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CONTRACTION SEMIGROUPS IN LEBESGUE SPACE

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Let $(T_t: t>0)$ be a strongly continuous semigroup of linear contractions on $L_1(X, \Sigma, \mu)$, where (X, Σ, μ) is a σ-finite measure space. Without assuming the initial continuity of the semigroup it is shown that $(T_t: t>0)$ is dominated by a strongly continuous semigroup $(S_t: t>0)$ of positive linear contractions on $L_1(X, \Sigma, \mu)$, i.e., that $|T_t f| \leqq S_t|f|$ holds a.e. on X for all $f \in L_1(X, \Sigma, \mu)$ and all $t>0$. As an application, a representation of $(T_t: t>0)$ in terms of $(S_t: t>0)$ is obtained, and the question of the almost everywhere convergence of $1/b \int_0^b T_t f dt$ as $b \to +0$ is considered.

Introduction. Let (X, Σ, μ) be a σ-finite measure space and let $L_p(X) = L_p(X, \Sigma, \mu)$, $1 \leqq p \leqq \infty$, be the usual Banach spaces of real or complex functions on (X, Σ, μ). For a set $A \in \Sigma$, $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish a.e. on $X - A$. If $f \in L_p(X)$, we define supp f to be the set of all $x \in X$ at which $f(x) \neq 0$. Relations introduced below are assumed to hold modulo sets of measure zero. A linear operator T on $L_p(X)$ is called a *contraction* if $\|T\|_p \leqq 1$, and *positive* if $f \geqq 0$ implies $Tf \geqq 0$.

Let $(T_t: t > 0)$ be a strongly continuous semigroup of linear contractions on $L_1(X)$, i.e.,

(i) each T_t is a linear contraction on $L_1(X)$,

(ii) $T_t T_s = T_{t+s}$ for all $t, s > 0$,

(iii) for every $f \in L_1(X)$ and every $s>0$, $\lim_{t \to s} \|T_t f - T_s f\|_1 = 0$.

Under the additional hypothesis of strong-$\lim_{t \to +0} T_t = I$ (I denotes the identity operator), Kubokawa [6] proved that there exists a strongly continuous semigroup $(S_t: t > 0)$ of positive linear contractions on $L_1(X)$ such that $|T_t f| \leqq S_t|f|$ a.e. on X for all $f \in L_1(X)$ and all $t > 0$. The main purpose of this paper is to prove the same result, without assuming any additional hypothesis. We then obtain a representation of $(T_t: t > 0)$ in terms of $(S_t: t > 0)$ which is a continuous extension of Akcoglu-Brunel's representation ([1], Theorem 3.1), and a decomposition of the space X for $(T_t: t>0)$ which asserts the existence of a set $Y \in \Sigma$ such that $T_t f \in L_1(Y)$ for all $f \in L_1(X)$ and all $t > 0$ and also such that if $f \in L_1(Y)$ then $T_t f$ converges in the norm topology of $L_1(X)$ as $t \to +0$ and $1/b \int_0^b T_t f dt$ converges a.e. on X as $b \to +0$.

Existence theorem. Our main result is the following existence theorem.

THEOREM 1. *If* $(T_t: t > 0)$ *is a strongly continuous semigroup of linear contractions on* $L_1(X)$, *then there exists a strongly continuous semigroup* $(S_t: t > 0)$ *of positive linear contractions on* $L_1(X)$, *called the semigroup modulus of* $(T_t: t > 0)$, *such that*

$$(1) \qquad |T_t f| \leqq S_t |f| \qquad (f \in L_1(X),\ t > 0)\ .$$

If $0 \leqq f \in L_1(X)$, S_t *is given by*

$$(2) \qquad S_t f = \sup\left\{\tau_{t_1}\cdots\tau_{t_n} f\colon \sum_{i=1}^{n} t_i = t,\ t_i > 0,\ n \geqq 1\right\}$$

where τ_t *denotes the linear modulus of* T_t *in the sense of Chacon-Krengel* ([3]).

