PACIFIC JOURNAL OF MATHEMATICS
Vol. 78, No. 1, 1978

MEAN VALUE THEOREMS FOR A CLASS OF DIRICHLET SERIES

DON REDMOND

In this paper we are concerned with mean value theorems for the summatory functions of a class of Dirichlet series. This class of Dirichlet series is a class of Dirichlet series satisfying functional equations involving multiple gamma factors. If $f(s) = \sum a(n)\lambda_n^{-s}$ is a Dirichlet series satisfying such a functional equation and $E(x)$ is the associated error term (see (1.2) and (1.4), respectively), then we prove 0-estimates for

$$\int_0^x |E(y)|^2 dy \tag{1}$$

and

$$\sum_{\lambda_n \leqq x} |a(n)|^2 , \tag{2}$$

in the latter case when $\lambda_n = n$. The results we get for (1) improve known results in some cases. Also the general result (1) is applicable in cases where a similar result of Chandrasekharan and Narasimhan is not.

1. Introduction and historical survey. In this paper we shall obtain a mean square estimate for the error term of the summatory function of a class Dirichlet series. We shall also obtain an estimate for the sum of the squares of the coefficients of these Dirichlet series. The class of Dirichlet series we are concerned with consists of those satisfying a functional equation involving multiple gamma factors such as was considered by Chandrasekharan and Narasimhan in [4].

Let $\{a(n)\}$ and $\{b(n)\}$, $1 \leqq n < +\infty$, be two sequences of complex numbers, not all zero, and let $\{\lambda_n\}$ and $\{\mu_n\}$, $1 \leqq n < +\infty$, be two sequences of positive real numbers increasing to $+\infty$. Suppose that

$$f(s) = \sum_{n=1}^{\infty} a(n)\lambda_n^{-s} \quad \text{and} \quad g(s) = \sum_{n=1}^{\infty} b(n)\mu_n^{-s}$$

each converge in some half plane with finite abcissas of absolute convergence $\sigma_a(f)$ and $\sigma_a(g)$, respectively. Let

$$\Delta(s) = \prod_{k=1}^{N} \Gamma(\alpha_k s + \beta_k) , \tag{1.1}$$

where $\alpha_k > 0$ and β_k is complex, $1 \leqq k \leqq N$. Then $f(s)$ and $g(s)$ are said to satisfy the functional equation

$$\Delta(s)f(s) = \Delta(r - s)g(r - s) \tag{1.2}$$

if there exists in the s plane a domain D, which is the exterior of a compact set S, in which there exists a holomorphic function $G(s)$ with the properties:

$$(1) \qquad \lim_{|t|\to\infty} G(\sigma + it) = 0 \, ,$$

uniformly in every strip $-\infty < \sigma_1 \leqq \sigma \leqq \sigma_2 < +\infty$, and

$$(2) \qquad G(s) = \begin{cases} \Delta(s)f(s) & \text{for } \operatorname{Re}(s) > \sigma_a(f) \\ \Delta(r-s)g(r-s) & \text{for } \operatorname{Re}(s) < r - \sigma_a(g) \, . \end{cases}$$

If

$$(1.3) \qquad Q(x) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s} x^s ds \, ,$$

where C is a curve enclosing all the singularities of the integrand, let

$$(1.4) \qquad E(x) = \sum_{\lambda_n \leqq x}{}' a(n) - Q(x) \, ,$$

where the dash indicates that if $\lambda_n = x$, then we add only $a(n)/2$. $E(x)$ is called the error term for the summatory function of the coefficients of the Dirichlet series $f(s)$.

With the notation as above, in this paper we will obtain estimates for

$$(1.5) \qquad \int_0^x |E(y)|^2 dy$$

and

$$(1.6) \qquad \sum_{\lambda_n \leqq x} |a(n)|^2 \, ,$$

in the latter case when $\lambda_n = n$. Both of there estimates can be used to obtain information on the size of the error term by use of the Cauchy-Schwarz inequality. Estimates for (1.5) imply estimates for the average size of the error term,

$$\frac{1}{x}\int_0^x |E(y)| dy \, ,$$

and estimates for (1.6) imply estimates for the summatory function,

$$\sum_{\lambda_n \leqq x} |a(n)| \, ,$$

and so also on the size of the error term.

