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ON THE EXPANSION IN JOINT GENERALIZED EIGENVECTORS

KLAUS GERO KALB

Let $\mathscr{A}$ be a family commuting selfadjoint of (normal) operators in a complex (not necessarily separable) Hilbert space H. A natural triplet $\phi \subset H \subset \phi'$ is described, such that (1) $\mathscr{A}$ possesses a complete system of joint generalized eigenvectors in ϕ'; (2) the joint generalized point spectrum of $\mathscr{A}$ essentially coincides with the joint spectrum of $\mathscr{A}$; (3) the generalized point spectra, generalized spectra and spectra essentially coincide for all $A \in \mathscr{A}$; (4) the simultaneous diagonalization of $\mathscr{A}$ in H by means of its spectral measure extends to ϕ'. Also the multiplicity of the joint generalized eigenvectors of $\mathscr{A}$ is discussed.

Let ϕ be a locally convex space, which is embedded densely and continiously into H, such that $A\phi \subset \phi$ and $\dot{A} = A|\phi \in \mathscr{L}(\phi)$ for all $A \in \mathscr{A}$. Consider the triplet $\phi \subset H \subset \phi'$. A joint generalized eigenvector of $\mathscr{A}$ with respect to the joint generalized eigenvalue $(\lambda_A)_{A \in \mathscr{A}} \in \prod_{A \in \mathscr{A}} C$ is a continuous linear form $x' \in \phi'$ such that

$$x' \neq 0 \quad \text{and} \quad \dot{A}'x' = \lambda_A \cdot x' \quad \text{for all} \quad A \in \mathscr{A}. \tag{1.1}$$

The system $\mathfrak{E}$ of all joint generalized eigenvectors of $\mathscr{A}$ is called complete, if $\langle \varphi, e' \rangle = 0$ for all $e' \in \mathfrak{E}$ implies $\varphi = 0$ ($\varphi \in \phi$). For H separable there is a number of conditions on ϕ, under which $\mathfrak{E}$ is complete (cf. e.g., [14], [3]), and there also are effective constructions of ϕ with respect to a given family $\mathscr{A}$ (cf. [13], [14] for $\mathscr{A}$ countable; [15]). The fact that especially in the case of a single normal operator there generally exist many more joint generalized eigenvalues and eigennvectors than necessary (and reasonable in physical applications) has led to recent investigations ([15], [16]; [1]; [2]; [5]; [8], [9]). Let $\sigma_P(\dot{\mathscr{A}}')$ be the joint generalized point spectrum of $\mathscr{A}$ (i.e., the set of all joint generalized eigenvalues of $\mathscr{A}$), let $\sigma(\mathscr{A})$ be the joint spectrum of $\mathscr{A}$ as defined in Gelfand theory (cf. § 2). Let $\mathscr{B}$ be the (commutative) C^*-algebra generated by $\mathscr{A}$ and 1. In the present work we propose the construction of a natural triplet $\phi \subset H \subset \phi'$, by which the following is achieved:

(a) $\sigma_P(\dot{\mathscr{A}}') \subset \overline{\sigma_P(\dot{\mathscr{A}}')} = \sigma(\mathscr{A})$;

(b) $\sigma_P(\dot{B}') \subset \overline{\sigma_P(\dot{B}')} = \sigma(\dot{B}') = \sigma(B)$ for all $B \in \mathscr{B}$;

(c) the simultaneous diagonalization of $\mathscr{B}$ by means of its spectral measure can be transferred to $\dot{\mathscr{B}}'$.

For H separable we can even attain $\sigma_P(\dot{\mathscr{A}}') = \sigma(\mathscr{A})$ and $\sigma_P(\dot{B}') = \sigma(B)$ for all $B \in \mathscr{B}$, and also have a description of the multiplicity of the joint generalized eigenvalues.

In the case of a single selfadjoint operator our method reduces to that of [9] (cf. also [11]) and for $\mathscr{A} = \mathscr{B}$ is similar to that of [15] where for H separable the equation $\sigma(\mathscr{B}) = \sigma_P(\dot{\mathscr{B}}')$ is realized. The basic idea of the construction, due to R. A. Hirschfeld [7], is to choose (by means of an appropriate spectral representation of $\mathscr{B}$) the space ϕ as a space of continuous functions with compact support on a locally compact space R (or as a space of continuous vector fields, if the theory of R. Godement [6] is used), such that the joint generalized eigenvectors essentially are the point masses (characters).

