopies $g_t: (A, T_A) \to E^2 (t \in [0, 1])$ and $h_t: (B, T_B) \to E^2 (t \in [0, 1])$ such that*

(1) $g_0 = \mathrm{id}$ *and* $h_0 = \mathrm{id}$,

(2) $g_1(A) = h_1(B)$ *as subsets of* E^2,

(3) *for all* $t \in [0, 1]$ *both* h_t *and* g_t *leave both* v *and* w *fixed, and*

(4) *for* $0 \leqq s \leqq u \leqq 1$, $g_u(A) \cup h_u(B)$ *misses the unbounded component of* $E^2 - (g_s(A) \cup h_s(B))$.

The proof of Theorem 2.1 uses the following lemma to reduce bends.

LEMMA 2.2. *Let (A, T_A) be a triangulated arc linearly embedded in E^2 from v to w, a be a vertex of A, and x be another point of A such that*

(1) *the segment ax meets A only at its ends and*

(2) *the disk D_{ax} bounded by $ax \cup A_{ax}$ (where A_{ax} denotes the subarc of A from a to x) contains neither v nor w in its interior. Then there is a push $h_t: (A, T_A) \to E^2 (t \in [0, 1])$ so that*

(1) $h = \mathrm{id}$,

(2) *each h_t is fixed on each of a, v, and w,*

(3) *for $0 \leqq s < u \leqq 1$, $h_u(A) \subset (A - A_{ax}) \cup D^s_{ax}$ where D^s_{ax} is the disk bounded by $ax \cup h_s(A)_{ax}$, and*

(4) $h_1(A) = (A - A_{ax}) \cup ax$.

Proof of lemma. Hypotheses 1 and 2 imply that $D_{ax} \cap A = A_{ax}$. If A_{ax} contains only one vertex a_j of A other than a, push a_j straight to x, shortening the edge of A containing x. If x were a vertex of A and the open arc (A_{ax}) contains only one vertex a_j, a_j could be pushed to the center of ax. If the open arc (A_{ax}) contains more than one vertex of A, triangulate D_{ax} without adding interior vertices. Find a shelling of the triangulation which leaves the 2-simplex containing ax until last. Let the shelling guide a desired push of A.

The proof of Theorem 2.1 uses the following lemma to reduce the number of components of $A \cap B$.

LEMMA 2.3. *Suppose that (A, T_A) and (B, T_B) are triangulated arcs linearly embedded in E^2 from v to w such that each of v and w is accessible from the unbounded component of $E^2 - (A \cup B)$. Then there is a linear isotopy $h_t: (A, T_A) \to E^2 (t \in [0, 1])$ such that*

(1) $h_0 = \mathrm{id}$,

(2) *each h_t is fixed on v and w,*

(8) *for $0 \leqq s \leqq u \leqq 1$, $h_u(A)$ misses the unbounded component of $E^2 - (B \cup h_s(A))$, and*

(4) *for each edge b_jb_{j+1} of B, $b_jb_{j+1} \cap h_1(A)$ is either empty, connected, or consists of exactly two points x and y so that x and y are interior points of two consecutive edges of A.*

Proof of lemma. Let b_jb_{j+1} be an edge of B.

Case 1. If $b_jb_{j+1} \cap A$ contains a vertex of A, then by repeated applications of Lemma 2.2, A can be pushed to make $b_jb_{j+1} \cap A$ connected.

Case 2. Suppose that $(b_jb_{j+1} \cap A) = \{x_1, x_2, \cdots, x_r\}$ where each x_i is an interior point of an edge of A and $x_1 < x_2 < x_3 < \cdots < x_r$ on b_jb_{j+1}.

If, for some i, $A_{x_ix_{i+1}}$ contains two or more vertices of A, triangulate $D_{x_ix_{i+1}}$ without adding interior vertices. As in the proof of Lemma 2.2, follow a shelling that leaves the 2-simplex containing x_ix_{i+1} until last to push A to $(A - A_{x_ix_{i+1}}) \cup x_ix_{i+1}$. Now apply Case 1.

Suppose each $A_{x_ix_{i+1}}$ contains one vertex of A and $b_jb_{j+1} \cap A$ contains at least three points x_1, x_2, and x_3. Let a_i and a_{i+1} be respectively the vertex of A between x_1 and x_2 and the vertex of A between x_2 and x_3. By moving a_i toward x_1 and a_{i+1} toward x_3, pivoting about x_2, a_i and a_{i+1} can be linearly isotoped down to x_1 and x_3 respectively and Case 1 can again be applied to make the image of A intersect b_jb_{j+1} in a connected set.

The movements we used did not complicate the intersection of the image of A and the other edges of B so we can finish Lemma 2.3 by considering the edges of B one at a time.

EXAMPLE 6.1. We note that Lemma 2.3 cannot be strengthened to require a push rather than a linear isotopy as demonstrated by Figure 2.1.

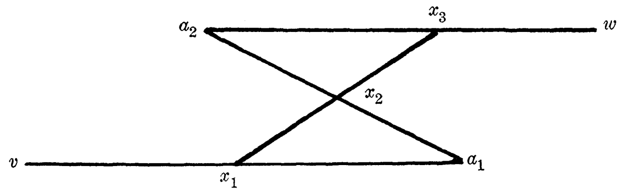

FIGURE 2.1

Note that no interior vertex of A or B can be pushed at all without moving A or B partially into the unbounded component of $E^2 - (A \cup B)$.

Proof of Theorem 2.1. The proof uses induction on n, the sum of the number of bends in A and the number in B. If $n = 0$, A and B coincide already.

We assume the theorem for k less than n and suppose that the sum of the bends in A and B is n. Let $\{a_i\}_{i=0}^{m}$ and $\{b_i\}_{i=0}^{p}$ be vertices of A and B in order so that $a_0 = b_0 = v$, $a_m = b_p = w$, and $(m - 1) + (p - 1) = n$.

Case 1. Suppose v or w (say v) is not accessible from the unbounded component of $E^2 - (A \cap B)$. In this case find a point x on A or B (say A) such that $vx \cap a_1a_2 = \emptyset$, $vx \cap b_1b_2 = \emptyset$, and vx meets $A \cup B$ only at its ends.

If $w \notin \operatorname{Int} D_{vx}$, apply Lemma 2.2 to push A onto $vx \cup A_{xw}$ thereby reducing the number of bends in A.