Estimates for (1.5) were first obtained in 1922 by Cramér [7] for the special cases $E(x) = \Delta_2(x)$ and $P(x)$, the error terms of the divisor problem and the sums of two squares problem, respectively. In 1933 Walfisz [20] gave estimates for the integral in the case of cusp forms. In 1964 Chandrasekharan and Narasimhan [6] gave estimates for the integral (1.5) in the general case. All of these methods used identities between the summatory functions being studied and series involving Bessel functions or integrals that are generalizations of Bessel functions. They also used the differencing methods developed by Landau in his work on lattice point problems.

In 1938 Walfisz [21] gave estimates for (1.5) in the case $E(x) = P_4(x)$, the error term associated to the problem of counting lattice points in four dimensional ellipsoids. In 1940 Jarnik [12] gave estimates for general $P_k(x)$. Both of these methods used the modular relations between theta series defined from quadratic forms.

We generalize the method of Walfisz [21] to obtain our estimates on the mean value integral (1.5). The result we obtain improves a result of Chandrasekharan and Narasimhan [6] in those cases where they do not get an asymptotic estimate.

The sums (1.6) have been studied in many special cases. Ramanujan stated in [16] and Wilson proved in [22] asymptotic estimates for the case $a(n)=d(n)$, the divisor function. In [9] Hardy gave an 0-estimate for the case $a(n) = \tau(n)$, Ramanujan's function, and later Rankin, in [15], sharpened this to an asymptotic result, as well as giving similar estimates for the coefficients of cusp forms in general. In [21] Walfisz gave asymptotic formulas for $a(n) = r_3(n)$ and $r_4(n)$, which generalize immediately to the general case $a(n) = r_k(n)$, $k \geqq 5$, where $r_k(n)$ is the number of ways of representing n by a given positive definite quadratic form in k variables.

Estimates for the sum (1.6) also appear in the hypotheses to several theorems. For example, in Apostol's work on approximate functional equations for Hecke series [2, Corollary 2] such estimates are used for estimating the error terms that arise (see also [5]). Also the mean value theorem of Chandrasekharan and Narasimhan that was mentioned above [6, Theorem 1] has as one of its main hypotheses an estimate on the sum (1.6).

We generalize the method of Walfisz [21] to obtain an 0-estimate in the general case when $\lambda_n = n$. Because we can no longer appeal to any special properties of the coefficients $a(n)$ our general result does not give asymptotic results in the above mentioned cases, though it does apply to a wide class of arithmetical functions.

In the sequel we shall use the following notation:

(1) $\int_{(a)}$ will denote the integral $\int_{a-i\infty}^{a+i\infty}$,

(2) $\int_{(a,T)}$ will denote the integral $\int_{a-iT}^{a+iT}$

and

(3) $\sum$ will denote the sum $\sum_{n=1}^{\infty}$.

Also c_j, $j = 1, , \cdots 2$, will denote positive absolute constants.

2. **Statement of results.** Suppose $f(s) = \sum a(n)\lambda_n^{-s}$ and $g(s) = \sum b(n)\mu_n^{-s}$ satisfy the functional equation (1.2) with $r > 0$. Let

$$I(u) = \frac{1}{2\pi i}\int_C \frac{\Delta(s)}{\Delta(r-s)} u^{r-1-s} ds \, , \tag{2.1}$$

where $\Delta(s)$ is defined by (1.1) and C is a curve enclosing all the singularities of the integrand. Let

$$F(s) = \sum a(n) \exp(-\lambda_n s) \, , \tag{2.2}$$

for $\mathrm{Re}(s) > 0$. Then it is known [3, Theorems 3 and 5] that

$$F(s) = \int_0^\infty Q'(x)e^{-sx}dx + \sum b(n)\mu_n^{1-r}\int_0^\infty I(\mu_n x)e^{-sx}dx \, , \tag{2.3}$$

if we assume all the singularities of $f(s)$ are in the right half plane. Note that if $f(s)$ is entire, then from (1.3) $Q(x) = f(0)$, a constant.