2. Simultaneous diagonalization and spectral decomposition. In this section we summarize the spectral and multiplicity theory of [17], [18], [19]. Let S be the spectrum of $\mathscr{B}$, i.e., the set of all (continuous) homomorphisms of $\mathscr{B}$ onto $\boldsymbol{C}$, endowed with the usual topology. Let $\hat{B}(\cdot)\colon S \to \boldsymbol{C}$, defined by $\hat{B}(s) = s(B)$ $(s \in S)$, be the Gelfand transform of $B \in \mathscr{B}$. The application $\mathscr{B} \ni B \mapsto \hat{B}(\cdot) \in C(S)$ is an isometrical $*$-isomorphism of $\mathscr{B}$ onto $C(S)$. Let $E(\cdot)$ be the spectral measure of $\mathscr{B}\colon B = \int_S \hat{B}(s)dE(s)$ $(B \in \mathscr{B})$. The joint spectrum (cf. [18], p. 150) of $\mathscr{A}$, denoted $\sigma(\mathscr{A})$, is defined by $\sigma(\mathscr{A}) = \{(\hat{A}(s))_{A \in \mathscr{A}}\colon s \in S\}$. $\sigma(\mathscr{A}) \subset \prod_{A \in \mathscr{A}} \sigma(A)$ is homeomorphic to S under the application

$$\kappa\colon S \ni s \longmapsto (A(s))_{A \in \mathscr{A}} \in \sigma(\mathscr{A})\ . \tag{2.1}$$

Choose a decomposition $H = \bigoplus_{i \in I} H_i$, such that $\mathscr{B} H_i \subset H_i$ and $\mathscr{B}_i = \mathscr{B}|_{H_i}$ possesses a cyclic vector x_i $(i \in I)$. Let S_i be the spectrum of $\mathscr{B}_i$ $(i \in I)$. Then there is a family $(m_i)_{i \in I}$ of positive Borel measures on S_i with support S_i inducing a spectral representation $H \leftrightarrow \bigoplus_{i \in I} L^2(S_i, m_i)$. Thereby H_i is transferred in $L^2(S_i, m_i)$, especially x_i in 1_{S_i} $(i \in I)$; an operator $B \in \mathscr{B}$ is converted in the multiplication by $(\hat{B}_i(\cdot))_{i \in I}$, where $\hat{B}_i(\cdot)$ $(=\hat{B}(\cdot)|_{S_i}$ if S_i is considered as a subset of $S)$ denotes the Gelfand transform of $B|_{H_i}$ $(i \in I)$; a spectral projection $E(\mathfrak{b})$, $\mathfrak{b}$ a Borel subset of S, is transferred in the multiplication by $(\chi_{\mathfrak{b} \cap S_i})_{i \in I}$. Finally we have $m_i(\cdot) = (E(\cdot)x_i, x_i)$ $(i \in I)$. When H is separable, we can choose $I = \boldsymbol{N}$ and achieve by a normalization (cf. [17], [10]) that (in an essentially unique manner) $m_1 > m_2 > \cdots$, particularly $S = S_1 \supset S_2 \supset \cdots$. The (well defined) function

$$m_H(s) = \#\{n \in \boldsymbol{N}\colon s \in S_n\} \quad (s \in S) \tag{2.2}$$

is called the Hellinger-Hahn multiplicity function of $\mathscr{B}$.

We return to the general case, in which, for the sake of simplification of notation, we formulate the affirmations concerning spectral decompositions in a somewhat different way (cf. [19]): We consider the sets S_i $(i \in I)$ as pairwise disjoint sets $\tilde{S}_i$ $(i \in I)$ and define $R = \dot{\bigcup}_{i \in I} \tilde{S}_i$. A set $V \subset R$ is defined to be open, if for all $i \in I$ the set $V \cap \tilde{S}_i$ (interpreted as a subset of S_i) is open in S_i. With that R is a locally compact topological Hausdorff space; each S_i is open and compact in R. A function $f: R \to \boldsymbol{C}$ belongs to $C_c(R)$ if and only if $f|_{\tilde{S}_i} \in C(S_i)$ for all $i \in I$ and $f|_{\tilde{S}_i} = 0$ for all but finitely many $i \in I$. Define a Radon measure μ on R by

$$\mu(f) = \int_R f \cdot d\mu = \sum_{i \in I} \int_{S_i} f \cdot dm_i \quad (f \in C_c(R)) .$$

