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# A HOMEOMORPHISM CLASSIFICATION OF WILDLY IMBEDDED TWO-SPHERES IN $S^3$

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**If a two-manifold $M$ is wildly imbedded $S^3$, it gives rise to a pair of noncompact three-manifolds of a special type. This type is considered in detail and a homeomorphism classification theorem for it is derived. This result is then used to decide whether there is a homeomorphism of $S^3$ to itself, which takes $M$ onto another, given two-manifold.**

**Introduction.** In recent years much work has been done on wild imbeddings of two-spheres in three-spheres. The survey article [5] gives an excellent bibliography of this work. Relatively little of this work, however, involves the homotopy type of such imbeddings or algebraic characterizations of them. Two papers which make some use of these are [3] and [6]. In this paper we develop some of this theory. An imbedding of $S^2$ in $S^3$ gives rise to a pair of noncompact three-manifolds with boundary. If these three-manifolds are trail irreducible (this term will be defined in §1) and the set of wild points of $S^2$ is totally disconnected, we can develop a useful uniform homotopy theory which allows us to determine the homeomorphism type of the manifold from the homotopy type. Homeomorphisms of the three-manifolds may then be sewn together to form a homeomorphism theorem for the imbedding. Although the case of two-spheres imbedded in $S^3$ is historically of greatest interest, there is nothing restricting our methods to that case, so we shall develop the theory for finitely many closed two-manifolds imbedded in $S^3$. We also derive a method for showing that an imbedding is trail irreducible. Using this method we show the existence of uncountably many distinct imbeddings for which the theorems hold. In particular we show that they hold for Alexander's horned sphere.

**1. Notation and definitions.** Suppose $M_1, \cdots, M_k$ are closed two-manifolds and that they are disjointly (and possibly wildly) imbedded in $S^3$. Let $M = M_1 \cup \cdots \cup M_k$. Then $S^3 - M$ has $k+1$ components. Let $T$ be one component. Let $I = \{(x, y, z) \in R^3 : z \geqq 0\}$.

DEFINITION. A point $p$ of $M \cap \bar{T}$ is *tame from* $T$ if there is a homeomorphism $h: (I, \partial I) \to (T, M)$ with $p \in h(\partial I)$. A point of $M \cap \bar{T}$ is *wild from* $T$ if it is not tame from $T$.

It is easy to show that a point $p \in M_i$ is tame in the usual sense if and only if it is tame from each component of $S^3 - M_i$. Except for points of $M$ which are wild from $T$, $\bar{T}$ would be a three-manifold. Thus it is useful to make the following definition.

DEFINITION. Let $M$ be a finite set of closed two-manifolds, imbedded in $S^3$. Let $T$ be one component of $S^3 - M$. Let $W$ be the set of all points of $M$ which are wild from $T$. $\bar{T} - W$ is a *manifold imbedding*.

Alternatively $\bar{T} - W$ is the largest three-manifold in $\bar{T}$.

A manifold imbedding inherits a metric from $S^3$ and this metric will be maintained throughout. It is convenient to deal with a particular kind of map between manifold imbeddings.

DEFINITION. Suppose $X$ and $Y$ are metric spaces. A continuous map $f: X \to Y$ is proper if the inverse image of a compact set is compact. A map is $p$ if it is proper and uniformly continuous in the respective metrices. A *p-homotopy* is a homotopy which is $p$-uniform.

We shall be concerned with manifold imbeddings with the following two restrictions.

DEFINITION. Let $A$ be a manifold imbedding. It is *trail irreducible* if given a sequence $\{l_n\}$ of loops which are null homotopic in $A$ and have diam $(l_n) \to 0$, then the loops are null homotopic in sets $\{X_n\}$ with diam $(X_n) \to 0$. If $A$ is a manifold imbedding, we shall refer to $\bar{A} - A$ as the wild point set of $A$. Our second restriction is that the wild point set of $A$ be totally disconnected. (If the wild point set is totally disconnected, trail irreducibility is equivalent to end irreducibility as defined in [4]. However, trail irreducibility is more appropriate in our context.)

We shall generally follow Waldhausen's terminology for three-manifolds, [12]. A surface $K$ is *properly imbedded* in a three-manifold $A$ if $K \cap \partial A = \partial K$. A *system of surfaces* in $A$ is the union of surfaces imbedded disjointly in $A$. A surface $K$, piecewise linearly imbedded in $A$, is *compressible* if it is either a two-sphere bounding a three-cell in $A$, or if the homomorphism $\pi_1(K) \to \pi_1(A)$ induced by inclusion, is not injective. It is *incompressible* if it is not compressible. A system of surfaces is incompressible if each component is incompressible. A three-manifold, $A$, is *irreducible* if any two-sphere piecewise linearly imbedded in $A$ bounds a three-cell. It is *boundary irreducible* if $\partial A$ is incompressible in $A$. Let $K$ be a system of surfaces properly imbedded in $A$. Then $K$ has a regular neighborhood, $U$, which may be coordinatized as $K \times I$ with $K = K \times \{1/2\}$. Let

$X$ be an arbitrary space. A map $f: X \to A$ is *transverse* with respect to $K$ if $f$ induces in $f^{-1}(U)$ the structure of a line bundle and $f$ maps each fiber homeomorphically onto a fiber. If $X$ is a subset of $A$ which is a submanifold of $S^3$, $\bar{X}$ will denote the closure of $X$ in $S^3$, while $\mathrm{cl}(X)$ will denote the closure of $X$ in $A$, i.e., $\bar{X} \cap A$. The frontier of $X$, $\mathrm{Fr}(X)$, is $\mathrm{cl}(X) \cap \mathrm{cl}(A - X)$. Also $X^0 = S^3 - \overline{(S^3 - X)}$.

DEFINITION. An *exhausting sequence* $\{C_n\}$ for a three-manifold $A$, is a sequence of compact, piecewise linear submanifolds of $A$ with:
(1) $\mathrm{Fr}(C_n)$ a system of surfaces, properly imbedded in $A$,
(2) $UC_n = A$,
(3) $C_n \subset C_{n+1} - \mathrm{Fr}\, C_{n+1}$.