If $\hat{Q}(s)$ denotes the Laplace transform of $Q(x)$, then we have

$$s\hat{Q}(s) = \int_0^\infty Q'(x)e^{-sx}dx \, . \tag{2.4}$$

In what follows we shall assume that $f(s)$ has a finite number of singularities in the right half plane and that these singularities are poles lying in the strip $0 < \mathrm{Re}(s) \leqq r$. If the poles are $\{\xi_1, \cdots, \xi_n\}$ and r_ξ is the order of the pole at ξ, then we can explicitly evaluate $Q(x)$, namely,

$$Q(x) = \sum_{j=1}^{n} \zeta_j x^{\xi_j} \log^{r_{\xi_j}-1} x \, , \tag{2.5}$$

where ζ_j is the residue of $f(s)$ at the pole $s = \xi_j$. Suppose that β is the real part of a pole with maximal real part and ρ is the maximal order of a pole with real part β. Then, from (2.4) and (2.5), we have

$$s\hat{Q}(s) = 0(|s|^{-\beta} \log^{\rho-1}|s|) \, , \tag{2.6}$$

as $|s| \to \infty$.

Finally we assume that as $|s| \to +\infty$

$$\sum b(n)\mu_n^{1-r} \int_0^\infty I(\mu_n x) e^{-sx} dx \sim s^{-m} \sum e(n) \exp\{-k(\mu_n/s)^\alpha\}\,, \tag{2.7}$$

where m is a nonnegative real number, k and α are positive real numbers and the $e(n)$ are complex numbers. Also we assume that the series on the right hand side of (2.7) converges absolutely for $\operatorname{Re}(s) > 0$. In §5 we shall prove a theorem that establishes (2.7) for a subclass of Dirichlet series satisfying the functional equation (1.2). We remark now that (2.7) is known as an equality for $\operatorname{Re}(s) > 0$ in the case $\Delta(s) = \Gamma(s)$, with $m = r$ (see [3, p. 152]).

Define the real numbers δ and η by

$$\delta = \min(m, \beta) \quad \text{and} \quad \eta = \begin{cases} 0 & \text{if} \quad \delta = m \quad \text{and} \quad m \neq \beta \\ 1 & \text{if} \quad \delta = \beta \quad \text{or} \quad m = \beta\,. \end{cases} \tag{2.8}$$

Further define the function $M(x)$ by

$$\begin{aligned} M(x) = {} & x^{m+1/2} + x^{m/2+\delta+1} \log^{(\rho-1)\eta} x + x^{(m+\beta)/2} \log^{\rho-1} x \\ & + x^{\beta+1/2} \log^{2\rho-1} x + x^{2\delta+1/2} \log^{2(\rho-1)\eta+1} x \\ & + x^{\delta+\beta/2+1} \log^{(\rho-1)(\eta+1)} x\,. \end{aligned} \tag{2.9}$$

We shall prove the following results with the notation as above.

THEOREM 1. *Assume the hypotheses as above. If $f(s)$ is not an entire function, then as $x \to \infty$*

$$\int_0^x |E(y)|^2 dy \ll M(x)\,.$$

If $f(s)$ is an entire function, then as $x \to \infty$

$$\int_0^x |E(y)|^2 dy \ll x^{2m+1/2} \log x + x^{3m/2+1}\,.$$

COROLLARY 1. *Suppose the hypotheses of Theorem 1 hold with $\Delta(s) = \Gamma(s)$. Then we have, if $f(s)$ is not entire function,*

$$\int_0^x |E(y)|^2 dy \ll x^{r+1/2} \log x + x^{r/2+\beta+1} \log^{(\rho-1)\eta} x + x^{2\beta+1/2} \log^{2(\rho-1)\eta+1} x\,,$$

as $x \to \infty$. If $f(s)$ is an entire function, then as $x \to \infty$

$$\int_0^x |E(y)|^2 dy \ll x^{3r/2+1} + x^{2r+1/2} \log x\,.$$

THEOREM 2. *Suppose the hypotheses of Theorem 1 hold with $\lambda_n = n$. Let $h(x)$ be defined by*

$$(2.10) \qquad h(x) = \begin{cases} x^{(m+\beta-1)/2}\log^{\rho+1} x + x^{m-1} & \textit{if} \quad m = 2\alpha - 1 \\ x^{m-\alpha+\beta/2}\log^{\rho} x + x^{2(m-\alpha)} & \textit{if} \quad m \neq 2\alpha, 2\alpha - 1 \\ x^{(m+\beta)/2}\log^{\rho} x + x^{m}\log x & \textit{if} \quad m = 2\alpha\,. \end{cases}$$