Then there is a spectral representation $H \leftrightarrow L^2(R, \mu)$ of $\mathscr{B}$ by which $\mathscr{B}$ is converted in a subalgebra of the multiplication algebra $BC(R)$ (: = algebra of bounded continuous numerical functions on R) on $L^2(R, \mu)$: $\mathscr{B} \ni B \mapsto$ multiplication by $\tilde{B}(\cdot) \in BC(R)$, where $\tilde{B}(r) := \hat{B}(\lambda r)$ $(r \in R)$. Here $\lambda: R \to \bigcup_{i \in I} S_i \subset S$ is the natural surjection. Finally we shall need:

(2.3) $\quad E(\cdot)$ is concentrated on $\bigcup_{i \in I} S_i$; particularly $\overline{\bigcup_{i \in I} S_i} = S$;

(2.4) $$\|B\| = |\hat{B}(\cdot)|_{C(S)} = |\tilde{B}(\cdot)|_{BC(R)} \qquad (B \in \mathscr{B});$$

(2.5) $$\sigma(B) = \hat{B}(S) = \overline{\tilde{B}(R)} \qquad (B \in \mathscr{B}) .$$

($|\cdot|$ denotes the supremum norm.)

3. Expansion in joint generalized eigenvectors. We proceed now to the construction of the triplet $\phi \subset H \subset \phi'$. We assume without loss of generality that $H = L^2(R, \mu) \leftrightarrow \bigoplus_{i \in I} L^2(S_i, m_i)$ and $\mathscr{B} \subset CB(R)$. Let $\phi := C_c(R)$. It is easy to see that ϕ is topologically isomorphic to the locally convex direct sum $\dot{\sum}_{i \in I} C(S_i)$ (considered in [9]). ϕ satisfies with respect to $\mathscr{B}$ (and $\mathscr{A}$) all the prerequisites listed in the introduction. For $r \in R$ define $e'(r) \in \phi'$ by $\langle \varphi, e'(r) \rangle = \varphi(r)$ $(\varphi \in \phi)$.

THEOREM (3.1). (i) $\dot{B}'e'(r) = \tilde{B}(r) \cdot e'(r)$ $(B \in \mathscr{B}, r \in R)$.

(ii) $(\varphi, \psi) = \int_R \langle \varphi, e'(r) \rangle \overline{\langle \psi, e'(r) \rangle} d\mu(r)$ $(\varphi, \psi \in \phi)$ [(i) *and* (ii) *mean that* $\mathfrak{E} = \{e'(r): r \in R\}$ *is a complete system of joint generalized eigenvectors of* $\mathscr{B}$].

(iii) $\sigma_P(\dot{B}') = \tilde{B}(R)$ $(B \in \mathscr{B})$.

(iv) $\sigma(\dot{B}') = \overline{\sigma_{c1}(\dot{B}')} = \sigma(B)$ $(B \in \mathscr{B})$.

Here $\sigma(\dot{B}')$ denotes the spectrum of $\dot{B}'$ in the sense of Waelbroeck (cf. e.g., [12]) and $\sigma_{c1}(\dot{B}')$ is defined as the set of those $z \in \boldsymbol{C}$, for which $\dot{B}' - z$ is not invertible in $\mathscr{L}(\phi')$. Thereby on ϕ' always is considered the strong topology and on $\mathscr{L}(\phi')$ the topology of uniform convergence on bounded subsets of ϕ.