If $w \in \operatorname{Int} D_{vx}$, find a point y on $\operatorname{Bd} D_{vx}$ so that $wy \cap vx = \emptyset$ and $wy \cap a_{m-2}a_{m-1} = \emptyset$. Let y' be the nearest point of $wy \cap A_{va_{m-2}}$ to w. Since $(A_{xw} \cup wy') \subset D_{vx}$, then $D_{wy'}$ is a subset of D_{vx} and hence does not intersect vx. Since w is not in $\operatorname{Int} D_{wy'}$ either, apply Lemma 2.2 to push A to $(A - A_{y'w}) \cup y'w$ and thereby reduce the number of bends in A. One can use the facts that the movement occurs in $D_{wy'}$, $D'_{wy'} \subset D_{vx}$, and D_{vx} misses the unbounded component of $E^2 - (A_{vx} \cup B)$ to show that Condition 4 of Theorem 2.1 is satisfied.

Case 2. Suppose v and w are both accessible from the unbounded component of $E^2 - A \cup B$. By isotoping A according to Lemma 2.3 and then changing the roles of A and B and isotoping B according to the lemma, we can linearly isotope A and B to a position where the conclusion of Lemma 2.3 is satisfied by both A and B. If this adjustment reduced the number of bends in $A \cup B$, then Theorem 2.1 follows by induction. We assume that it did not and proceed to consider five subcases of Case 2.

Subcase 2a. Suppose $a_1 = b_1$. We apply induction to the two arcs A_{a_1w} and B_{b_1w}.

Subcase 2b. Suppose $va_1 \subset vb_1$ and $a_1a_2 \cap B = \{a_1\}$. Then push a_1 toward b_1, pivoting on a_2, until the moved a_1a_2 hits a point of $A_{a_3w} \cup B_{b_1w}$. Let a_1' be the position of a_1 at that moment. If $a_1'a_2 \cap A_{a_3w}$ contains a vertex a_j, push A to $va_1 \cup a_1a_j \cup A_{a_jw}$ by an application of Lemma 2.2 and thereby reduce the number of bends in A.

If $a_1'a_2$ contains a vertex b_j of B_{b_2w}, push B to $va_1 \cup a_1b_j \cup B_{b_jw}$. This push moves the vertex b_1 to the point a_1 and, therefore, throws us into Subcase 2a which was already considered.

If $a_1' = b_1$ and $a_1'a_2 \cap (A_{a_3w} \cup B_{b_2w}) = \varnothing$, then push a_1 to b_1 which puts us into Subcase 2a.

Subcase 2c. Suppose $va_1 \subset vb_1$ and $a_1a_2 \cap B \neq \{a_1\}$. By Lemma 2.3, $a_1a_2 \cap B = \{a_1\} \cup \{y\}$ where y is an interior point of b_1b_2.

Several things could happen. First, if b_1b_2 intersects A in a point z other than y, then z is an interior point of a_2a_3 by conclusion 4 of Lemma 2.3. Furthermore $z \in yb_2$. This is true because A_{a_2w} cannot intersect the triangle a_1b_1w since such an intersection would violate either Lemma 2.3 or the fact that w is accessible from the unbounded component of $E^2 - (A \cup B)$. We move a_1 to b_1 and a_2 to z, pivoting about y. This procedure takes us to Subcase 2a which has already been considered.

Second, if b_1b_2 misses A_{a_2w}, it may be that the segment b_1b_2 can be extended slightly beyond b_2 without meeting the unbounded domain of $E^2 - (A \cup B)$. In this case, we let x be the first point at which this extension of b_1b_2 intersects $A \cup B$. If $x \in B$, apply Lemma 2.2 to push B to $(B - B_{b_2x}) \cup b_2x$ and thereby reduce the bends in B. If x is a vertex of A push A to $(A - A_{yx}) \cup yx$ and then to $(A - A_{vy}) \cup vb_1 \cup b_1y$ to reduce the number of bends in A. If x is not a vertex of A but $x \in a_2a_3$, move a_2 to x and a_1 to b_1 while pivoting about y. This puts us in Subcase 2a. If $x \in A_{a_3w}$, triangulate disk bounded by $A_{yx} \cup yx$ without adding interior vertices. As in the proof of Lemma 2.2, find a shelling of this disk which leaves the 2-simplex containing yx until last. Let this shelling guide a linear isotopy of A onto the set $va_1 \cup a_1y \cup yx \cup A_{xw}$. Now a_1 can be moved to b_1 putting us in Case 2a.

Third, and last, if $b_1b_2 \cap A_{a_2w} = \emptyset$ and b_1b_2 cannot be extended beyond b_2 without intersecting the unbounded component of $E^2 - (A \cup B)$ then move b_2 toward b_3 as you move b_1 toward a_1, pivoting about y. If the pivoting segment b_1b_2 never meets a point of A_{a_3w} nor B_{b_3w} while b_1 is moved onto a_1, then this linear isotopy puts us in Subcase 2a. If not, then the first moment at which b_1b_2 meets A_{a_3w} or B_{b_3w} locates a vertex a_j or $b_j (j \geqq 3)$ such that the segment ya_j or yb_j satisfies the hypotheses of Lemma 2.2. An application of Lemma 2.2 would then reduce the bends in A or B.

Subcase 2d. Suppose $vb_1 \subset va_1$. This case is identical to Subcase 2b and 2c with the roles of A and B exchanged.

Subcase 2e. Suppose $va_1 \cap B = v$,

Perhaps the segment from v through a_1 can be extended beyond a_1 without going into the unbounded domain of $E^2 - (A \cup B)$. If it can, extend it until the extension hits a point x of $A \cup B$. If $x \in A$, use Lemma 2.2 to push A to $vx \cup A_{xw}$, reducing the number of bends in A. If $x \in B$, use Lemma 2.2 to push B to $vx \cup B_{xw}$. This either reduces the number of bends in B or carries us to a previous case.

If va_1 cannot be extended as considered in the last paragraph, examine the segment vx as x moves from a_1 to a_2 and find the first x_0 at which vx_0 meets $B \cup A_{a_2w}$ in a point other than v.

If vx_0 meets A_{a_2w}, let a_j be the point of $vx_0 \cap A_{a_2w}$ nearest v. Lemma 2.2 implies that A can be pushed to $va_j \cup A_{a_jw}$, thereby reducing the number of bends in A.

