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# (*hnp*)-RINGS OVER WHICH EVERY MODULE ADMITS A BASIC SUBMODULE

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**The structure of those bounded (*hnp*)-rings over which every module admits a basic submodule, is determined. It is shown that such rings are precisely the block lower triangular matrix rings over $D\backslash M$ where $D$ is a discrete valuation ring with $M$ as its maximal ideal.**

In [12], the author generalized some well known results on decomposability of torsion abelian groups to torsion modules over bounded (*hnp*)-rings. Let $R$ be a bounded (*hnp*)-ring and $M$ be a (right) $R$-module. A submodule $N$ of $M$ is called a *basic* submodule of $M$ if it satisfies the following conditions:

( i ) $N$ is decomposable in the sense that it is a direct sum of uniserial modules and finitely generated uniform torsion free modules.

(ii) $N$ is a pure submodule of $M$.

(iii) $M/N$ is a divisible module.

The following result has been proved by the author (see [9] for details):

THEOREM 1. *Any torsion module $M$ over a bounded (hnp)-ring has a basic submodule and any two basic submodules of $M$ are isomorphic.*

In general an $R$-module need not have a basic submodule. However Marubayashi [8, Theorem (3.6)] showed that every module over a $g$-discrete valuation ring has a basic submodule. In this paper we determine the structure of those bounded (*hnp*)-rings, over which every (right) module admits a basic submodule (Theorems 3 and 4).

As defined by Marubayashi [8, p. 432], a prime, right as well as left principal ideal ring $R$, such that its Jacobson radical $J(R)$ is the only maximal ideal, and idempotents modules $J(R)$ can be lifted, is called a $g$-discrete valuation ring; further if $R/J(R)$ is a division ring, then $R$ is called a discrete valuation ring. In view of [8, Lemma (3.1)] and [7, Lemma (2.1)], $g$-discrete valuation rings are precisely the matrix rings over discrete valuation rings. Modules considered will be unital right modules and the notations and terminology of [12, 13] will be used without comment.

Henceforth in all lemmas, $R$ is a bounded (*hnp*)-ring over which every module admits a basic submodule. Further $Q$ stands for the classical quotient ring of $R$.

LEMMA 1. *A submodule $N$ of a torsion free module $M$ over an $(hnp)$-ring $S$ is pure if and only if $M/N$ is torsion free.*

*Proof. Necessity.* Let for some $x \in M$ and a regular element $b$ in $S$, $xb = y \in N$. As $N$ is pure, for some $z \in N$, $xb = zb$. This in turn gives $x = z \in N$. This proves that $M/N$ is torsion free.

SUFFICIENCY. Let $M/N$ be torsion-free. Consider a finite system of equations $\sum_i x_i r_{ij} = s_j$, $s_j \in N$, having a solution $\{x_i\}$ in $M$. If $K = \sum x_i S + N$, then $K/N$ being finitely generated and torsion free, is projective. Hence $K = K_1 \oplus N$. This gives that the above system of equations have a solution in $N$. Hence $N$ is pure in $M$.

LEMMA 2. *If $U$ is a uniform torsion free right $R$-module, then either $U$ is finitely generated, or divisible.*

*Proof.* Since by Lemma 1, 0 and $U$ are only pure submodules of $U$, so 0 or $U$ is the basic submodule of $U$. Hence $U$ is divisible or finitely generated.

LEMMA 3. *Every over-ring of $R$ different from $Q$ is finitely generated as an $R$-module.*

*Proof.* Consider an over ring $S$ of $R$ such that $S \neq Q$. Now $S = \oplus \sum U_i$, $U_i$ are uniform as right $S$-modules, since $S$ is an $(hnp)$-ring [6]. If any $U_i$ is divisible as a right $R$-module, then $S = Q$, otherwise by Lemma 2, $S_R$ is finitely generated.

Let $L$ be any ring and $J$ be an ideal of $L$. Let $n$ be a positive integer and $(k_1, k_2, \cdots, k_r)$ be an ordered $r$-tuple of positive integers such that $k_1 + k_2 + \cdots + k_r = n$. In the notations of Reiner [10, Chapter 8], we can form a block matrix ring of the type:

$$\begin{bmatrix} (L) & (J) \cdots (J) \\ (L) & (L) \cdots (J) \\ (L) & (L) \cdots (L) \end{bmatrix} (k_1, k_2, \cdots, k_r) .$$

In the terminology of Robson [11], any such matrix ring is said to be a block lower triangular matrix ring over $L \backslash J$.