Proof. For $0 \leqq f \in L_1(X)$ and $t > 0$, put

$$M(t, f) = \left\{\tau_{t_1}\cdots\tau_{t_n} f\colon \sum_{i=1}^{n} t_i = t,\ t_i > 0,\ n \geqq 1\right\}.$$

Since $\|\tau_t\|_1 = \|T_t\|_1 \leqq 1$ and $\tau_t\tau_s f \geqq \tau_{t+s} f$ for all $t,\ s > 0$, we see that if g_1 and g_2 are in $M(t, f)$, then there exists a function h in $M(t, f)$ such that

$$\max(g_1, g_2) \leqq h \text{ and } \|h\|_1 \leqq \|f\|_1\ .$$

Thus it is possible to define a function $S_t f$ in $L_1(X)$ by the relation:

$$S_t f = \sup\{g\colon g \in M(t, f)\}\ .$$

It is clear that $\|S_t f\|_1 \leqq \|f\|_1$ and $S_t f \geqq 0$. It is easily seen that if c is a positive constant and f and g are nonnegative functions in $L_1(X)$, then

$$S_t(cf) = cS_t f \text{ and } S_t(f + g) = S_t f + S_t g\ .$$

Therefore S_t may be regarded as a positive linear contraction on $L_1(X)$. By the definition of S_t it follows that

$$S_t S_s = S_{t+s} \qquad (t,\ s > 0)\ .$$

It is now enough to prove the strong continuity of $(S_t: t > 0)$. To do this, we first show the following result:

$$(3) \qquad \lim_{t \to s+0} \|\tau_t f - \tau_s f\|_1 = 0 \quad (f \in L_1(X),\ s > 0)\ .$$

To see this, we may and do assume without loss of generality that f is nonnegative. Let $\varepsilon > 0$ be given. By [3] there exist functions $g_i \in L_1(X)$, $1 \leqq i \leqq n$, such that

$$|g_i| \leqq f \text{ and } \|\tau_s f - \max_{1\leqq i\leqq n} |T_s g_i|\,\|_1 < \varepsilon .$$

Since $(T_t: t > 0)$ is strongly continuous on $(0, \infty)$, we can take a $\delta > 0$ so that $|s - t| < \delta$ implies $\| T_s g_i - T_t g_i\|_1 < \varepsilon/n$ for each $1 \leqq i \leqq n$. Fix a $t > 0$ so that $|s - t| < \delta$. We then have $\| \, |T_s g_i| - |T_t g_i| \, \|_1 \leqq \| T_s g_i - T_t g_i\|_1 < \varepsilon/n$ for each $1 \leqq i \leqq n$, and so it follows that

$$\|\tau_s f - \max_{1\leqq i\leqq n} |T_t g_i|\,\|_1 < 2\varepsilon .$$

By this and the fact that $\tau_t f \geqq \max_{1\leqq i\leqq n} |T_t g_i|$, we get

$$\|(\tau_t f - \tau_s f)^-\|_1 \leqq \|(\max_{1\leqq i\leqq n} |T_t g_i| - \tau_s f)^-\|_1 < 2\varepsilon .$$

Therefore

$$\lim_{t\to s} \|(\tau_t f - \tau_s f)^-\|_1 = 0 . \tag{4}$$

Next, let $t > s$ and write $t = s + a$. Since

$$\|(\tau_a \tau_s f - \tau_s f)^-\|_1 \leqq \|(\tau_t f - \tau_s f)^-\|_1 ,$$

it follows that

$$\lim_{a\to +0} \|(\tau_a \tau_s f - \tau_s f)^-\|_1 = 0 . \tag{5}$$

On the other hand,

$$\tau_a \tau_s f = (\tau_a \tau_s f - \tau_s f)^+ - (\tau_a \tau_s f - \tau_s f)^- + \tau_s f .$$

Thus, by (5), we have that

$$\begin{aligned} \|(\tau_{a+s} f - \tau_s f)^+\|_1 &\leqq \|(\tau_a \tau_s f - \tau_s f)^+\|_1 \\ &\leqq \|(\tau_a \tau_s f - \tau_s f)^-\|_1 + \|\tau_a \tau_s f\|_1 - \|\tau_s f\|_1 \\ &\leqq \|(\tau_a \tau_s f - \tau_s f)^-\|_1 \to 0 \end{aligned}$$

as $a \to +0$, because $\|\tau_a\|_1 \leqq 1$. This and (4) establish (3).