If vx_0 misses A_{a_1w}, use Lemma 2.2 to push A to $vx_0 \cup A_{x_0w}$. The first edge vx_0 of $vx_0 \cup A_{x_0w}$ contains a point of B other than v and we are in a previous subcase.

The following theorem is used in the proof of Theorem 4.2.

THEOREM 2.4. *Let $(J \cup A, T)$ be a triangulated θ-curve linearly embedded in E^2 so that A is a spanning arc of the disk bounded by the simple closed curve J. Let k be a homeomorphism of E^2 such that $k|J = \mathrm{id}$ and $k|A$ is a linear embedding of $(A, T|A)$. Then there is a linear isotopy $h_t: (J \cup A, T) \to E^2 (t \in [0, 1])$ such that $h_0 = \mathrm{id}$, $h_1 = k|J \cup A$, and for each t in $[0, 1]$, $h_t|J = \mathrm{id}$.*

Proof. Let $(A, T|A)$ and $(k(A), k(T|A))$ be the two arcs in the hypothesis of Theorem 2.1. Let $f_t': (A, T|A) \to E^2 (t \in [0, 1])$ and $g_t: (k(A), k(T|A)) \to E^2 (t \in [0, 1])$ be linear isotopies satisfying the conclusion of Theorem 2.1. Each of these linear isotopies can be extended to J by the identity. We do so and abuse the notation slightly by letting f_t' and g_t now denote those linear isotopies of $(J \cup A, T)$ and $(k(J \cup A), k(T))$ respectively.

Now $f_1'(A) = g_1 \circ k(A)$ as sets; however, it may not be the case that $f_1' | A$ equals $g_1 \circ k | A$. (See Figure 2.2.) In order to rectify this situation find a linear isotopy $f_t'': (J \cup A, T) \to E^2 (t \in [0, 1])$ such that $f_0'' = f_1'$, $f_1'' | A = g_1 \circ k | A$, and for each t in $[0, 1]$, $f_t'' | J = \text{id}$. The linear isotopy f_t'' simply moves the vertices of $f_1'(A)$ until they are in the position to which they are mapped by $k \circ g_1$. Note that for any $\varepsilon > 0$ f_t'' can be chosen so that for each t in $[0, 1]$, $f_t''(A)$ lies in the ε-neighborhood of $f_1'(A)$; however, as illustrated in Figure 2.2, it may not be possible to have $f_t''(A) = f_1'(A)$ for each t.

A linear isotopy h_t satisfying the conclusion of Theorem 2.4 can now be obtained by performing three linear isotopies in succession. First perform $f_t'(t \in [0, 1])$, second perform $f_t''(t \in [0, 1])$, and finally perform $g_{1-t} \circ k(t \in [0, 1])$.

Note that for any $\varepsilon > 0$, h_t could be chosen so that for t, $h_t(A)$ almost misses the unbounded component C of $E^2 - (A \cup k(A))$ where "almost" means that $h_t(A)$ misses C except for an ε-neighborhood of $f_1'(A)$.

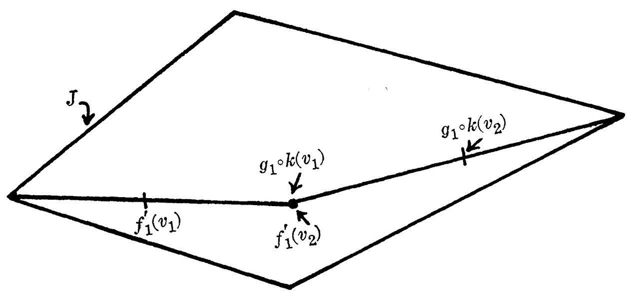

FIGURE 2.2

3. Super triangulations. A triangulation T of a disk P is *super* if and only if it has the following three properties.

(1) Every linear embedding of $\text{Bd}\, P$ in E^2 can be extended to a linear embedding of (P, T).

(2) If f and g are two linear embeddings of (P, T) which agree on $\text{Bd}\, P$, then there is a linear isotopy $h_t: (P, T) \to E^2 (t \in [0, 1])$ such that $h_0 = f$, $h_1 = g$, and for all $t \in [0, 1]$, $h_t | \text{Bd}\, P = f | \text{Bd}\, P = g | \text{Bd}\, P$.

(3) If h_0 and h_1 are two linear embeddings of (P, T) into E^2 and f_t is a linear isotopy of $\text{Bd}\, P$ into E^2 from $h_0 | \text{Bd}\, P$ to $h_1 | \text{Bd}\, P$, then f_t can be extended to a linear isotopy of P from h_0 to h_1.

It may be noted that Properties 1 and 2 imply Property 3. To see that this is true one could use Property 1 and the compactness of $[0, 1]$ to cover $[0, 1]$ with subintervals $[t_0, t_1], [t_1, t_2], \cdots, [t_{n-1}, t_n]$ such that for each i there is a linear isotopy $h_t^i: (P, T) \to E^2$, $t \in [t_i, t_{i+1}]$ with $h_t^i|\mathrm{Bd}\, P = f_t|\mathrm{Bd}\, P$. Although $h_{t_{i+1}}^i$ and $h_{t_{i+1}}^{i+1}$ need not agree on interior vertices of T, Property 2 can be used to adjust them so they do.

In this section we produce (in Theorem 3.4) for each integer $n (n \geqq 3)$ a super triangulation T_n of a disk P_n which has n 1-simplexes on the boundary. For $n = 3$, T_3 could be chosen to be a single 2-simplex. For $n = 4$ or 5, T_n could be chosen to be obtained by coning from an interior point to $\mathrm{Bd}\, P_n$. This T_n is super since any linear embedding of $\mathrm{Bd}\, P_n (n = 4, 5)$ would bound a star-like disk. To produce T_6, we add an annulus A to P_5 so that A has five 1-simplexes in the boundary component which is $\mathrm{Bd}\, P_5$, but six in the other. This annulus is given a specific, simple triangulation. It is shown in Theorem 3.3 that if one begins with a super triangulation of a disk with n sides and enlarges the disk and triangulation by the addition of an annulus which is triangulated as specified in Theorem 3.3, then the new triangulation of the new disk is also super. Thus to produce T_7 another annulus is added on to P_6. This process is continued to produce triangulations T_n where each T_n has a bull's-eye pattern.

Theorems 3.1 and 3.2 have the same general form as Theorem 3.3 except where different triangulations of the added annulus are considered. They are included in this section because their proofs contain techniques used in the proof of Theorem 3.3. They are used explicitly in the next section.