Then as $x \to \infty$, if $f(s)$ is not an entire function,

$$\sum_{n \leqq x} |a(n)|^2 \ll x^{\rho}\log^{2\rho-1} x + x^{2\delta}\log^{2(\rho-1)\eta+1} x + h(x)$$

for $0 < \beta \leqq 1$ and

$$\sum_{n \leqq x} |a(n)|^2 \ll x^{2\beta}\log^{2\rho-2} x + h(x)$$

for $\beta > 1$. If $f(s)$ is an entire function, then as $x \to \infty$

$$\sum_{n \leqq x} |a(n)|^2 \ll x^{2m}\log x\,.$$

COROLLARY 2. *Suppose the hypotheses of Theorem 2 hold with $\Delta(s) = \Gamma(s)$. Let $h(x)$ be defined by*

$$h(x) = \begin{cases} x^{r-1+\beta/2}\log^{\rho} x + x^{2(r-1)} & \textit{if} \quad r \neq 1, 2\,, \\ x^{\beta/2}\log^{\rho+1} x & \textit{if} \quad r = 1 \\ x^{1+\beta/2}\log^{\rho} x + x^{2}\log x & \textit{if} \quad r = 2\,. \end{cases}$$

Then as $x \to \infty$, if $f(s)$ is not an entire function,

$$\sum_{n \leqq x} |a(n)|^2 \ll x^{2\beta}\log^{2(\rho-1)\eta+1} x + h(x)$$

for $0 < \beta \leqq 1$ and

$$\sum_{n \leqq x} |a(n)|^2 \ll x^{2\beta}\log^{2\rho-2} x + h(x)$$

for $\beta > 1$. If $f(s)$ is an entire function, then as $x \to \infty$

$$\sum_{n \leqq x} |a(n)|^2 \ll x^{2r}\log x\,.$$

We shall prove these results only in the case when $f(s)$ is not an entire function and indicate the changes to be made if $f(s)$ is entire. The proofs of these results involves a series of lemmas and the sections devoted to their proofs will be divided into two parts: the first part for the proofs of the lemmas and the second part for the proofs of the theorems themselves.

The methods of proof of Theorems 1 and 2 are similar. They both involve an identity relating the integral or sum to be estimated to a double integral. The double integral is then rewritten as a sum of integrals over short intervals by means of Farey fractions. We give the definition of these intervals now.

DEFINITION 2.1. Let $(h, k) = 1$ and $0 < k \leqq x^{1/2}$. Let r_1 and r_2 be the Farey fractions of the Farey sequence of order $[x^{1/2}]$ that immediately proceed and succeed h/k, respectively. We denote by $B(h, k)$ the interval $[r_1, r_2]$. By $R(h, k)$ we denote the right hand endpoint of $B(h, k)$ and by $L(h, k)$ the left hand endpoint.

By Theorem 35 of [10], we have

$$\begin{aligned} &B(h, k) \\ &= \{u: h/k - \theta_1/k\sqrt{x} \leqq u \leqq h/k + \theta_2/k\sqrt{x}, 1/2 \leqq \theta_1, \theta_2 \leqq 1\} . \end{aligned} \tag{2.11}$$

3. The mean value integral.

3.1. Preliminary lemmas.

LEMMA 3.1. *If $s = 1/x + 2\pi ui$ and $t = 1/x + 2\pi vi$, where u and v are real and x is a fixed number greater than 1, then*

$$\begin{aligned} &\int_0^x |E(y)|^2 dy \\ &= \frac{1}{4\pi^2} \int_{(1/x)} \int_{(1/x)} \{F(s) - s\hat{Q}(s)\}\overline{\{F(t) - t\hat{Q}(t)\}} \\ &\quad \times \frac{\exp(x(s + \bar{t})) - 1}{s\bar{t}(s + \bar{t})} dsdt . \end{aligned} \tag{3.1}$$