We next show that

$$\lim_{t\to s+0} \|S_t f - S_s f\|_1 = 0 \quad (f \in L_1(X),\ s > 0) . \tag{6}$$

To see this, we may and do assume without loss of generality that f is nonnegative. Let $\varepsilon > 0$ be given, and choose a function $g \in M(s, f)$ so that

$$||S_s f - g||_1 < \varepsilon ,$$

where g is of the form

$$g = \tau_{t_1} \cdots \tau_{t_n} f, \; \sum_{i=1}^{n} t_i = s, \text{ and } t_i > 0 \; (1 \leqq i \leqq n) .$$

Let $s_n > t_n$. Then

$$\begin{aligned} ||g - \tau_{t_1} \cdots \tau_{t_{n-1}} \tau_{s_n} f||_1 &= ||\tau_{t_1} \cdots \tau_{t_{n-1}} (\tau_{t_n} f - \tau_{s_n} f)||_1 \\ &\leqq ||\tau_{t_n} f - \tau_{s_n} f||_1 , \end{aligned}$$

and hence, by (3),

$$\lim_{s_n \to t_n + 0} ||g - \tau_{t_1} \cdots \tau_{t_{n-1}} \tau_{s_n} f||_1 = 0 . \tag{7}$$

Let us write $t = t_1 + \cdots + t_{n-1} + s_n (> s)$. Since

$$S_t f - S_s f \geqq (\tau_{t_1} \cdots \tau_{t_{n-1}} \tau_{s_n} f - g) + (g - S_s f) ,$$

it follows that

$$(S_t f - S_s f)^- \leqq |\tau_{t_1} \cdots \tau_{t_{n-1}} \tau_{s_n} f - g| + |g - S_s f| .$$

This and (7) yield that

$$\limsup_{t \to s+0} ||(S_t f - S_s f)^-||_1 \leqq \varepsilon .$$

Since ε is arbitrary,

$$\lim_{t \to s+0} ||(S_t f - S_s f)^-||_1 = 0 .$$

Hence

$$\begin{aligned} ||(S_t f - S_s f)^+||_1 &= ||(S_t f - S_s f)^-||_1 + ||S_t f||_1 - ||S_s f||_1 \\ &\leqq ||(S_t f - S_s f)^-||_1 \longrightarrow 0 \end{aligned}$$

as $t \to s + 0$, because $||S_t f||_1 \leqq ||S_s f||_1$ for all $t > s$. This proves (6).

Using (6), it is now direct to show that the semigroup $(S_t: t>0)$ is strongly continuous on $(0, \infty)$, and we omit the details.

THEOREM 2. *Let $(T_t: t > 0)$ and $(S_t: t > 0)$ be as in Theorem 1. Then T_t converges strongly as $t \to +0$ if and only if S_t converges strongly as $t \to +0$.*

Proof. If $T_0 = \text{strong-lim}_{t \to +0} T_t$ exists, then $(T_t: t \geqq 0)$ is a semigroup and strongly continuous on $[0, \infty)$. Hence we can apply the same arguments as in the proof of Theorem 1 to obtain that $\lim_{t \to +0} ||S_t f - \tau_0 f||_1 = 0$ for all $f \in L_1(X)$, where τ_0 denotes the linear modulus of T_0.

Conversely, if $S_0 = \text{strong-lim}_{t\to+0} S_t$ exists, then, for all $f \in L_1(X)$, the set $\{T_t f: 0 < t < 1\}$ is weakly sequentially compact in $L_1(X)$, since $|T_t f| \leqq S_t|f|$ and $\lim_{t\to+0} \| S_t|f| - S_0|f| \|_1 = 0$ (cf. Theorem IV. 8.9 in [4]). Thus, by Lemma 1 of the author [8], T_t converges strongly as $t \to +0$.

The hypothesis of being a contraction semigroup can not be weakened in Theorem 1. To see this, we give the following example, motivated by S. Tsurumi.