DEFINITION. Let J be a PL simple closed curve in E^2. A point x in $\mathrm{Int}\, J$ *can see* J if and only if for each point y in J the segment xy meets J only at y.

THEOREM 3.1. *Suppose* (P, T) *is a triangulated disk,* A *is a subcomplex of* T *which is an annulus containing* $\mathrm{Bd}\, P$*, the closure of* $P - A$ *is a disk* D*, and* A *has the following triangulation. Namely,* A *is the union of* n *4-sided disks* $v_i v_{i+1} w_{i+1} w_i (i = 1, 2, \cdots, n,$ *counting is* $\mathrm{mod}\, n)$ *where for each* i, $v_i \in \mathrm{Bd}\, P$ *and* $w_i \in \mathrm{Bd}\, D$ *and the 2-simplexes of* T *in* A *are precisely those of the form* $v_i w_i v_{i+1}$ *or* $w_i v_{i+1} w_{i+1}$. *Then* T *super if* T *is restricted to* D *is super.*

Proof. Let h be a linear embedding of $\mathrm{Bd}\, P$ in E^2 with $h(v_i) = v_i'$. To show that (P, T) has Property 1 we can pick the image of w_i to be w_i', a point in $\mathrm{Int}\, h(\mathrm{Bd}\, P)$ near v_i' on the bisection of angle $v_{i-1}' v_i' v_{i+1}'$. This linearly embeds A. The fact that T restricted to D is super ensures that the embedding can be extended.

Next, we show that (P, T) has Property 2. For convenience we suppose that f is the identity and $g(w_i) = w_i'$ is as described in the preceding paragraph. Our plan is to push each w_i to w_i' and use the fact that D has Property 3 to show that (P, T) has Property 2.

Special Case. If $v_i v_{i+1} w_{i+1} w_i$ is convex we could push w_i to a point near v_i along $w_i v_i$ and then push it to a point near v_i so that if w_i'' is the new w_i, both $v_{i-1} v_i w_i'' w_{i-1}$ and $v_i v_{i+1} w_{i+1} w_i''$ are convex. Similarly, w_{i-1} can be pushed to w_{i-1}'', a point near v_{i-1} so that both $v_{i-2} v_{i-1} w_{i-1}'' w_{i-2}$ and $v_{i-1} v_i w_i'' w_{i-1}''$ are convex. Continuing back through the w_i's (and counting mod n) each w_i can be pushed to w_i'' a point near v_i so that each quadrilateral $v_j v_{j+1} w_{j+1}'' w_j''$ is convex. Now we can push each w_i'' onto the bisector of $v_{i-1} v_i v_{i+1}$ and then to w_i'.

General Case. Finally we show that some w_{i+1} can be pushed to w_{i+1}''' so that the resulting $v_i v_{i+1} w_{i+1}''' w_i$ is convex. To do this, we pick an i so that $w_i w_{i+2}$ is a spanning arc of D. (If $n = 3$ there is no spanning arc and we use $w_1 w_3$ for $w_i w_{i+2}$.) We note that w_{i+1} can see $\mathrm{Bd}\, v_{i+1} v_{i+2} w_{i+2} w_i$ so it can be pushed in a straight line to a point w_{i+1}''' near the side $w_i v_{i+1}$ so that w_{i+1}''' can see $\mathrm{Bd}\, v_i v_{i+1} v_{i+2} w_{i+2} w_i$. The resulting $v_i v_{i+1} w_{i+1}''' w_i$ is convex and we proceed as in the Special Case.

THEOREM 3.2. *Suppose (P, T) is a triangulated disk and A is a subcomplex of (P, T) such that A is an annulus containing* Bd P, *the closure of $P - A$ is a disk D, and A has the following triangulation. Namely, A is the union of n 4-sided disks $v_i v_{i+1} w_{i+1} w_i$ ($i = 1, 2, \cdots, n$, counting is* mod n) *where for each j, $v_j \in \mathrm{Bd}\, P$ and $w_j \in \mathrm{Bd}\, D$, and T restricted to A is determined by coning over the boundary of each of these 4-sided disks from an interior point.* (*See Figure* 3.1.) *Then T is super if T restricted to D is super.*

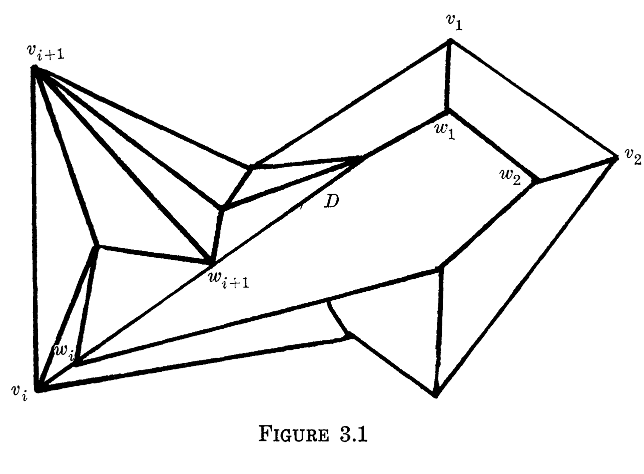

FIGURE 3.1

Proof. To show that (P, T) has Property 1 we let h be a linear embedding of Bd P in the plane and place the image of w_i near $h(v_i)$

in Int $h(\operatorname{Bd} P)$ and on the bisector of angle $h(v_{i-1})h(v_i)h(v_{i+1})$. The linear embedding of P can be completed since T restricted to D is super.

To show that (P, T) has Property 2 we suppose that f is the identity and $g(w_i) = w'_i$ is near v_i and on the bisector of angle $v_{i-1}v_iv_{i+1}$. We wish to push w_i to w'_i.

Let X_i be an interior diagonal of the disk $v_iv_{i+1}w_{i+1}w_i$. If the disk is convex, there are two choices of X_i, but if it is concave, there is only one choice.

If the X's can be chosen so that some w_j is not the end of any X, push this w_j along w_jv_j to a point near v_j. Then push it to a point w''_j near v_j so that $v_{j-1}v_jw''_jw_{j-1}$ and $v_jv_{j+1}w_{j+1}w''_j$ are convex. Now their diagonals can be chosen in two ways. We pick them to contain w''_j and find that there is another w_k with no X containing it. It in turn is moved into a position w''_k analogous to w''_j above. Continuing, we find that if we can get started and can move one w_i to a good position, the others can be pushed into position also. Once each w_j has been moved to a point w''_j near v_j so that each quadrilateral $v_jv_{j+1}w''_{j+1}w''_j$ is convex, each w''_j is pushed to a point on the bisector of angle $v_{j-1}v_jv_{j+1}$. Finally, each w_j is moved along the bisector to w'_j. As the boundary of each quadrilateral disk $v_iv_{i+1}w_{i+1}v_i$ is moved it is a minor matter to move the vertex that is on the interior of the quadrilateral.