**2. Some basic lemmas.** First we shall establish some basic lemmas about manifold imbeddings.

LEMMA A. *A manifold imbedding $A$ is irreducible if and only if $\pi_2(A) = 0$.*

This is a simple application of the sphere theorem, [10].

LEMMA B. *Suppose $R$ and $Q$ are closed $n$-cells and $f: (R, \partial R) \to (Q, \partial Q)$ is a map with $f \mid \partial R$ a homeomorphism of $\partial R$ onto $\partial Q$. Then $f$ is homotopic* rel $\partial R$ *to a homeomorphism of $R$ onto $Q$.*

Since $f \mid \partial R$ is a homeomorphism, we may coordinatize $R$, and $Q$ as $n$-cubes by homeomorphisms $u: R^n \to R$ and $v: R^n \to Q$ such that $f \circ u \mid \partial R^n = v \mid \partial R^n$. Then we define the homotopy by:

$$H_t(\underline{x}) = (1 - t) f(\underline{x}) + t\underline{x} .$$

LEMMA C. *Let $A$ be trail irreducible manifold imbedding with totally disconnected wild point set. Then $A$ has an exhausting sequence $\{C_n\}$ such that:*
(1) *$\mathrm{Fr}\, C_n$ intersects no compact component of $\partial A$,*
(2) *$\mathrm{Fr}\, C_n$ is incompressible in $A$,*
(3) *No component of* $\mathrm{cl}(A - C_n)$ *(the closure in $A$) is compact,*
(4) *$C_n$ is connected,*
(5) *For any $\varepsilon > 0$, there is an $n$ such that every component of $A - C_n$ has diameter less than $\varepsilon$.*

Let $W$ be the set of wild points of $A$. Let $\{C_n\}$ be an exhausting sequence for $A$. The wild point set of $A$, $W$, is totally disconnected, so by [8] (2.94) for any $\varepsilon > 0$ there is a finite open covering $\{U_1, \cdots, U_k\}$ of $W$ by disjoint subsets of $S^3$ of diameter less than $\varepsilon$. Since $A - (U_1 \cup \cdots \cup U_k)$ is compact, for $n$ sufficiently large it is contained in

$C_n$. Each component of $A - C_n$ is contained in some $U_i$ and so has diameter less than $\varepsilon$; this is condition (5). Also this shows that for $W$ totally disconnected, trail irreducible is equivalent to the term referred to as end irreducible in [4]. Therefore, $A$ satisfies the conditions of [4] Lemma 3.1, so it has an exhausting sequence satisfying (1)-(4). We have shown that any exhausting sequence satisfies (5).

LEMMA D. *A trail irreducible manifold imbedding with totally disconnected but nonempty wild point set can be neither simply connected, nor have a boundary component which is a two-sphere minus a single point.*

Let $A$ be trail irreducible manifold imbedding with totally disconnected wild point set. Pick a wild point $w \in \bar{A}$. Suppose $A$ is simply connected. Let $\{l_n\}$ be a sequence of loops in $A$ converging to $\omega$. Since $A$ is simply connected, the $l_n$ are null homotopic in $A$. Since $A$ is trail irreducible and diam $(l_n) \to 0$, there is a sequence, $\{X_n\}$, of subsets of $A$ with $l_n$ null homotopic in $X_n$ and diam $(X_n) \to 0$. Since this holds for any sequence of loops converging to any point of $\bar{A} - A^0$, $\bar{A}$ is $1 - ULC$; therefore, by [3] Theorem 6, the wild point set of $A$ is empty, which is a contradiction.

Suppose $L$ is a component of $\partial A$ which is a two-sphere minus a single point $w$. Let $\{l_n\}$ be a sequence of simple loops in $L$, converging to $w$, with $w$ in the smaller component of $\bar{L} - l_n$. Then $l_n$ is null homotopic in $L$. Since $A$ is trail irreducible, there is a sequence of singular, piecewise linear disks $\{D'_n\}$ converging to $w$ with $\partial D'_n = l_n$. By the loop theorem, [10], $l_n$ bounds a simple disk, $D_n$, contained in a closed regular neighborhood of $D'_n$, (in $A$). We may assume $D_n \cap L = l_n$. Let $L_n$ be the larger component of $L - l_n$. Since $D_n \cup L_n$ is a piecewise linear two-sphere in $A$, there is a piecewise linear two-sphere $S_n \subset A^0$ which is parallel to $D_n \cup L_n$ with the corresponding points no more than $1/n$ apart. Then $\{S_n\}$ is a sequence of two-spheres homeomorphically approximating $\bar{L}$, so by [2] Theorem 11.1, $\bar{L}$ is tame from $A$; i.e., $w$ was not a wild point at all.

The next two results are of some independent interest. The first is a generalization of Waldhausen's homeomorphism theorem ([12], Theorem 6.1) to the case of manifolds which are not boundary irreducible but are imbedded in $S^3$. The second says that the homology of a manifold imbedding is what we would expect it to be.

LEMMA E. *Let $A$ and $B$ be compact, connected three-manifolds, piecewise linearly imbedded in $S^3$. Let $f\colon (A, \partial A) \to (B, \partial B)$ be a map such that:*

(1) $f_*: \pi_1(A) \to \pi_1(B)$ *is injective,*

(2) *For any component $J$ of $\partial A$, $f$ takes $J$ homeomorphically onto a component of $\partial B$,*

(3) $\pi_2(A) = 0$, $\pi_2(B) = 0$, *and* $\partial A \neq \varnothing$.

*Then $f$ is homotopic* rel $\partial A$ *to a map, $g$, such that either:*

(A) *$g$ is a homeomorphism of $A$ onto $B$, or*

(B) *$A$ is homeomorphic to $KxI$ for a closed surface $K$, and $g(A) \subset \partial B$.*

By [11] Theorem 1, if $f: (A, \partial A) \to (B, \partial B)$ is a map satisfying the conditions of this theorem, it is homotopic rel $\partial A$ to a map $g: A \to B$ with either:

(a) $g$ a covering map, or

(b) conclusion (B).