Proof. (i), (ii) are direct consequences of our construction. (iii): Let $B \in \mathscr{B}$. Because of (i) we only have to show that $\sigma_P(\dot{B}') \subset \tilde{B}(R)$. Let $z \in \sigma_P(\dot{B}')$ and suppose that $z \notin \tilde{B}(R)$. Choose $x' \in \phi'$ such that $x' \neq 0$ and $\dot{B}'x' = zx'$. Let $\varphi \in \phi$ be arbitrary. Then there exists $\psi \in \phi$ such that $\varphi(r) = (\tilde{B}(r) - z) \cdot \psi(r)$ $(r \in R)$. Hence $\langle \varphi, x' \rangle = \langle (\tilde{B}(\cdot) - z) \cdot \psi(\cdot), x' \rangle = \langle \psi, (\dot{B}' - z)x' \rangle = 0$, i.e., $x' = 0$. Contradiction. (iv): By (iii) we have $\sigma(B) = \overline{\tilde{B}(R)} = \overline{\sigma_P(\dot{B}')} \subset \overline{\sigma_{c1}(\dot{B}')} \subset \sigma(\dot{B}')$. It remains to show that $\sigma(\dot{B}') \subset \overline{\tilde{B}(R)}$: Let $z \notin \overline{\tilde{B}(R)}$. To demonstrate that $z \notin \sigma(\dot{B}')$, the two cases $z = \infty$ and $z \in \boldsymbol{C}$ have to be treated seperately. Let $z = \infty$. Choose $C > 0$ such that $|\tilde{B}(r)| \leqq C$ $(r \in R)$. Then $U := \{\infty\} \cup \{w \in \boldsymbol{C} : |w| \geqq 2 \cdot C\}$ is a neighborhood of ∞, and $|(\tilde{B}(r) - w)^{-1}| \leqq 1/C$ $(r \in R)$ for $w \in U \cap \boldsymbol{C}$. For $w \in U \cap \boldsymbol{C}$ define $Q(w) \in \mathscr{L}(\phi')$ by

$$\langle \varphi, Q(w)x' \rangle = \langle (\tilde{B}(\cdot) - w)^{-1} \cdot \varphi(\cdot), x' \rangle \qquad (\varphi \in \phi,\ x' \in \phi') .$$

It is clear that $Q(w)(\dot{B}' - w) = (\dot{B}' - w)Q(w) = 1$ for all $w \in U \cap \boldsymbol{C}$ and easy to see that $\{Q(w) : w \in U \cap \boldsymbol{C}\}$ is bounded in $\mathscr{L}(\phi')$. Hence $\infty \notin \sigma(\dot{B}')$. If $z \in \boldsymbol{C}$, choose a neighbourhood V of z such that $\bar{V} \cap \overline{\tilde{B}(R)} = \emptyset$ and proceed similarity.

We shall show now that the spectral measure $E(\cdot)$ of $\mathscr{B}$ can be extended to a spectral measure of $\dot{\mathscr{B}}'$.

THEOREM (3.2). *There is an (unique) spectral measure $P(\cdot)$ on S with values in $\mathscr{L}(\phi')$ such that $\dot{B}' = \int_S \hat{B}(s) \cdot dP(s)$ $(B \in \mathscr{B})$ and $P(\cdot)|_H = E(\cdot)$.*

Proof. ϕ' is the space of Radon measures on R. Define $P(\mathfrak{b})x' = \chi_{\lambda^{-1}(\mathfrak{b})} \cdot x'$ ($\mathfrak{b}$ a Borel subset of S, $x' \in \phi'$), i.e., $\langle \varphi, P(\mathfrak{b})x' \rangle = \int_{\lambda^{-1}(\mathfrak{b})} \varphi \cdot dx'$ for $\varphi \in \phi$. It is easily chequed that $P(\cdot)$ is a bounded σ-additive spectral measure in $\mathscr{L}(\phi')$ and that $P(\cdot)|_H = E(\cdot)$. Since ϕ' is complete and barrelled, the integral $\int_S \hat{B}(s) \cdot dP(s)$ $(B \in \mathscr{B})$ exists in the strong sense. An easy calculation shows that $\left\langle \varphi, \int_S \hat{B}(s)\cdot dP(s)x' \right\rangle = \int_S \hat{B}(s) d\langle \varphi, P(s)x' \rangle = \langle B\varphi, x' \rangle$ for all $\varphi \in \phi$, $x' \in \phi'$, i.e., $\int_S \hat{B}(s)\cdot dP(s) = \dot{B}'$.

We now discuss the relations between the joint spectrum and the joint generalized point spectrum of $\mathscr{A}$:

THEOREM (3.3). $\sigma_P(\dot{\mathscr{A}}') \subset \overline{\sigma_P(\dot{\mathscr{A}}')} = \sigma(\mathscr{A})$.