If the X's are chosen so that each w_j is the end of an X then each w_j is the end of precisely one X since there are the same number of X's as w's. By considering a new triangulation T' of A whose 2-simplexes are those into which the X's divide the quadrilaterals, we note that the new triangulation of A makes it satisfy the hypothesis of Theorem 3.1. The result then follows from Theorem 3.1.

The next theorem is a generalization of Theorem 3.2 in which we allow the annulus to have a slightly different triangulation.

THEOREM 3.3. *Suppose (P, T), A, D, are as in Theorem 3.2 except that $\operatorname{Bd} P$ has an extra vertex v_{n+1}, counting is $\operatorname{mod} n$ in subscripting the w's but $\operatorname{mod} n + 1$ in subscripting the v's and T restricted to A has an additional 2-simplex $v_{n+1}v_1w_1$. Then T is super if T restricted to D is super.*

Figure 3.2 gives a schematic view of the quadrilaterals in A. It does not show the vertices of T in the interiors of these quadrilaterals since these interior vertices can be dragged along as the boundaries of the quadrilaterals are moved.

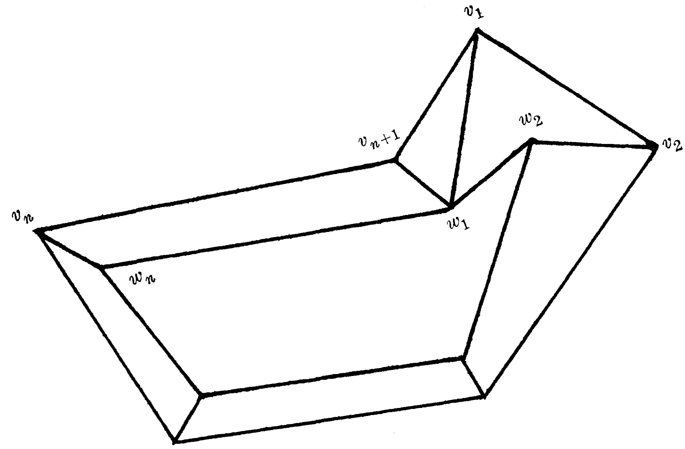

FIGURE 3.2

Proof that (P, T) *has Property* 1. Let H be a linear embedding of $\text{Bd}\,P$ in E^2. For convenience we denote $h(v_i)$ by v_i and the disk bounded by $h(\text{Bd}\,P)$ by P. We cannot hope to put the images of the w's near the corresponding v's if $v_n v_1$ is not a spaninng arc of P.

Let k be the largest of $1, 2, \cdots, n+1$ for which $v_{k-1}v_{k+1}$ is a spanning arc of P. The image of w_i is denoted by w_i' and is located as follows: w_i' is in $\text{Int}\,P$ near v_i and on the bisector of angle $v_{i-1}v_iv_{i+1}$ if $i = 1, 2, \cdots, k-1$; w_i' is near v_{i+1} and on the bisector of angle $v_iv_{i+1}v_{i+2}$ if $i = k, \cdots, n$.

Proof that (P, T) *has Property* 2. We suppose f is the identity map and $g(w_i) = w_i'$ is as described in the preceding paragraph. The cases where $n = 3, 4$ need a slightly different approach so we do not include them.

We start by retriangulating A as was done in the proof of Theorem 3.2. This is done by removing the vertex inside each quadrilateral $v_iv_{i+1}w_{i+1}w_i$ and using a diagonal X_i to divide the quadrilateral into two 2-simplexes. If the quadrilateral is convex there two choices for X but if it is concave there is only one choice. Note that with the new triangulation T' of A, the sum of the orders of the vertices on $\text{Bd}\,D$ is $4n + 1$.

The proof is now broken into steps with Steps *a*, *b*, and *c* being used to push certain w's near their corresponding v's, Steps d and e to push all w's near $\text{Bd}\,P$, and Step f to push the w's to the w''s.

Step a. *Pushing one* w. If some $w_i (i \neq 1)$ is not on any X, it is of order 3 in T' and we push w_i along w_iv_i to a point near v_i and then to a point w_i'' near v_i so that both $v_{i-1}v_iw_i''w_{i-1}$ and $v_iv_{i+1}w_{i+1}w_i''$

are convex. We do not claim that w''_i is near w'_i since w'_i may be near v_{i+1} rather than near v_i.

If no w_i is of order 3, then one w is of order 5 and the others are of order 4. In this case we pick a w_j of order 4 such that $w_{j-1}w_{j+1}$ is a spanning arc of D. Figure 3.3 shows one possibility where $j \neq 1$ and dotted