Suppose $g$ is an $n$-sheeted covering map. Let $V_1, \cdots, V_m$ be the components of $\overline{S^3 - B}$. Let $K$ be a component of $\partial V_i$. Since $K$ is a closed two-manifold, $\overline{S^3 - K}$ has two components. Since $B^\circ$ and $V_i^\circ$ are locally separated by $K$, they are in different components of $S^3 - K$. Therefore, $\partial V_i = B \cap V_i$ is contained in the intersection of the closures of the components of $S^3 - K$, that is in $K$. Consequently, $\partial V_i$ is connected. Each $\partial V_i$ is the image of $n$ components of $\partial A$. Sewing a copy of $V_i$ to $A$ at each component of $f^{-1}(\partial V_i)$ gives an $n$-sheeted covering space of $S^3 = B \cup V_1, \cdots, V_m$. Thus $m = 1$ and $g$ is a homeomorphism.

LEMMA F. *Let $A$ be a manifold imbedding. Let $g =$ genus $(\bar{A} - A^\circ)$, and let $m$ be the number of components of $\bar{A} - A^\circ$. Then*

$$H_1(A) = Z^g \quad and \quad H_2(A) = Z^{m-1}\,.$$

Let $M = \bar{A} - \bar{A}^\circ$, which has $m$ components denoted $M_1, \cdots M_m$, having genuses $g_1, \cdots, g_m$ respectively; let $g = g_1 + \cdots + g_m$. Applying the Poincare and Alexander duality theorems to $S^3 - M$, $S^3 - M_1, \cdots, S^3 - M_m$ gives $H_2(A^\circ) = Z^{m-1}$, and $H_1(S^3 - M) = Z^{2g}$.

Let $B_1, \cdots, B_m$ be handle bodies with boundaries of genuses $g_1, \cdots, g_m$ respectively. Let $X$ be the union of $A$ and $B_1, \cdots, B_m$ with $\partial B_i$ associated homeomorphically to $M_i$; $X$ is compact. If $x$ is an interior point of $A$ or some $B_i$, it has an open neighborhood homeomorphic to $R^3$. Suppose $x \in M_i$. By [3] Theorem 5, $x$ is contained in an open disk in $M_i$ which is contained in some two-sphere, $S$, imbedded in $S^3$. Let $U$ be the component of $S^3 - S$ containing points in $A^0$ which are close to $x$. Let $Y$ be the union of $\bar{U}$ with a closed three-cell, whose boundary is $S$. By [9] Theorem 2, $Y$ is a three-sphere. However, a small neighborhood of $x$ in $Y$ is homeomorphic to a small neighborhood of $x$ in $X$. Therefore, $X$ is a closed three-manifold, so we may apply the Poincare and Alexander duality theorems to it. Since we know the homology of the $B_i$, we obtain $H_1(A)$.

## 3. Homeomorphism type of manifold imbeddings.

THEOREM 1. *Let $A$ and $B$ be irreducible and trail irreducible manifold imbeddings with totally disconnected but nonempty wild point sets. Let $f: (A, \partial A) \to (B, \partial B)$ be a p-unifold map such that:*

(1) $f_*: \pi_1(A) \to \pi_1(B)$ *is injective,*

(2) *For $I$, a component of $\partial A$, and $J$, the component of $\partial B$ containing $f(I)$, $f_*: \pi_1(I) \to \pi_1(J)$ is an isomorphism.*

*Then $f$ is p-homotopic to a homeomorphism of $A$ onto $B$ by a homotopy which takes $\partial A$ into $\partial B$. If $f|\partial A$ is already a local homeomorphism, we may choose the homotopy fixed on $\partial A$.*

Let $I_1, \cdots, I_k$ be the components of $\partial A$, and $J_1, \cdots, J_l$ be the components of $\partial B$. Then there is a function, $z$, such that $f(I_i) \subset J_{z(i)}$. Let $f^i = f|I_i$. Since $A$ is irreducible, none of the $I_i$ is a two-sphere. By Lemma D no $I_i$ is simply connected. Thus $f^i$ is $p$-homotopic to a map $f_0^i$ such that either:

(a) $f_0^i$ is a homeomorphism of $I_i$ onto $J_{z(i)}$, or

(b) $I_i \cong S^1 \times R^1$ and there is a simple loop $l \subset J_{z(i)}$ with one component of $J_{z(i)} - l$ homeomorphic to $S^1 \times R^+$ and $f_0^i(s, u) = (f_0^i(s, 0), |u|)$ in the respective coordinatizations, and $f_0^i|S^1 \times 0$ is a homeomorphism onto $l$. (In neither case is the metric of $S^1 \times R^1$ or $S^1 \times R^+$ the product metric. Although $d((s, u), (-s, u)) \to 0$ as $u \to \infty$, the two-manifolds are topologically equivalent.)

This is the two dimensional analogue of [4] Theorem 3.4. Doing this for each component of $\partial A$ and using regular neighborhoods of $\partial A$ and $\partial B$, we may extend the homotopies to a $p$-homotopy of all of $A$. We want $f_0$ to be a local homeomorphism on $\partial A$. (Since it induces an isomorphism on fundamental groups, this implies that it is a homeomorphism on components of $\partial A$.) If $f$ were already a local homeomorphism on $\partial A$, we could have skipped this first step. All succeeding homotopies will be fixed on $\partial A$, so the final comment in the statement of the theorem is justified. The restriction $f|\partial A$ can fail to be a local homeomorphism only if (b) occurs for one or more components of $\partial A$. We shall first prove the theorem assuming (a) occurs for all components of $\partial A$.

Since $A$ and $B$ are trail irreducible and have totally disconnected wild point sets, we may construct exhausting sequences $\{C'_n\}$ for $A$ and $\{D_n\}$ for $B$ satisfying the conditions of Lemma C. By choosing subsequences we may also assume that:

$$f_0(C'_n) \subset D_n - \operatorname{Fr} D_n\,,$$
$$f_0^{-1}(D_n) \subset C'_{n+1} - \operatorname{Fr} C_{n+1}\,.$$

By [12] (1.3) there is a map $f_1$ which is homotopic rel $\partial A$ to $f_0$ by a homotopy which moves $(C'_{n+1} - C'_n)$ only within $(D_{n+1} - D_{n-1})$, and such that $f_1$ is transverse to $\operatorname{Fr} D_n$ and $f_1^{-1}(\operatorname{Fr} D_n)$ is a system of incompressible surfaces properly imbedded in $A$. By (5) of Lemma C, the maximal diameter of the components of $B - D_n$ goes to zero, so the homotopy from $f_0$ to $f_1$ is $p$-uniform. Let $C_n = f_1^{-1}(D_n)$. Using (5) of Lemma C, we may choose subsequences so that each component of $\operatorname{Fr}' D_n$ has diameter so small that it can neither intersect two components of $\partial B$, nor separate two components of $\partial B$. (Since a manifold imbedding is constructed by using finitely many disjoint closed two-manifolds, the minimal distance between components of $\partial B$ and the minimal diameters of components of $\partial B$ are both greater than 0.) This gives each component of $\operatorname{Fr} D_n$ or $\partial D_n$ intersecting exactly one component of $\partial B$. (This technique is used in [4] Theorem 3.4.)