Proof. For $r \in R$ we have by Theorem (3.1) (i) that $(\tilde{A}(r))_{A\in\mathscr{A}} = (\hat{A}(\lambda r))_{A\in\mathscr{A}} \in \sigma_P(\dot{\mathscr{A}}')$ $(r \in R)$. Hence $\kappa(\lambda(R)) = \kappa(\bigcup_{i\in I} S_i) \subset \sigma_P(\dot{\mathscr{A}}')$, where κ is the homeomorphism of (2.1). Because of (2.3) we obtain $\sigma(\mathscr{A}) = \kappa(S) \subset \overline{\kappa(\bigcup_{i\in I} S_i)} \subset \overline{\sigma_P(\dot{\mathscr{A}}')}$. It remains to show that $\sigma_P(\dot{\mathscr{A}}') \subset \sigma(\mathscr{A})$. Let $(\lambda_A)_{A\in\mathscr{A}} \in \sigma_P(\dot{\mathscr{A}}')$; let $x' \in \phi' = C'_c(R)$ be a joint generalized eigenvector of $\mathscr{A}$, i.e., (1.1) holds. Choose $i \in I$ such that $x'_i = x'|_{C(S_i)} \neq 0$. Consider the triplet $\phi_i \subset H \subset \phi'_i$, where $\phi_i = C(S_i)$, $H = L^2(S_i, m_i)$. We then have $(A|_{\phi_i})' x'_i = \lambda_A \cdot x'_i$ $(A \in \mathscr{A})$. We shall show that there exists an (unique) $s_i \in S_i$, such that $\lambda_A = \hat{A}(s_i)$ $(A \in \mathscr{A})$. For the sake of simplification of notation we suppress the index i, i.e., we consider the case of total multiplicity 1 without loss of generality. We first extend the function

$$\mathscr{A} \ni A \longmapsto \lambda_A \in \boldsymbol{C} \tag{3.4}$$

to $\mathscr{B}$ such that (1.1) remains valid. To do this, let $\mathscr{P}(\mathscr{A})$ be the algebra of polynomials in elements of $\mathscr{A}$ and 1. The closure of $\mathscr{P}(\mathscr{A})$ in $\mathscr{L}(H)$ equals $\mathscr{B}$. If $p = p(\alpha_1, \cdots, \alpha_n)$ is a polynomial in n variables, we define $\lambda_B = p(\lambda_{A_1}, \cdots, \lambda_{A_n})$ for $B = p(A_1, \cdots, A_n) \in \mathscr{P}(\mathscr{A})$. By (1.1) we conclude that the function

$$\mathscr{P}(\mathscr{A}) \ni B \longmapsto \lambda_B \in \boldsymbol{C} \tag{3.5}$$

is well defined, constitutes an extension of (3.4) and satisfies

$$\dot{B}'x' = \lambda_B \cdot x' \qquad (B \in \mathscr{P}(\mathscr{A}))\,. \tag{3.6}$$

Observing that $\lambda_B \in \sigma_P(\dot{B}') \subset \sigma(B)$ (cf. (3.1) (iii), hence $|\lambda_B| \leqq \|B\|$, we obtain that the (linear) function (3.5) is continuous. Hence it possesses an unique extension as a continuous function on $\mathscr{B}$, which we again denote by $B \mapsto \lambda_B$ and which satisfies for reasons of continuity the relations

$$\dot{B}'x' = \lambda_B \cdot x' \qquad (B \in \mathscr{B})\,. \tag{3.7}$$

Using this it is easily chequed that $B \mapsto \lambda_B$ is an homomorphism of $\mathscr{B}$ onto $\boldsymbol{C}$ (cf. [15]), i.e., defines an element $s \in S$ such that $\lambda_B = s(B) = \hat{B}(s)$ $(B \in \mathscr{B})$.

The proof shows particularly that a joint generalized eigenvector of $\mathscr{A}$ is automatically one of $\mathscr{B}$.