THEOREM 2. *Let $R$ be a bounded $(hnp)$-ring over which every module admits a basic submodule. Then there exists a discrete valuation ring $D$ with maximal ideal $M$ such that $R$ is a block lower triangular matrix ring over $D \backslash M$.*

*Proof.* First of all we show that $R$ has only one maximal invertible ideal. Let $A$ be a maximal invertible ideal of $R$. If $A$ is not the only maximal invertible ideal, then in the notations of [13] $R < R_A < Q$. There exists a non unit regular element $a$ in $R$ such that $a$ is a unit modulo $A$. Then $U_n a^{-n} R \subset R_A$ and $U_n a^{-n} R$ is not finitely generated as a right $R$-module. This contradicts Lemma 3. Hence $A$ is the only maximal invertible ideal of $R$ and $R = R_A$. Then $J(R) = A$. This then gives $R$ has only finitely many idempotent ideals. Let $B$ be a minimal nonzero idempotent ideal of $R$. Then $O_l(B) = \{x \in Q: xB \subset B\}$ is a Dedekind prime ring [3, Proposition (1.8)]. As for $R$, every torsion free uniform $O_l(B)$-module is either finitely generated or divisible. As a consequence $O_l(B)$ has only one maximal ideal $P$ and $O_l(B) = O_l(B)_P$. So by [7, Lemma (2.1)] $O_l(B) = D_n$ for some discrete valuation ring $D$. By Jacobson [5, p. 120], $R$ is equivalent to $O_l(B)$. Hence by Robson [11, Theorem (6.3) and Corollary (2.8)], $R$ is a block lower triangular matrix ring over $D\backslash M$, where $D$ is a discrete valuation ring with $M$ as its maximal ideal.

It is clear that any non block lower triangular matrix ring over $D\backslash M$ where $D$ is a discrete valuation ring with $M$ is its maximal ideal, is equivalent to $D_n$. So to prove the converse of the above theorem it is enough to prove the following:

THEOREM 3. *Let $R$ be a bounded ($hnp$)-ring such that $R$ is equivalent to $S$, for some g-discrete valuation ring $S$, which is an overring of $R$, then every $R$-module admits a basic submodule.*

*Proof.* First of all we show that any uniform torsion free $R$-module $U$ is either divisible or finitely generated. Suppose $U$ is not divisible. Now $S = D_n$. There exist regular elements $a$ and $b$ in $R$ such that $aSb \subset R$. Since $S$ is bounded there exists a nonzero ideal $\mathscr{I}$ of $S$ such that $\mathscr{I} \subset Sb$. Then $a\mathscr{I} \subset R$ and the fact that $S_S$ is embeddable in $a\mathscr{I}$ gives that $S_R$ is finitely generated. Similarly ${}_RS$ is finitely generated. So using [3, Theorem (1.6)], we get $S = O_l(A) = A^* = AA^*$ for some idempotent ideal $A$ of $R$. We can suppose that $U \subset Q$, the classical quotient ring of $R$. If $US = eQ$, then $UAA^*A = eQA = eQ$. However $UAA^*A \subset U$. Thus in this case $U$ is divisible. Hence $US$ is finitely generated as $S$-module [8, Lemma (3.2)]. This gives $U_R$ is finitely generated, since as proved above $S_R$ is finitely generated.

Thus every uniform torsion free right $R$-module is injective or projective. Consider any right $R$-module $M$ and let $T$ be its torsion submodule. $T$ admits a basic submodule $B$ by Theorem 1. Then $B$ is a pure submodule of $M$ and $T/B$ is divisible; further $T/B$ is the torsion submodule of $M/B$. So we can write

$$M/B = L/B \oplus T/B \oplus K/B$$

where $L/B$ is torsion free, divisible $R$-module and $K/B$ is a torsion free reduced $R$-module. If $K/B = 0$, we get $B$ itself is a basic submodule. So let $K/B \neq 0$. We can find a maximal uniform submodule $U/B$ of $K/B$. By what has been proved above $U/B$ is finitely generated and hence projective. So by Lemma 1, $U$ is a pure submodule of $K$, and $U = U_1 \oplus B$, where $U_1$ is a finitely generated uniform torsion submodule. By Zorns lemma, we can find a maximal direct sum $E = B \oplus \sum \oplus U_i$, in $K$ such that $E$ is a pure submodule of $K$, $U_i$ are finitely generated uniform, torsion free $R$-modules. By Lemma 1, $K/E$ is torsion free. If $K/E$ is not divisible, then as before we get a nonzero finitely generated uniform submodule $V/E$ of $K/E$ such that $V/E$ is pure in $K/E$. Then $V = V_1 \oplus E$ and $V$ is a pure submodule of $K$. This contradicts the maximality of $E$. Hence $K/E$ is divisible. $E$ is clearly decomposable and is a basic submodule of $M$. This completes the proof.

We remark that any two basic submodules of a module over the ring of the above theorem, can be shown to be isomorphic.

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