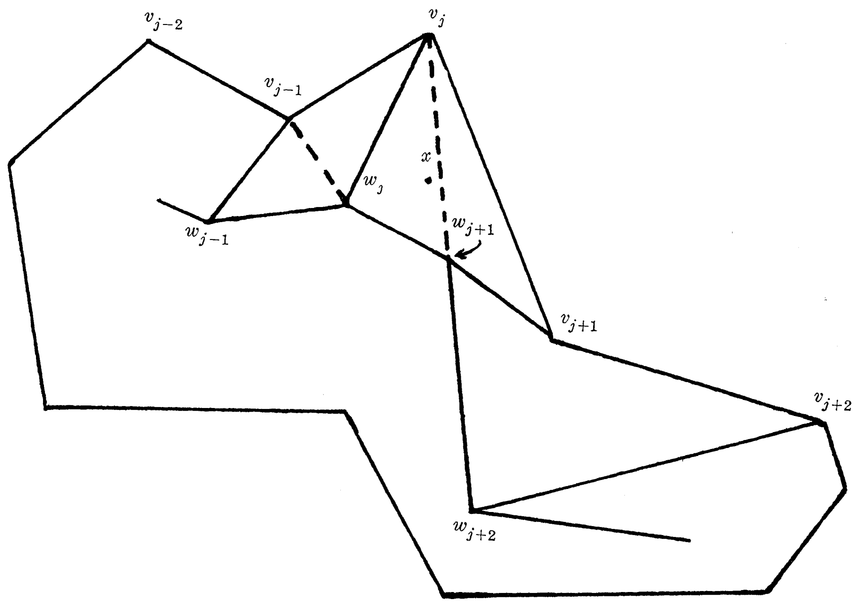

FIGURE 3.3

X's lean one way. For any spanning segment $(w_{j+1}v_j)$ of a 5-sided planar disk $(w_{j-1}w_{j+1}v_{j+1}v_jv_{j-1})$ one end or the other is in the closure of the points which can see the whole boundary. Let x be a point which can see the boundaries of both $w_{j-1}w_{j+1}v_jv_{j-1}$ and $w_{j-1}w_{j+1}v_{j+1}v_jv_{j-1}$ and push w_j to x. For convenience suppose that $x = w_j$. Note that $w_jw_{j+1}v_{j+1}v_j$ is convex and its triangulation can be changed by replacing X_j by the other diagonal. This makes w_j of order 5 in the new triangulation and causes some other w_i to be of order 3. While the above argument is based on Figure 3.3 the argument is similar if the X's lean the other way or if $j = 1$.

Since the new triangulation made w_i of order 3, it can be pushed near v_i as previously described. We say that w_i was crushed. Since each of the quadrilaterals containing this crushed w_i is convex, we suppose that the triangulation of the adjusted A is such that the crushed w_i is of order 5.

Step b. *Pushing another* w. By considering another spanning segment w_iw_{i+2} of D, it can be shown that we can move other w's (leaving the one fixed considered in Step a) and get a new triangulation of the resulting A such that another w_j is of order 3 in the new triangulation. This w_j is pushed to a point w_j'' near v_j so that the quadrilaterals of A containing w_j'' are both convex.

Step c. *Pushing a string of* w's. We continue pushing the w_i's $(i \neq 1)$ to points near their corresponding v_i's as long as this can be done. It can be shown that if both w_i and w_j have been crushed with $i \neq j$ then all w's between them have been crushed. Hence a string of adjacent w's have been crushed and the string has at least two crushed w's. Figure 3.4 shows the situation. The diagonals in the quadrilaterals with a crushed vertex are not shown since they are convex and the X's can lean either way. The quadrilaterals which do not have a crushed vertex are concave and have their X's as drawn. (Figure 3.4 is schematic and shows them as convex rather than as concave.) Let w_f and w_l be the first and last w's respectively that are crushed.

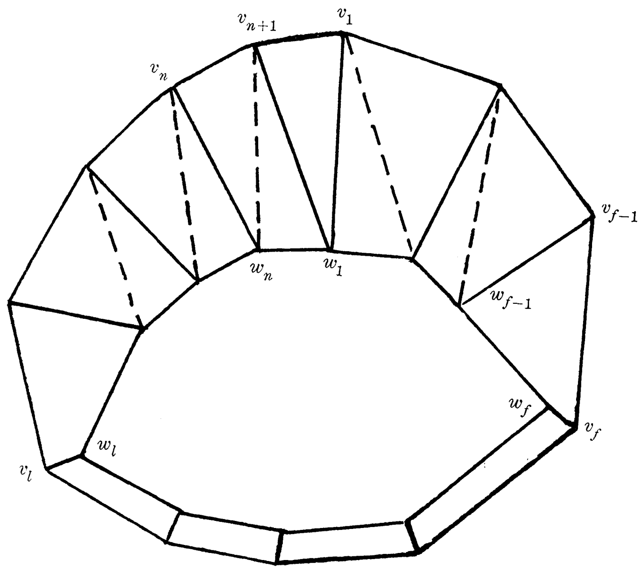

FIGURE 3.4 (Schematic)

Step d. *Leaning* w's. Let $w_{j-1}w_{j+1}$ be a spanning segment of D. Since w_j can be adjusted and the triangulation adjusted to make w_j of order 5, we note that $j \in \{f-1, f, \cdots, l, l+1\}$. We suppose j was selected to be minimal. It is not l or $l+1$.

If $j = f-1$, we consider the 5-sided disk $w_{f-2}w_fv_fv_{f-1}v_{f-2}$ and note that w_{f-1} can either be pushed to a point near w_f or to a point near v_{f-1}. Since we are assuming that $v_{f-1}w_{f-1}w_{f-2}v_{f-2}$ is not convex, we know that w_{f-1} cannot be pushed to a point near v_{f-1}. Therefore we push w_{f-1} to a point near w_f and say that w_{f-1} leans forward.

After w_{f-1} is moved to a point near w_f, w_f is moved to a point near v_{f+1}, w_{f+1} to a point near $v_{f+2}, \cdots, w_n$ to a point near v_{n+1}, w_1 to a point near $v_1, \cdots$, and w_{f-2} to a point near v_{f-2}. In this case we have pushed all the w's near the v's but not necessarily to the corresponding v''s.

Similarly, if $j = f + 1, \cdots, l - 1$ we can lean certain w's forward and send others near their corresponding v's.

If $j = f$, we are faced with a different situation. See Figure 3.5. Although $w_{f-1}w_{f+1}$ is a spanning arc of D, $v_{f-1}v_{f+1}$ need not be a spanning arc of P. However, we push w_f along w_fw_{f+1} to a point

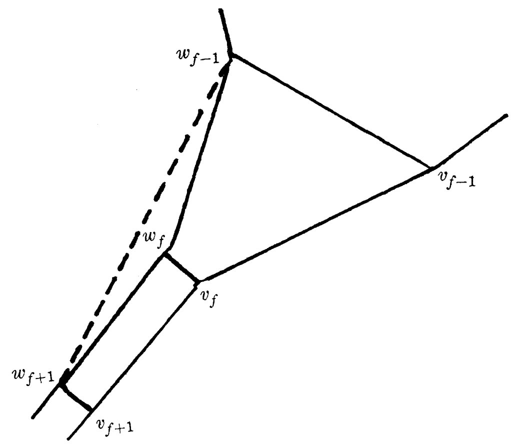

FIGURE 3.5

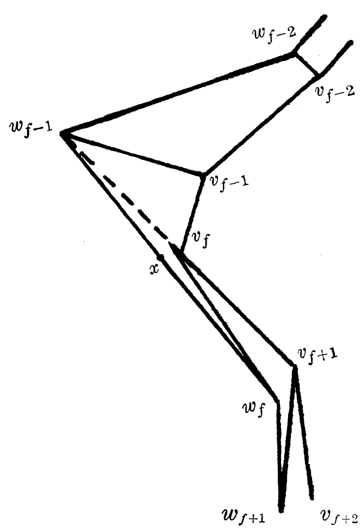

FIGURE 3.6

near w_{f+1}, which is close to v_{f+1}. This move may destroy the convexity of $v_f w_f w_{f-1} v_{f-1}$. However we push w_{f+1} to a point near $v_{f+2}, \cdots, w_n$ to a point near v_{n+1}, w_1 to a point near $v_1, \cdots,$ and w_{f-2} to a point near v_{f-2}. Figure 3.6 shows the difficulty of proceeding to push w_{f-1} down to a point near v_{f-1}.

