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LINEAR OPERATORS FOR WHICH T^*T AND $T+T^*$ COMMUTE III

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Let Θ denote the set of all linear operators T acting on a separable Hilbert space $\mathscr{H}$ for which T^*T and $T+T^*$ commute. It will be shown that if $T\in\Theta$ and T^* is hyponormal, then T is normal. Also if $T\in\Theta$ and T is hyponormal, then T is subnormal.

1. Introduction. Operators in Θ need not be hyponormal [4], but have many hyponormal-like properties [1]-[4], [7], [8]. Therefore our first result is not surprising.

THEOREM 1. *If $T\in\Theta$ and T^* is hyponormal, then T is normal.*

Let $(QA)=\{T\mid T=Q+A, [Q, Q^*Q]=0, A=A^*, [A; Q]=0\}$ where $[X, Y]=XY-YX$. Then $(QA)\subset\Theta$ [2] and all operators in (QA) are subnormal. In [4] an example of a hyponormal operator in Θ, that is not in (QA), is given. That operator is a block weighted shift. Given that it is much "easier" for a shift to be hyponormal instead of subnormal, our second result is, at least to us, surprising.

THEOREM 2. *If $T\in\Theta$ and T is hyponormal, then T is subnormal.*

2. *Proof.* The proofs of Theorems 1 and 2 are closely related. If A is a positive linear operator with spectral resolution $A=\int\lambda dE(\lambda)$, then A^+ is defined by $A^+=\int\lambda^+dE(\lambda)$, where $\lambda^+=1/\lambda$ if $\lambda\neq 0$ and $0^+=0$. Note that A^+, while possibly unbounded, is self-adjoint, and $\mathscr{D}(A^+)=R(A)$. Here $\mathscr{D}$, R denote domain and range. The null space is denoted N.

Proof of Theorem 2. Suppose $T\in\Theta$ and $[T^*T-TT^*]\geqq 0$. Without loss of generality assume $\|T\|<1$. Let $A=[T^*T-TT^*]^{1/2}$ be the positive square root of $[T^*T-TT^*]$. Then $T^*A^2=A^2T$ since $T\in\Theta$ [1]. Thus $A^+T^*A^2=AT$. Hence, $A^+T^*Ax=ATA^+x$ for all $x\in\mathscr{D}(A^+)$. Let $B=ATA^+$. Since AT is bounded, $B^*=A^+T^*A$, and $B\subseteqq B^*$. But $\lambda-A^+T^*A=A^+(\lambda-T^*)A+\lambda(I-A^+A)$. Since $(i+T^*)$, $(i-T^*)$ are both invertible, both deficiency indices of B are zero. Thus $\bar{B}=B^*$ where $\bar{B}$ is the closure of B [5, p. 1230]. Now on $\hat{\mathscr{H}}=\mathscr{H}\oplus\mathscr{H}\oplus\mathscr{H}$, define

$$N = \begin{bmatrix} T & A & 0 \\ 0 & \bar{B} & A \\ 0 & 0 & T^* \end{bmatrix}.$$

But for all $x \in \mathscr{D}(B) = \mathscr{D}(A^+)$, $AB = T^*A$. Hence $A\bar{B} = T^*A$ for all $x \in (\bar{B})$. Since A, T^* are bounded, we also have $\bar{B}^*A = \bar{B}A = AT$. But then N is closed and $N^*N = NN^*$. Hence N is normal [5, 1258-1259] and

$$(1) \qquad Nx = \lim_{n\to\infty} \int_{|\lambda|\leq n} \lambda F(d\lambda)x\,, \quad x \in \mathscr{D}(N)$$

for a resolution of the identity $F(\cdot)$ defined on the complex plane. $\mathscr{D}(N)$ is just those x for which the limit in (1) exists. Note that $N - N^*$ is bounded and hence the support of $F(\cdot)$ lies in a horizontal strip. Let $\Delta = \{\lambda \,\|\, \lambda\,| \leqq \|T\|\}$. We now wish to show that $F(\Delta)\mathscr{H} = \mathscr{H}$ when $\mathscr{H}$ is imbedded into $\hat{\mathscr{H}}$ by $\mathscr{H} \to \mathscr{H} \oplus 0 \oplus 0$. But $x \in R(F(\Delta))$ if and only if both

(i) $x \in \mathscr{D}(N^m)$ for all $m \geqq 0$

and

(ii) $\|N^m x\|/\|T\|^m \leqq \|x\|$ for all $m \geqq 0$.

Since $\mathscr{H}$ clearly satisfies both (i) and (ii), we have $F(\Delta)\mathscr{H} = \mathscr{H}$. But then $NF(\Delta)$ is a bounded normal extension of T and T is subnormal as desired.

Proof of Theorem 1. Suppose that $T \in \Theta$ and T^* is hyponormal. We shall first show that T^* is subnormal. Let $A = [TT^* - T^*T]^{1/2}$ be the positive square root of $[TT^* - T^*T]$. Again,

$T^*A^2 = A^2T$. Define $B, \bar{B}$ as in the proof of Theorem 2. This time let

$$N = \begin{bmatrix} T & 0 & 0 \\ A & \bar{B} & 0 \\ 0 & A & T^* \end{bmatrix}.$$

Again N is a possibly unbounded normal operator, and one can argue that $N^*F(\Delta)$ is a normal extension of T^*. Hence T^* is subnormal. The remainder of the proof is a modification of the proof of Lemma 2 in [9].

Let $M = \begin{bmatrix} T^* & C \\ 0 & B \end{bmatrix}$ be the normal extension of T^*. Let $L = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$, where $D = [TT^* - T^*T] \geqq 0$. Then $ML = LM^*$ since $T \in \Theta$. Hence by the Fuglede-Putnam theorem $M^*L = LM$ and $LM = M^*L$. Thus

$$DT^* = TD\,,$$
$$DC = 0\,.$$

But $T^*D = DT$ since $T \in \Theta$. Hence

$$DTT^* = T^*TD\,,$$

or equivalently,

$$(TT^* - T^*T)(TT^*) = T^*T(TT^* - T^*T)\,.$$

Simplifying gives

$$(TT^*)^2 + (T^*T)^2 = 2(T^*T)(TT^*)\,.$$

Hence $[T^*T, TT^*] = 0$. But $T \in \Theta$ and $[T^*T, TT^*] = 0$ implies T is quasinormal [6]. Hence T is subnormal. But then T is normal since T and T^* are both subnormal.

It should be noted that one has to consider the extensions of B in the proofs since A^+ may be unbounded. Examples can easily be constructed by taking direct sums of multiples of the block shift in [4].

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