The sequence $\{C_n\}$ satisfies (1), (2), and (5) of Lemma C. By choosing a subsequence, we may assume that each component of $\operatorname{Fr} C_n$, or $\partial C_n$ intersects exactly one component of $\partial A$. Let $R$ be the largest component of $C_0$. Pick $p_i \in J_i$ for each component $J_i$ of $\partial B$. By choosing a subsequence, we may assume all the $f_0^{-1}(p_i)$ are contained in a single component of $C_0$, and that no component of $A - R$ intersects two components of $\partial A$.

Suppose $H$ is a component of $\operatorname{Fr} C_n$, and $K$ is the component of $\operatorname{Fr} D_n$ containing $f_1(H)$. If $K$ is a disk, we may use Lemma B to find a homeomorphism of $H$ onto $K$, homotopic rel $\partial H$ to $f_1|H$. If $K$ is not a disk, its nonempty boundary, $\partial K$, is taken homeomorphically onto $\partial H$, so we may apply [12] (1.4.3) to get a homeomorphism, which is homotopic rel $\partial H$ to $f_1|H$. $K^\circ$ and $H^\circ$ have neighborhoods $K^* \cong K^\circ \times I$ and $H^* \cong H^\circ \times I$ such that diam $(\{y\} \times I) \to 0$ as $y$ approaches $\partial K$ or $\partial H$ respectively, and $K^*$ and $H^*$ are contained in regular neighborhoods of $K$ and $H$. Using these neighborhoods, we may extend the homotopies over all of $A$ to get a map $f_2$, homotopic to $f_1$ rel $\partial A$, with $f_2$ a homeomorphism on each component of $\operatorname{Fr} C_n$.

Let $P$ be a component of $\overline{C_{n+1} - C_n}$ and $Q$ be the component of $\overline{D_{n+1} - D_n}$ containing $f_2(P)$. Then $\operatorname{Fr} P$ and $\operatorname{Fr} Q$ are incompressible, so $P$ and $Q$ are irreducible, and $\pi_1(P) \to \pi_1(A)$ is injective. Since $f_{2*}$

is injective, $\pi_1(P) \to \pi_1(Q)$ is also injective. By the choice of $\{C_n\}$ each component of $\partial P$ intersects exactly one component of $\partial A$, so $f_2$ is a homeomorphism on components of $\partial P$. Therefore we may apply Lemma E to $f_2|P$ to get a map $g_P$ homotopic rel $\partial P$ to $f_2|P$ such that either:

(A) $g_P$ is a homeomorphism of $P$ onto $Q$, or

(B) $P \cong K \times I$ for some closed surface $K$, and $g_P(P) \subset \partial Q$.

We may do this same construction for $P$ and $Q$ components of $C_0$ and $D_0$ respectively. Let $R$ be the largest component of $C_0$. If $P \neq R$, $P$ can intersect only one component of $\partial A$, so $f_2$ is a homeomorphism on $\partial P$, and $g_P$ is a homeomorphism. Patching all the $g_P$ together, we get a map $g: A \to B$ which is homotopic rel $\partial A$ to $f_2$ and agrees with $g_P$ on $P$. Since the diameter of components of $B - D_n$ goes to zero, the homotopy is $p$-uniform. If (A) applies to $R$, $g$ is a homeomorphism and we are done.

Suppose (B) applies to $R$. Then $\partial R$ has two components, so $\partial A$ has two components, which are taken to the same component of $\partial B$. Let $R_0 = R$, and for each $n$ let $R_n$ be the component of $C_n$ containing $R_{n-1}$. Then $\{R_n\}$ is an exhausting sequence for $A$. We may apply Lemma E to get maps $g_n: R_n \to D_n$ which are homotopic rel $\partial R_n$ to $f_2|R_n$ and satisfy either (A) or (B). Since $f_2$ is not a homeomorphism on $\partial A \cap R_0$, it can not be a homeomorphism on any $\partial R_n$, so (B) applies to each of them. Therefore, there are closed surfaces $\{K_n\}$ such that $R_n \cong K_n \times I$. Since $R_n$ is a component of $C_n$, each component of $\operatorname{Fr} R_n$ is a component of $\operatorname{Fr} C_n$. Accordingly $\operatorname{Fr} R_n$ is incompressible in $A$, so $\pi_1(R_n) \to \pi_1(A)$ is injective. This gives $K_n \times \{1/2\}$ incompressible in $A$. Therefore, the inclusion map $R_n \to R_{n+1}$ induces an injection $\pi_1(K_n) \to \pi_1(K_{n+1})$. This injection can be induced by a covering map of $K_n$ onto $K_{n+1}$. Since a closed two-manifold can not cover another closed two-manifold of greater genus, genus $K_{n+1} \leqq$ genus $K_n$. By dropping some initial terms we may assume that genus $K_n$ is a fixed constant, $c$, for all $n$.

Since genus $K_n$ is constant, $\pi_1(K_n) \to \pi_1(K_{n+1})$ is an isomorphism for all $n$. Therefore, $\pi_1(K_0) \to \pi_1(R_n)$ is an isomorphism, and so also $\pi_1(K_0) \to \pi_1(A)$. By the Hurewic theorem $H_1(K_0) \to H_1(A)$ is an isomorphism, so $H_1(A) = Z^{2c}$. Then by Lemma F, genus $(\bar{A} - A^\circ) = 2c$.