Step e. *Pushing the last w close to* Bd P. If $w_f w_{f-1} v_{f-1} v_f$ were convex we could push w_{f-1} to a point near v_{f-1}, but if it is concave, v_f may block the edge $w_{f-1} w_f$. We suppose this is the case and push w_{f-1} toward v_{f-1} but stop before $w_{f-1} w_f$ hits v_f. Now there is a point x on $w_{f-1} w_f$ very close to v_f so that x can see v_{f-1} and v_{f+1}.

Let $w_{j-1} w_{j+1}$ be a spanning arc of the new D such that $j \neq f-2$ or $f-1$. If $j = f$, consider $w_{f+1} v_{f+1} v_f w_{f-1}$ and push w_f to x. Now w_f is near v_f and w_{f-1} can be pushed along $w_{f-1} v_{f-1}$ until it is near v_{f-1}.

If $j = f+1, \cdots, n$, we push w_j to a point near v_j, then w_{j-1} to a point near $v_{j-1}, \cdots, w_f$ to a point near v_f and w_{f-1} to a point near v_{f-1}. If $j = 1, 2, \cdots, f-3$ we push w_j to a point near w_{j+1}, then w_{j+1} to a point near $v_{j+2}, \cdots,$ and finally w_{f-1} to a point near v_f. It is to be noted that each w_i is now near either v_i or v_{i+1}, that a string (perhaps null) of w's lean forward and the rest are near their corresponding v's.

Step f. *The final moves.* We recall that k is the largest of $1, 2, \cdots, n+1$ for which $v_{k-1} v_k$ is a spanning arc of P. Hence it is not 1 or 2.

If there is no w near v_k, the w's are now in their standard position.

Suppose some w is near v_k. This w is either w_{k-1} or w_k. We let v_r be the v with no w near it.

We now show that w_k is not near v_k. If it were, $w_1, w_2, \cdots, w_k$ would be near $v_1, v_2, \cdots, v_k$ respectively and $r = k+1, k+2, \cdots,$ or $n+1$. But then there would be a spanning arc $v_{r-1} v_r$ of P and this violates the definition of k.

If w_{k-1} is near v_k, we push w_{k-1} to a point near v_{k-1}, w_{k-2} to a point near $v_{k-2}, \cdots,$ and w_r to a point near v_r. Each w is now near the correct v and only a small adjustment is necessary to move each vertex w to standard position.

THEOREM 3.4. *For each integer $n \geqq 3$, there is a triangulated disk (P_n, T_n) such that P_n has n sides and T_n is super.*

Proof. For $n = 4$ or 5 we can produce T_n by coning over Bd P_n.

Once a (P_{n-1}, T_{n-1}) is obtained, one can construct a (P_n, T_n) by putting an annulus A as described in Theorem 3.3 about P_{n-1}.

Question 3.1. It may be noted that for $n > 4$, the T_n we described has $2n^n - n - 40$ 2-simplexes. How low could one go?

Question 3.2. Let T be a triangulation of a disk which satisfies Property 1 of a super triangulation. Is T super?

4. Super subdivisions. In this section we prove (Theorem 4.2) that every triangulation of a disk has a subdivision which is super and does not subdivide the boundary. Theorem 4.1 is the principal tool used in the proof of Theorem 4.2 and is of the same type as Theorems 3.1, 3.2, and 3.3.

THEOREM 4.1. *Suppose (P, T) is a triangulated disk and A is a subcomplex of (P, T) such that A is an annulus containing* Bd P, *the closure of $P - A$ is a disk D, and A has the following triangulation. Namely, A is the union of n 4-sided disks $v_i v_{i+1} w_{i+1,1} w_{i,k_i}$ $(i = 1, 2, \cdots, n$, counting is* mod n, *and for each j and k, $v_j \in$* Bd P *and $w_{j,k} \in$* Bd D) *together with 2-simplexes $v_i w_{i,j} w_{i,j+1}$ $(j = 1, 2, \cdots, k_i - 1)$ where T restricted to A contains the 2-simplexes $v_i w_{i,j} w_{i,j+1}$ together with those determined by coning over the boundary of each 4-sided disk from an interior point.* (*See Figure* 4.1.)

Then T is super if T restricted to D is super.

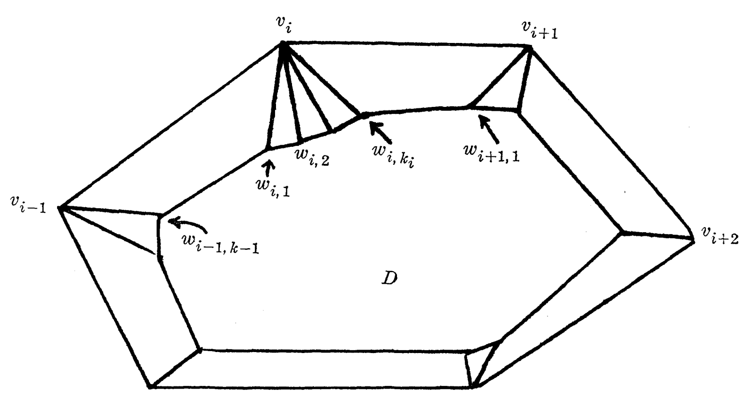

FIGURE 4.1

Proof. Theorem 3.2 is a special case of this theorem and is actually the most difficult case.

The proof that (P, T) has Property 1 is essentially identical to that in Theorem 3.2 so we do not do it.

If $k_i = 1$ for each i, then (P, T) satisfies the hypotheses of Theorem 3.2 and, therefore, T is super.