EXAMPLE. Let X be the positive integers, Σ all possible subsets of X, and μ the counting measure. Let $\varepsilon > 0$ be given. By an elementary computation, there exists a real constant r, with $1/e < r < 1$, such that

$$1 < \sup\{r^t(|\cos t| + |\sin t|): t \geqq 0\} < 1 + \varepsilon \,. \tag{8}$$

For $f \in L_1(X)$ and $t > 0$, define

$$T_t f(2n-1) = r^{nt}[f(2n-1)\cos nt - f(2n)\sin nt] \quad (n \geqq 1)$$

and

$$T_t f(2n) = r^{nt}[f(2n-1)\sin nt + f(2n)\cos nt] \quad (n \geqq 1)\,.$$

It is easily seen that $(T_t: t > 0)$ is a strongly continuous semigroup of linear operators on $L_1(X)$ satisfying $\| T_t \|_1 \leqq 1 + \varepsilon$ for all $t>0$. Furthermore

$$\lim_{m\to\infty} \|(\tau_{1/m})^m\|_1 = \infty \,. \tag{9}$$

To see this, let 1_n denote the indicator function of $\{n\}$. Then

$$\begin{aligned} \|(\tau_{1/m})^m\|_1 &\geqq \|(\tau_{1/m})^m(1_{2n-1} + 1_{2n})\|_1 / \|1_{2n-1} + 1_{2n}\|_1 \\ &= \left[r^{n/m}\left(\left|\cos\frac{n}{m}\right| + \left|\sin\frac{n}{m}\right| \right) \right]^m \quad (n \geqq 1)\,, \end{aligned}$$

as has been pointed out by S. Koshi. Hence (8) implies (9).

By (9) it is now immediate to see that $(T_t: t > 0)$ can not be dominated by a semigroup of positive linear operators on $L_1(X)$.

Representation theorem. Let $(T_t: t > 0)$ be a strongly continuous semigroup of linear contractions on $L_1(X)$. It is well known that given an $f \in L_1(X)$ there exists a scalar function $g(t, x)$ on $(0, \infty) \times X$, measurable with respect to the product of Lebesgue measure and μ, such that for each $t > 0$, $g(t, x)$, as a function of x, belongs to the equivalence class of $T_t f$. In the sequel $g(t, x)$ will be denoted by $T_tf(x)$. Using Fubini's theorem, we see that there exists a set $E(f) \in \Sigma$, with $\mu(E(f)) = 0$, such that if $x \notin E(f)$ then the scalar function $t \mapsto T_tf(x)$ is Lebesgue integrable on every finite interval (a, b) and the integral $\int_a^b T_tf(x)dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_tfdt$, where $\int_a^b T_tfdt$ denotes the Bochner integral of the vector valued function $t \mapsto T_tf$ with respect to Lebesgue measure on (a, b).

If $(S_t: t > 0)$ denotes the semigroup modulus of $(T_t: t > 0)$, then the ratio ergodic theorem holds for $(S_t: t > 0)$, i.e., for any f and g in $L_1(X)$, with $g \geqq 0$, the ratio ergodic limit

$$\lim_{b \to \infty} \left(\int_0^b S_tf(x)dt\right) \Big/ \left(\int_0^b S_tg(x)dt\right)$$

exists and is finite a.e. on the set $\left\{x: \int_0^\infty S_tg(x)dt > 0\right\}$ (cf. [5]). Thus Hopf's decomposition holds, i.e., X decomposes into two measurable sets C and D, called respectively the conservative and dissipative parts of X, such that if $0 \leqq g \in L_1(X)$ then $\int_0^\infty S_tg(x)dt = \infty$ or 0 a.e. on C and $\int_0^\infty S_tg(x)dt < \infty$ a.e. on D. A set $A \in \Sigma$ is called *invariant* (under $(S_t: t > 0)$), if $S_tL_1(A) \subset L_1(A)$ for all $t > 0$. It is immediate that A is invariant under $(S_t: t > 0)$ if and only if it is invariant under $(T_t: t > 0)$. It is known (cf. [7]) that C is invariant and the class Σ_i of all invariant subsets of C forms a σ-field in the class of all measurable subsets of C.

We are now in a position to state our representation theorem.