Let $I_0$ and $I_1$ be the components of $\partial A$ intersected by $K_n \times 0$ and $K_n \times 1$ respectively. Since $\bar{I}_i$ is a closed two-manifold and its set of wild points is totally disconnected, $I_i$ has a connected, compact submanifold $I_i'$ such that genus $I_i' =$ genus $\bar{I}_i$. For $n$ sufficiently large, $I_0' \cup I_0' \subset R_n - \operatorname{Fr} R_n$. Let $l$ be a component of $\partial I_i'$; it bounds a disk $D$ in $\bar{I}_i$. By Lemma F genus $(I_0' \cup I_1') =$ genus $\partial R_n$, so $l$ must bound a disk, $E_n$, in each $\partial R_n$. Therefore, $E_n \to D$. Since $E_n$

and $D$ are simply connected this is a homeomorphic approximation. By [2] Theorem 11.1, $D$ is tame from $A$. However, this is true for any component of $\partial I_i'$, so $I_0$ and $I_1$ are tame. This, however, is a contradiction.

Now let us lift the restriction that $f_0$ be a homeomorphism on each component of $\partial A$. Let $e(f)$ be the number of components of $\partial A$ for which (b) applies. We have shown that the theorem holds if $e(f) = 0$. Suppose $f: A \to B$ is such that $e(f)$ is minimal for which the theorem fails. Pick a component $I$ of $\partial A$ with $I \cong S^1 \times R^1$ and a loop $l'$ in a component $J$ of $\partial B$ such that one component of $\operatorname{cl}(J - l')$ is homeomorphic to $S^1 \times R^+$ with $f_0(s, t) = (s, |t|)$ in the respective coordinate systems. (Again the metrics are not the product metrics.) Let $\{D_n\}$ be an exausting sequence for $B$ satisfying (1)-(5) of Lemma C. By the method used above we may assume that $\{f_0^{-1}(D_n)\}$ is an exhausting sequence for $A$ satisfying (1), (2), and (5). For $n$ sufficiently large no component of $B - D_n$ (or $A - f_0^{-1}(D_n)$) intersects more than one component of $\partial B$ (or $\partial A$) and the two points of $\bar{I} - I$ are in different components of $A - f_0^{-1}(D_n)$. For $m$ sufficiently large $S^1 \times [m, \infty) \subset B - D_n$; let $l = S^1 \times \{m\}$. Let $B'$ be the component of $\operatorname{cl}(B - D_n)$ containing $l$; it is a manifold imbedding. The surjection $H_1(\bar{B}' - B'^\circ) \to H_1(B')$ of Lemma F commutes with the $H_1(\partial B') \to H_1(B)$ induced by inclusion, so $l$ is null homologous in $B$. Therefore, $l$ bounds a piecewise, linear incompressible surface $K$, properly imbedded in $B$. By [12] (1.3) $f_0$ is homotopic rel $\partial A$ to a map $f$, transverse to $K$ with $f_1^{-1}(K)$ a system of incompressible surfaces properly imbedded in $A$. (We may choose the homotopy fixed off some compact neighborhood of $f_0^{-1}(K)$, so the homotopy is $p$-uniform.) Let $S^1 \times \{-m\}$ and $S^1 \times \{m\}$ in $I$ be denoted $\lambda_0$ and $\lambda_1$ respectively. Since $\lambda_0$ and $\lambda_1$ are in distinct components of $A - f_1^{-1}(D_n)$, they must be in distinct components of $f_1^{-1}(K)$, which we denote $L_0$ and $L_1$ respectively. Since each component of $A - f_1^{-1}(D_n)$ intersects only one component of $\partial A$, $\partial L_i \subset I$; i.e., $\lambda_i = \partial L_i$.

The space $K \cup S^1 \times [m, \infty)$ is a closed two-manifold, so it separates $S^3$ into two pieces. Therefore, $B - K$ has two components, which we shall denote $B_0'$ and $B_1'$ with $\overline{f_1(I) - f_1(I)} \subset \overline{B_0'}$; $\partial B_0$ is connected. Similarly $A - L_0 - L_1$ has three components $A_0'$, $A_1'$, and $A_2'$. Two of these components (say $A_0$ and $A_2$) must be so small as to have connected boundaries. Let $A_i = \operatorname{cl}(A_i')$ and $B_i = \operatorname{cl}(B_i')$. We can index them so that:

$$f_1(A_0), f_1(A_2) \subset B_0 \,,$$
$$S^1 \times (-\infty, -m) \subset A_2, S^1 \times (m, \infty) \subset A_0 \,,$$
$$L_0 = A_0 \cap A_1 \,, \quad L_1 = A_1 \cap A_2 \,.$$

The restriction $f_1|A_i$ is a $p$-uniform map with

$$e(f_1|A_0) + e(f_1|A_1) + e(f_1|A_2) = e(f) - 1 .$$

$L_0$ and $L_1$ are incompressible, so $\pi_1(A_0)$ and $\pi_1(A_2)$ are mapped injectively into $\pi_1(B_0)$ by $f_{1*}$, and $A_0$ and $A_2$ are irreducible and trail irreducible. Thus we may apply the theorem for $e(f_1|A_i) < e(f)$ to get homeomorphisms $g_i: A_i \to B_0$ (for $i = 0, 2$) which are $p$-homotopic rel $\partial A_i$ to $f_1|A_i$. Pick $p \in \partial L_0$ and consider $\pi_1(A_0, p)$. Let $m$ be a loop in $A_0$ with $m(0) = p$. Then there is a loop $m'$ in $A_2$ such that $m$ and $m'$ are taken to the same loop in $B_0$ by $f_1$. In the coordinate system of $\partial A$, $p = p' \times \{1\}$, and $m'(0) = p' \times \{-1\}$. We may extend $m'$ along $p \times -[1, 1]$ to $p$, giving a loop $m''$ based at $p$. Then $g_0 \circ m$ and $g_2 \circ m''$ are homotopic loops in $B$. However, $g_0$ and $g_2$ are homotopic to $f_1|A_0$ and $f_1|A_2$, and $f_{1*}$ is injective, so $m$ is homotopic to $m''$ in $A$. Let $C = \{(x, y) \in R^2: x^2 + y^2 \leqq 1\}$. There is a piecewise linear map $r: C \to A$ with $m$ being the restriction of $r$ to the upper half circle and $m''$ being $r$ restricted to the lower half circle. We may assume $r$ is in general position, and $r^{-1}(L_0)$ has as few components as possible. In particular no component of $r^{-1}(L_0)$ is a contractible loop. Since $r^{-1}(L_0) \cap \partial C$ is just two points, $r^{-1}(L_0)$ must be a single arc in $C$ from $(1, 0)$ to $(-1, 0)$. The restriction of $r$ to the subdisk between that arc and the upper half circle is a homotopy of $m$ to a loop in $L_0$. Consequently the map $\pi_1(L_0) \to \pi_1(A_0)$ is surjective. Since $\pi_1(L_0) \to \pi_1(A_0)$ was already injective, it is an isomorphism. By Lemma F, if genus $L_0 = c$ and the genus of the components of $\partial A$ entirely contained in $A_0$ is $c'$, then $H_1(A_0) = Z^c + Z^{c'}$. By the Hurewizc theorem $i_*(H_1(L_0)) = Z^{2c}$. Since $i_*$ commutes with the map given in Lemma F, $i_*(H_1(L_0))$ is the first summand of $H_1(A_0)$, so $c = 0$. Therefore, $H_0$ is simply connected, and $A_0$ is, too. This contradicts Lemma D. Therefore, the case $e(f) > 0$ can not occur, which completes the proof of the theorem.