We assume that there is a j for which $k_j \neq 1$ and proceed to prove that T has Property 2. We assume that f is the identity and that for each i and j, $g(w_{i,j})$ is a point near v_i and near the bisector of angle $v_{i-1}v_iv_{i+1}$.

For each j where $1 < j < k_i$, push $w_{i,j}$ straight along $v_iw_{i,j}$ to a point near v_i.

In each quadrilateral $v_iv_{i+1}w_{i+1,1}w_{i,k_i}$ draw a diagonal X_i. We have added n diagonals. Since for some j, $k_j \neq 1$, there is a vertex $w_{i,1}$ or w_{i,k_i} which is not met by any X. Suppose $w_{i,1}$ is not met. (The other case is analogous.) That vertex $w_{i,1}$ can be pushed straight along $v_iw_{i,1}$ to a point near v_i and then to a point $w'_{i,1}$ so that $v_{i-1}v_iw'_{i,1}w_{i-1,k_{i-1}}$ is convex. Replace its diagonal by the one that contains $w'_{i,1}$. This change guarantees the existence of another vertex $w_{j,1}$ or w_{j,k_j} which is not met by any diagonal. This vertex can now be pushed toward v_j as was done before. Continuing in this fashion, all the vertices $w_{i,j}$ can be pushed near their corresponding vertices v_i. An additional slight adjustment will bring each $w_{i,j}$ to the desired location $g(w_{i,j})$.

THEOREM 4.2. *Every triangulation T of a disk P has a super subdivision which does not subdivide the boundary.*

Proof. The proof is by induction on n, the number of interior vertices of T.

Case $n = 0$. Suppose T is a triangulation of a disk P which contains no interior vertices. Suppose $\operatorname{Bd} P$ has k sides.

Let $\{A_i\}_{i=1}^{k-3}$ be the 1-simplexes of T which hit $\operatorname{Int} P$. Subdivide each A_i by adding k interior vertices. The super subdivision T' of T is now obtained by examining each 2-simplex σ of T, noting that $\operatorname{Bd} \sigma$ has been subdivided and giving σ a super triangulation without further subdivision of $\operatorname{Bd} \sigma$ using Theorem 3.4.

We now claim that T' is super. First notice that any linear embedding of $\operatorname{Bd} P$ can be extended to a linear embedding of the A_i's since they each have so many bends that they can be laid along the embedded $\operatorname{Bd} P$. Since each subdisk has a super triangulation, this embedding of $\operatorname{Bd} P \cup (\bigcup_{i=1}^{k-3} A_i)$ can be extended over each subdisk into which the A_i's divide P.

Next we show that T' has Property 2. Suppose $g, h\colon (P, T') \to E^2$ are two linear embeddings which agree on $\operatorname{Bd} P$. The plan is to push both $g(A_i)$ and $h(A_i)$ to a common arc for each i and then use the properties of the super triangulations of each subdisk of P into which the A_i's divide P to complete the proof.

Suppose $A_1 = v_1v_3$ is a 1-simplex of T belonging to a shellable 2-simplex $v_1v_2v_3$ of T. Use Theorem 2.4 to push $g(A_1)$ to an arc $g'(A_1)$ with a push that leaves $g(\mathrm{Bd}\, P \cup (\bigcup_{i=2}^{k-3} A_i))$ fixed and makes $g'(A_1)$ lie smoothly near $g(v_1v_2 \cup v_2v_3)$ so that $g'(A_1) \cap h(v_1v_2v_3) = g(v_1) \cup g(v_3)$.

Next use Theorem 2.4 to push $h(A_1)$ to the arc $h'(A_1)$ with a push that leaves $h(\mathrm{Bd}\, P \cup (\bigcup_{i=2}^{k-3} A_i))$ fixed and makes $h'(A_1) = g'(A_1)$. Following a shelling of T and repeating the above process of moving $g(A_i)$ first and then moving $h(A_i)$ to agree, the $g(A_i)$'s can be made to agree with the $h(A_i)$'s.

Using the fact that each subdisk into which the A_i's divide P has a super triangulation, these pushes of arcs can be extended to make the pushed g agree with the pushed h.

The inductive step. Suppose T has n interior vertices and the theorem is true for triangulations with fewer than n interior vertices.

Let w be an interior vertex of T so that $\mathrm{Lk}\,(w) \cap \mathrm{Bd}\, P$ contains a vertex v. Let A be a tight annular neighborhood of $\mathrm{Bd}\, P$ which contains no interior vertices of P. Let $D = \mathrm{Cl}\,(P - A)$. Let $z = vw \cap \mathrm{Bd}\, D$. Let z' and z'' be two points on $\mathrm{Bd}\, D$ on either side of z and very close to z. Let A^+ be the larger annulus whose inner boundary component contains $z'w$ and $z''w$ rather than $z'z$ and $z''z$. Find a triangulation $T(A^+)$ of A^+ which is a subdivision of T and makes A^+ into an annulus as described in Theorem 4.1. Now $\mathrm{Cl}\,(P - A^+)$ can be given a triangulation $T(\mathrm{Cl}\,(P - A^+))$ which is a subdivision of T, which has no additional interior vertices, and so that $T(A^+) \,|\, \mathrm{Bd}\,(\mathrm{Cl}\,(P - A^+))$ is a subcomplex. By induction $T(\mathrm{Cl}\,(P - A^+))$ has a super subdivision T' which does not subdivide $\mathrm{Bd}\,(\mathrm{Cl}\,(P - A^+))$. By Theorem 4.1, $T' \cup T(A^+)$ is a super triangulation of P.

The following result is an immediate corollary of Theorem 4.2. It appears with a different proof in [1, Theorem 5.2].

COROLLARY 4.3. *Let f be a* PL *homeomorphism of a* PL *disk P in E^2 which is fixed on* Bd P. *Then there is a triangulation T of P and a push of (P, T) which takes the identity to f and leaves* Bd P *fixed throughout.*

Question 4.1. Let T be a super triangulation of a disk P. Is every subdivision of T which does not subdivide $\mathrm{Bd}\, P$ also super?

Note. Example 2 in the introduction can be constructed with only three interior vertices. No such example could have only one interior vertex. In a preprint of this paper we posed the question of whether such an example could be constructed with only two interior vertices. C. W. Ho has recently answered this question in the negative by proving that the space of all linear homeomorphisms of an n-cell (C, T) which agree on Bd C and where T has only two interior vertices is a contractible space given the compact-open topology [3, p. 2].

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