4. **The multiplicity of the joint generalized eigenvalues.** First we give a supplement to the second part of the proof of Theorem (3.3):

LEMMA (4.1). *x' is a multiple of point mass in s.*

Proof. Recall that $R = S$ (according to our reduction to the cyclic case). (3.7) then means that

$$\langle \hat{B}(\cdot)\cdot\varphi(\cdot), x'\rangle = \hat{B}(s)\cdot\langle\varphi, x'\rangle \qquad (\varphi \in C(S),\ \hat{B}(\cdot)\in C(S)) .$$

This implies that the support of x' is contained in $\{s\}$. [When $\varphi \in C(S)$ is such that $\operatorname{supp}(\varphi) \subset S - \{s\}$, choose $\hat{B}(\cdot) \in C(S)$ such that $\hat{B}(s) = 1$ and $\operatorname{supp}(\hat{B}(\cdot)) \subset S - \operatorname{supp}(\varphi)$. Then $\hat{B}(\cdot)\varphi(\cdot) \equiv 0$ on S, hence $\langle\varphi, x'\rangle = \hat{B}(s)\cdot\langle\varphi, x'\rangle = \langle\varphi, \dot{B}'x'\rangle = \langle B\varphi, x'\rangle = \langle\hat{B}(\cdot)\cdot\varphi(\cdot), x'\rangle = 0$.] This proves the affirmation (since $x' \neq 0$; cf. [4], p. 70).

The lemma shows that the multiplicity of the joint generalized eigenvalues of $\mathscr{A}$ with respect to the triplet $\phi \subset H \subset \phi'$ constructed in § 3 is given by

$$\text{(4.2)} \qquad \operatorname{mult}((\hat{A}(s))_{A\in\mathscr{A}}) = \#\{i \in I: s \in S_i\} \qquad (s \in S) .$$

This formula illustrates the arbitrariness remaining in the selection of the spectral decomposition. Our construction is only well adapted to $\mathscr{A}$ with respect to the spectra.

When H is separable, we can base the construction of ϕ on the "canonical" spectral decomposition described in § 2. We then obtain:

THEOREM (4.3). (i) $\sigma_P(\dot{B}') = \sigma(\dot{B}') = \sigma(B)$ $(B \in \mathscr{B})$.
(ii) $\sigma_P(\dot{\mathscr{A}}') = \sigma(\mathscr{A})$.
(iii) $\operatorname{mult}((A(s))_{A\in\mathscr{A}}) = m_H(s)$ $(s \in S)$.

Proof. (i) and (ii) ensue from $S = S_1$, i.e., $\lambda R = S$, and the proofs of (3.1) and (3.3). (iii) is a consequence of formulas (2.2) and (4.2).

If $\mathscr{A}$ has simple spectrum (i.e., in the separable case: $\mathscr{A}$ possesses a cyclic vector, or, equivalently, $m_H(s) = 1$ $(s \in S)$) because of (4.3) (iii) the following formula holds:

$$\text{(4.4)} \qquad \operatorname{mult}((\lambda_A)_{A\in\mathscr{A}}) = 1 \quad \text{for all} \quad (\lambda_A)_{A\in\mathscr{A}} \in \sigma_P(\dot{\mathscr{A}}') .$$

In the nonseparable case we have the following result concerning multiplicity:

THEOREM (4.5). *If* $\mathscr{A} = \mathscr{B}$ *is maximal Abelian, then* (4.4) *holds.*

Proof. Then to $\mathscr{B}$ corresponds the full multiplication algebra $CB(R)$ on $L^2(R, \mu)$. As $CB(R)$ separates the points of $R = \dot{\bigcup}_{i \in I} \tilde{S}_i$, we obtain that $S_i \cap S_j = \emptyset$ for $i \neq j$. Now the affirmation ensues from (4.2).

The natural extension of the notion "$\mathscr{A}$ possesses simple spectrum" to the nonseparable case is that the von Neumann algebra generated by $\mathscr{A}$ and **1** is maximal Abelian (cf. [19]). Theorem (4.5) says that (4.4) holds, if $\mathscr{A}$ is a von Neumann algebra with simple spectrum. We conclude by formulating a problem: Let $\mathscr{A}$ be an arbitrary system with simple spectrum. How "must" the triplet $\phi \subset H \subset \phi'$ be constructed to obtain (4.4)?

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JOHANNES GUTENBERG-UNIVERSITÄT
D6500 MAINZ
GERMANY