DEFINITION. Let $M$ and $N$ be closed two-manifolds imbedded in $S^3$. A map $f: (S^3, M) \to (S^3, N)$ *preserves the imbedding* if $f(S^3 - M) \subset (S^3 - N)$, and each component of $S^3 - M$ goes into a distinct component of $S^3 - N$.

Let $M$ and $N$ be connected and $f: (S^3, M) \to (S^3, N)$ be a map which preserves the imbedding. The manifolds $M$ and $N$ each give two-manifold imbeddings which we shall denote $A_0$, $A_1$ and $B_0$, $B_1$ respectively, with $f(A_i^\circ) \subset B_i^\circ$. Let $W$ be the set of wild points of $(S^3, M)$ and $V$ the wild points of $(S^3, N)$. Let $V' = f(W) \cup V$ and $W' = f^{-1}(V')$. Then $f$ induces maps on the homotopy groups:

$$f_M\colon \pi_1(M - W') \longrightarrow \pi_1(N - V')\ ,$$
$$f_i\colon \pi_1(A_i^\circ) \longrightarrow \pi_1(B_i^\circ)\ .$$

These commute with the maps induced by inclusions:

$$M - W' \longrightarrow A_i \longleftarrow A_i^\circ\ ,$$
$$N - V' \longrightarrow B_i \longleftarrow B_i^\circ\ .$$

(The backward arrows induce isomorphisms.)

THEOREM 2. *Let $f\colon (S^3, M) \to (S^3, N)$ be a map preserving the imbedding such that:*

(1) *The $A_i$ and $B_i$ are trail irreducible,*

(2) *$f_*\colon \pi_1(A_i^\circ) \to \pi_1(B_i^\circ)$ is an injection for each $i$,*

(3) *$W'$ and $V'$ are totally disconnected,*

(4) *$f_M\colon \pi_1(M - W') \to \pi_1(N - V')$ is an isomorphism,*

(5) *If $\pi_1(M - W') = 0$, then $f_*\colon \pi_2(M) \to \pi_2(N)$ is an isomorphism.*

*Then there is a homeomorphism $g\colon (S^3, M) \to (S^3, N)$ which is homotopic to $f$ by a homotopy, $H$, such that each $H_t$ preserves the imbedding. If $f \mid M$ is a homeomorphism onto $N$, then the homotopy may be chosen fixed on $M$.*

We shall first dispose of the case $\pi_1(M - W') = 0$. If $W' = \varnothing$, then each $A_i$ is a closed three-cell. The map $\pi_2(M) \to \pi_2(N)$ is nontrivial so $N \subset f(M)$. Therefore, $V' = \varnothing$ also, and each $B_i$ is a closed three-cell. Since $f_*\colon \pi_2(M) \to \pi_2(N)$ is an isomorphism, $f \mid M$ is homotopic to a homeomorphism onto $N$. Using regular neighborhoods, we may extend the homotopy over all of $S^3$. Then we may use Lemma B to get a homotopy rel $M$ to a homeomorphism of all $S^3$. Suppose $W' \neq \varnothing$; then $W'$ has one point, and $M$ is a two-sphere. Applying Lemma D to each $A_i$, we see that $M$ must be a tame two-sphere. Since $N \subset f(M)$, $V'$ consists of one point also. By the same argument $N$ is a tame sphere, too. However, this contradicts the construction of $W'$ and $V'$.

Consider $f' = f \mid M - W'$. By the two dimensional analogue of of [3] Theorem 3.4, $f'$ is homotopic to a homeomorphism $f''\colon (M - W') \to (N - V')$. (If $f \mid M$ is already a homeomorphism, we may choose the homotopy fixed.) Using regular neighborhoods of $M - W'$ in $S^3 - W'$ and $N - V'$ in $S^3 - V'$, we may extend the $p$-homotopy over $W^3 - W'$. Since the homotopy is $p$-uniform, we may further extend it to a homotopy of $S^3$ into $S^3$. Let $f_0\colon S^3 \to S^3$ be the resulting map. Then $f_0(M) \subset N$. Since $V'$ is totally disconnected, for any $v \in V'$ there is a sequence of points $\{p_n\}$ in $N - V'$ converging to $v$. Since $f_0$ is a $p$-uniform homeomorphism on $M - W'$, $\{f_0^{-1}(p_n)\}$ is a sequence of points converging to some unique $w \in W'$. Therefore, $f_0|M$ is a homeomorphism. We may assume that $f_0$ is a homeomorphism on an open regular neighborhood of $M - W'$.

We wish to show that $V' = V$ and $W' = W$. Suppose $v \in V' - V$. Then there is a tame disk $E \subset N$ with $v \in E^\circ$ and $\partial E \cap V' = \varnothing$. Its inverse image, $D = f_0^{-1}(E)$, is a closed disk with $\partial D \cap W' = \varnothing$. Since $W'$ is totally disconnected, by [8] (2.94) we have a sequence of open sets $\{U_n\}$ such that:

(1) $U_n \subset D^\circ$,

(2) $W' \cap D \subset U_n$,

(3) $U_n$ has finitely many components, none having diameter greater than $1/n$,

(4) $D - U_n$ is a connected compact two-manifold.