THEOREM 3. *Let $(T_t: t > 0)$ be a strongly continuous semigroup of linear contractions on $L_1(X)$ and $(S_t: t > 0)$ denote the semigroup modulus of $(T_t: t > 0)$. Let C denote the conservative part of X with respect to $(S_t: t > 0)$ and let Σ_i be the σ-field of invariant subsets of C. Then there exists a (unique) set $\Gamma \in \Sigma_i$ and a function $u \in L_\infty(\Gamma)$ such that*

(i) $|u| = 1$ a.e. *on Γ and $T_tf = (1/u)S_t(uf)$ for all $f \in L_1(\Gamma)$ and all $t > 0$,*

(ii) *if $\Delta = C - \Gamma$, then the closed linear hull of $\{f - T_tf: f \in L_1(\Delta),\ t > 0\}$ is $L_1(\Delta)$,*

(iii) *a function $v \in L_\infty(\Gamma)$, with $|v| > 0$ a.e. on Γ, satisfies $T_tf = (1/v)S_t(vf)$ for all $f \in L_1(\Gamma)$ and all $t > 0$ if and only if there exists a function $r \in L_\infty(\Gamma)$ such that $|r| > 0$ a.e. on Γ, $S_t^*r = r$ a.e. on Γ for all $t > 0$, and $v = ru$.*

Proof. Let $h \in L_\infty(C)$ be such that $T_t^*h = h$ a.e. on C for all $t > 0$. Since $|h| = |T_t^*h| \leqq \tau_t^*|h| \leqq S_t^*|h|$ and the conservative part of X with respect to each single operator S_t is exactly C (cf. [7]), it follows that $|h| = S_t^*|h|$ a.e. on C for all $t > 0$, and hence supp $h \in \Sigma_i$. By this, we can find a function $h \in L_\infty(C)$ such that $T_t^*h = h$ a.e. on C for all $t > 0$ and also such that if $f \in L_\infty(C)$ satisfies $T_t^*f = f$ a.e. on C for all $t > 0$, then $\operatorname{supp} f \subset \operatorname{supp} h$. Put $\Gamma = \operatorname{supp} h$ and define $u \in L_\infty(\Gamma)$ by $u = h/|h|$ a.e. on Γ. If $0 \leqq f \in L_1(\Gamma)$ and $t > 0$, then, as in [1],

$$\int (S_t f)|h| d\mu = \int f S_t^*|h| d\mu = \int f|h| d\mu = \int (f/u) h d\mu$$
$$= \int (f/u) T_t^* h d\mu = \int T_t(f/u) u |h| d\mu .$$

Hence $S_t f = T_t(f/u)u$, since $S_t f \geqq |T_t(f/u)| = |T_t(f/u)u|$, and (i) is established.

To prove (ii), let $h \in L_\infty(\Delta)$ be such that $\int (f - T_t f) h d\mu = 0$ for all $f \in L_1(\Delta)$ and all $t > 0$. Then $T_t^* h = h$ a.e. on Δ (and hence on C) for all $t > 0$. Therefore, by the definition of Γ, $h = 0$ a.e. on Δ, and (ii) follows from the Hahn-Banach theorem.

To prove (iii), let $v \in L_\infty(\Gamma)$ and $|v| > 0$ a.e. on Γ. Put $r = v/u$. Then $T_t f = (1/v) S_t(vf)$ for all $f \in L_1(\Gamma)$ and all $t > 0$ if and only if $(1/ru) S_t(ruf) = (1/u) S_t(uf)$ for all $f \in L_1(\Gamma)$ and all $t > 0$, or equivalently, $S_t(rf) = r S_t f$ for all $f \in L_1(\Gamma)$ and all $t > 0$, since $\{uf : f \in L_1(\Gamma)\} = L_1(\Gamma)$. And this is equivalent to the fact that $S_t^* r = r$ a.e. on Γ for all $t > 0$, by Lemma 2.4 in [1].

The proof is complete.

Decomposition theorem. It is shown that, after eliminating an uninteresting subset of X, a strongly continuous semigroup $(T_t : t > 0)$ of linear contractions on $L_1(X)$ can be made strongly continuous at the origin and the local ergodic theorem holds.