Let $L$ be a component of $U_n$. Since $\partial \bar{L}$ is a simple loop, $f_0(\partial \bar{L})$ is a simple loop in $E$. Therefore, $f_0(\partial \bar{L})$ is null homotopic in both $B_0$ and $B_1$. Since $f_*: \pi_1(A_i^\circ) \to \pi_1(B_i^\circ)$ is injective for each $i$, $\partial \bar{L}$ is nullhomotopic in both $A_0$ and $A_1$. Since $A_i$ is trail irreducible, $\partial \bar{L}$ bounds a simple closed disk in $A_i$ of diameter $d(n)$, where $\lim d(n) = 0$. Therefore, we may replace $D$ by a disk $D_n$ in $A_i^\circ$ parallel to $D$ with no point more than $d(n)$ from the corresponding point in $D$. The sequences $\{D_n^0\}$ and $\{D_n^1\}$ homeomorphically approximate $D$ from each side so by [2] Theorem 11.1, $D$ is tame. However, this contradicts the choice of $v \in V'$. If $w \in W' - W$ is chosen, the argument proceeds in the same way except instead of $f_*$ being injective we use that it is a homomorphism. In like manner we can show that $f_0$ takes points which are wild from one side to points which are wild from only one side.

Let $f^i = f_0|A_i$. Then $f^i: (A_i, \partial A_i) \to (B_i, \partial B_i)$. Since $f_0$ is a homoeomorphism on $M$, $f^i|\partial A_i$ is also a homeomorphism, and $\pi_1(\partial A_i) \to \pi_1(\partial B_i)$ is an isomorphism. The conditions require $A_i$ and $B_i$ to be trail irreducible, so $f^i$ satisfies the conditions of Theorem 1. Let $H^i$ be a $p$-homotopy $\operatorname{rel} \partial A_i$ taking $f^i$ to a homeomorphism $g^i: A_i \to B_i$. Define a homotopy $H'$ rel $M$ by

$$H'_t(x) = \begin{cases} H_t^i(x) & x \in A_i \\ f_0(x) & x \in M\,. \end{cases}$$

Then $g = H_1$ is the desired homeomorphism from $S^3$ onto $S^3$.

**4. Some examples.** The theorems we have proven apply only to trail irreducible manifold imbeddings with small wild point sets. We would like to produce examples of such manifold imbeddings.

Therefore, we must develop some methods of showing that a given manifold imbedding satisfies these conditions. The next result is a converse to Lemma C.

LEMMA G. *Suppose $A$ is a manifold imbedding with an exhausting sequence $\{C_n\}$ such that:*

(1) $\mathrm{Fr}\, C_n$ *is incompressible in* $A$,

(2) *The maximum of the diameters of the components of $A - C_n$ goes to zero.*

*Then $A$ is trail irreducible with totally disconnected wild point set.*

Suppose $\{C_n\}$ is such an exhausting sequence for $A$. If $w$ and $v$ are wild points of $A$, the distance between them is greater than zero, so for $n$ sufficiently large, by (2), they are in different components of $\bar{A} - C_n$. The minimal distance between components of $\bar{A} - C_n$ is greater than zero, so we may contain each component of $\bar{A} - C_n$ in disjoint open subsets of $S^3$. Therefore, the wild point set of $A$ is totally disconnected.

Suppose $\{D_n\}$ is a sequence of singular disks, piecewise linearly imbedded in $A$, with $\partial D_n \to w$, a wild point of $A$. For each $n$ there is an $m(n)$ such that for $m \geqq m(n)$, $\partial D_n \cap C_m = \emptyset$; also $m(n) \to \infty$. Since $\mathrm{Fr}\, C_m$ is incompressible in $A$, we may replace $D_n$ by a disk with the same boundary but not intersecting $C_m$. Thus $D_n$ is contained in a single component of $A - C_m$, and so $\mathrm{diam}\,(D_n) \to 0$. Therefore, $A$ is trail irreducible.

This lemma has the following corollary, which is more convenient for calculations with specific manifold imbeddings.

THEOREM 3. *Suppose $A$ is a manifold imbedding with a sequence of compact submanifolds $\{R_n\}$ such that:*

(1) $\mathrm{Fr}\, R_n$ *is properly and piecewise linearly imbedded in* $A$,

(2) $A = U R_n$,

(3) $R_i \cap R_j = \emptyset$ *for* $|i - j| > 1$,

(4) $R_i \cap R_{i+1}$ *is a system of surfaces with one component for each component of* $R_{i+1}$,

(5) *If $H$ is a component of $R_i \cap R_{i+1}$ and $U$ and $V$ are the components of $R_i$ and $R_{i+1}$ containing $H$, then $H$ is incompressible in $U \cup V$,*

(6) $R_0$ *is connected,*

(7) *The maximal diameter of the components of $\bigcup_{m \geqq n} R_m$ goes to zero as $n$ goes to $\infty$.*

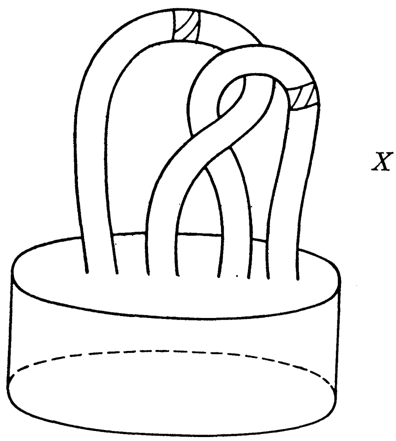


FIGURE 1

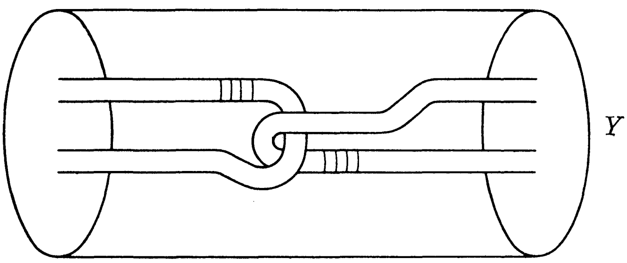


FIGURE 2

*Then A is a trail irreducible manifold imbedding with totally disconnected wild point set.*

Let $C_n = \bigcup_{i=0}^{n} R_i$. Then it is straightforward to show that $\{C_n\}$ satisfies the conditions of Lemma F.

Alexander's horned sphere is formed by taking the outside of Figure 1. The two shaded sections of the tubes are then replaced by a pair of tubes linked as in Figure 2. The new shaded areas are again replaced by pieces as in Figure 2. This process is repeated infinitely many times. The limiting surface is Alexander's horned sphere. The outside gives a manifold imbedding which is not a three-cell. Let $R_0$ be the outside of Figure 1. Let $R_1$ be the two pieces that are added in the shaded areas. Let $R_2$ be the four pieces that are added in the new shaded areas, and so forth. Let $U$ and $V$ be components of $R_n$ and $R_{n+1}$, respectively, with $U \cap V \neq \emptyset$. Then $\pi_1(U)$ is free on two generators, $a$ and $b$. Also $\pi_1(V)$ is free on two generators, $x$ and $y$. $\pi_1(U \cap V)$ is free on one generator, which goes to $b$ and $xyx^{-1}y^{-1}$. Then by van Kampen's theorem $\pi_1(U \cup V)$ is free on the three generators, $a, x$, and $y$. Thus $\{R_n\}$ satisfies

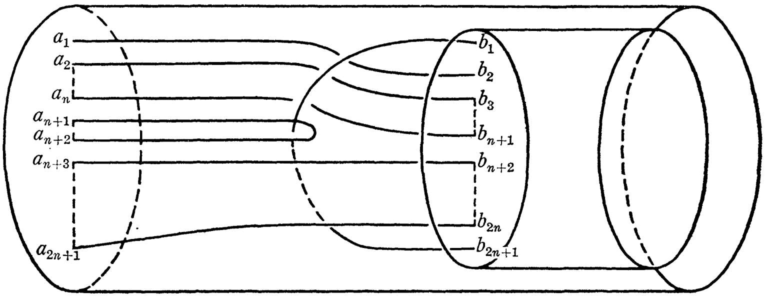


FIGURE 3

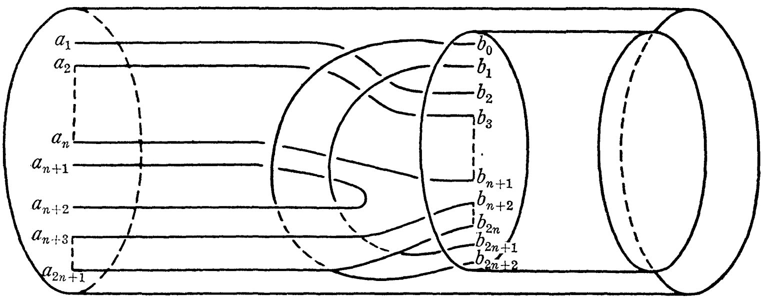


FIGURE 4

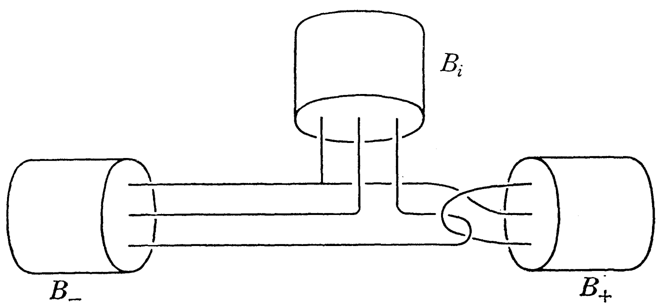


FIGURE 5

condition (5) of Theorem 3. The other conditions are easily verified. Therefore, the manifold imbedding formed by Alexander's horned sphere is trail irreducible.

Alford and Ball [1] give examples of two-spheres with one wild point and penetration index of any desired positive odd integer. They form these by joining together arcs as in Figures 3 and 4. To obtain one of penetration index $2n + 1$, piece together $n - 1$ copies of Figure 4. Each succeeding one is placed in the cavity on the right of the previous one and has the number of arcs entering from the left increased by two. When the number of arcs entering has been increased from three to $2n + 1$, and infinitely many copies of Figure 3. The author has shown by direct group manipulations that the intersection of two pieces is incompressible in their union. This whole construction is then inserted in $B_i$ of Figure 5. We can construct a copy of Figure 5 for each integer with a wild point in $B_i$ of any desired penetration index $p_i$ (odd and positive). These may be joined together along $B_+$ and $B_-$ and imbedded in $S^3$. The union of the one-complexes from each piece form a single one-complex, $\hat{a}$. Let $A$ be a closed regular neighborhood of $\hat{a}$ (in $S^3$ less the wild point in each $B_i$). Then $\bar{A} - A^\circ$ is a two-sphere imbedded in $S^3$ with a countable, totally disconnected set of wild points. We may choose any subset of the odd positive integers and make it the set of penetration indices of the isolated points of $\bar{A} - A^\circ$. $\bar{A}$ is a closed three-cell, so the distinction between these two-spheres is that they give rise to different manifold imbeddings containing $S^3 - A^\circ$. Using Theorem 3, we can show that these are trail irreducible and have totally disconnected wild point sets. Therefore, there are uncountably many trail irreducible manifold imbeddings with totally disconnected wild point sets.

4. **Concluding remarks.** Theorem 2 has only been stated for the case where $M$ and $N$ have one component each. This is the case of greatest interest. Although there is no technical difficulty in extending the theorem to the case where $M$ and $N$ have finitely many components, the statement of the conditions for the theorem would be exceeding complicated.

Theorem 2 determines if a given map from $(S^3, M)$ to $(S^3, N)$ is homotopic to a homeomorphism, under some restrictions on the imbeddings. In effect it says that if two imbeddings are of the same homotopy type, they are homeomorphic. The author has generalized [4] (4.6) to obtain necessary and sufficient algebraic conditions for a map preserving imbedding structure to exist between two given imbeddings, which induces the given homomorphisms of homotopy groups. In other words, the homeomorphism type of $(S^3, M)$ can be determined entirely algebraically.

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