THEOREM 4. *Let $(T_t : t > 0)$ be a strongly continuous semigroup of linear contractions on $L_1(X)$. Then X can be written as the union of two disjoint measurable sets Y and Z with the following properties*:

(i) *For every $f \in L_1(X)$ and every $t > 0$, $T_t f \in L_1(Y)$.*

(ii) *For every $f \in L_1(Y)$, $T_t f$ converges in the norm topology of $L_1(X)$ as $t \to +0$ and*

$$\lim_{b \to +0} \frac{1}{b} \int_0^b T_t f(x) dt$$

exists a.e. *on X.*

(iii) *For every $f \in L_1(Y)$ with $f > 0$* a.e. *on Y,*

$$Y = \bigcup_{n=1}^{\infty} \{x: \tau_{1/n} f(x) > 0\} .$$

Proof. Let $(S_t: t > 0)$ be the semigroup modulus of $(T_t: t > 0)$. Fix an $h \in L_1(X)$ with $h > 0$ a.e. on X, and put

$$Y = \bigcup_{n=1}^{\infty} \{x: S_{1/n} h(x) > 0\}$$

and $Z = X - Y$. It is easily seen that $S_t f \in L_1(Y)$ and hence $T_t f \in L_1(Y)$ for all $f \in L_1(X)$ and all $t > 0$. If we write $h_0 = \int_0^1 S_t h dt$, then $h_0 \in L_1(Y)$, $h_0 > 0$ a.e. on Y, and $\lim_{t \to +0} \|S_t h_0 - h_0\|_1 = 0$. Therefore, by approximation, the set $\{S_t f: 0 < t < 1\}$ is weakly sequentially compact in $L_1(X)$ for all $0 \leqq f \in L_1(Y)$, from which we observe that the set $\{T_t f: 0 < t < 1\}$ is also weakly sequentially compact in $L_1(X)$ for all $f \in L_1(Y)$, since $|T_t f| \leqq S_t |f|$ for all $t > 0$. Hence Lemma 1 of the author [8] implies that $T_t f$ converges in the norm topology of $L_1(X)$ as $t \to +0$ for all $f \in L_1(Y)$.

To prove the second part of (ii), we may and do assume without loss of generality that $X = Y$. Put $T_0 = \text{strong-lim}_{t \to +0} T_t$, and let $f \in L_1(X)$. Then f can be written as $f = g + h$, where $g = T_0 f$ and $T_t h = 0$ for all $t \geqq 0$, because $T_t T_0 = T_0 T_t = T_t$ for all $t \geqq 0$. It follows that

$$\lim_{a \to +0} \|(f - h) - \frac{1}{a}\int_0^a T_t g dt\|_1 = 0 .$$

If we write $f_a = h + 1/a \int_0^a T_t g dt$, then it is easily seen that

$$\lim_{b \to +0} \frac{1}{b}\int_0^b T_t f_a(x) dt = f_a(x) - h(x) \quad \text{a.e.}$$

on X. On the other hand, by Akcoglu-Chacon's local ergodic theorem ([2]),

$$\sup_{0<b<1} \left| \frac{1}{b}\int_0^b T_t f(x) dt \right| \leqq \sup_{0<b<1} \frac{1}{b}\int_0^b S_t |f|(x) dt < \infty \quad \text{a.e.}$$

on X. Thus, the second part of (ii) follows from Banach's convergence theorem (cf. Theorem IV. 11. 3 in [4]).

For the proof of (iii), let $f \in L_1(Y)$, $f > 0$ a.e. on Y. Put

$$P = \bigcup_{n=1}^{\infty} \{x: \tau_{1/n} f(x) > 0\} .$$

Clearly, $P \subset Y$, and by the definition of Y and (i),

$$Y = \bigcup_{n=1}^{\infty} \{x: S_{1/n} f(x) > 0\} .$$

Let $1/n < t$. Then $\tau_t f \leqq \tau_{1/n}\tau_{t-(1/n)}f$, and so $\operatorname{supp} \tau_t f \subset \operatorname{supp} \tau_{1/n}f$. Thus it follows that

$$\operatorname{supp} S_t f \subset P \qquad (t > 0) .$$

Therefore $Y \subset P$, and (iii) is established.

The proof is complete.

In conclusion, the author would like to remark that the question of whether the almost everywhere convergence of $1/b \int_0^b T_t f(x)dt$ as $b \to +0$ holds for all $f \in L_1(Z)$ remains an open problem.

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