Proof. On the left hand side of (3.1) we have, by (1.4),

$$\begin{aligned} \int_0^x |E(y)|^2 dy &= \int_0^x \{\sum_{\lambda_n \leqq y}{}' a(n) - Q(y)\}\{\sum_{\lambda_m \leqq y}{}' \overline{a(m) - Q(y)}\} dy \\ &= \int_0^x \{\sum_{\lambda_n, \lambda_m \leqq y} a(n)\overline{a(m)} - 2\,\mathrm{Re}\,[\overline{Q(y)} \sum_{\lambda_n \leqq y}{}' a(n)] + |Q(y)|^2\} dy \\ &= \sum_{\lambda_n, \lambda_m \leqq x} a(n)\overline{a(m)}\{x - \max(\lambda_n, \lambda_m)\} - 2\,\mathrm{Re}\left\{\sum_{\lambda_n \leqq x} a(n) \int_{\lambda_n}^x \overline{Q(y)} dy\right\} \\ &\quad + \int_0^x |Q(y)|^2 dy . \end{aligned} \tag{3.2}$$

Now, as $u \to \infty$, we have

$$\begin{aligned} |F(s) - s\hat{Q}(s)| &\leqq |F(s)| + |s\hat{Q}(s)| \\ &\ll \sum |a(n)| \exp(-\lambda_n \,\mathrm{Re}\,(s)) + |s|^{-\beta} \log^{\rho-1} |s| \\ &\ll \sum |a(n)| \exp(-\lambda_n/x) + x^{\beta} \log^{\rho-1} x . \end{aligned}$$

This estimate gives

$$\begin{aligned} &\left|\int_{(1/x)} \int_{(1/x)} \{F(s) - s\hat{Q}(s)\}\overline{\{F(t) - t\hat{Q}(t)\}} \frac{\exp(x(s + \bar{t})) - 1}{s\bar{t}(s + \bar{t})} dsdt\right| \\ &\quad \leqq \int_{(1/x)} \int_{(1/x)} |F(s) - s\hat{Q}(s)|\,|F(t) - t\hat{Q}(t)|\,|\exp(x(s + \bar{t})) - 1| \left|\frac{dsdt}{s\bar{t}(s + \bar{t})}\right| \end{aligned}$$

$$\leqq c_1\{\sum |a(n)| \exp(-\lambda_n/x) + x^\beta \log^{\rho-1} x\}^2 \int_{(1/x)}\int_{(1/x)} \left|\frac{dsdt}{s\bar{t}(s+\bar{t})}\right|$$

$$\leqq c_2 x\{\sum |a(n)| \exp(-\lambda_n/x) + x^\rho \log^{\rho-1} x\}^2$$

$$\times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{dudv}{(1+|u|)(1+|v|)(1+|u-v|)} .$$

Since the last double integral converges we see that the double integral on the right hand side of (3.1) converges absolutely.

Let

$$I = \frac{-1}{4\pi^2}\int_{(1/x)}\int_{(1/x)} \{F(s) - s\hat{Q}(s)\}\overline{\{F(t) - t\hat{Q}(t)\}}\frac{\exp(x(s+\bar{t})) - 1}{s\bar{t}(s+\bar{t})} dsdt$$

$$= \sum_{m,n=1}^{\infty} a(m)\bar{a}(n)\Big(\frac{-1}{4\pi^2}\Big)\int_{(1/x)}\int_{(1/x)} \exp(-\lambda_n s - \lambda_m \bar{t})\frac{\exp(x(s+\bar{t})) - 1}{s\bar{t}(s+\bar{t})} dsdt$$

$$(3.3)\quad -2\mathrm{Re}\Big\{\sum_{m=1}^{\infty} a(m)\Big(\frac{-1}{4\pi^2}\Big)\int_{(1/x)}\int_{(1/x)} \exp(-\lambda_n s)\overline{t\hat{Q}(t)}\frac{\exp(x(s+t))-1}{s\bar{t}(s+\bar{t})} dsdt\Big\}$$

$$+\Big(\frac{-1}{4\pi^2}\Big)\int_{(1/x)}\int_{(1/x)} s\hat{Q}(s)\overline{t\hat{Q}(t)}\frac{\exp(x(s+\bar{t})) - 1}{s\bar{t}(s+\bar{t})} dsdt$$

$$= \sum_{m=1}^{\infty} a(m)\overline{a(n)} J_{m,n} - 2\,\mathrm{Re}\Big\{\sum_{m=1}^{\infty} a(m) J_m\Big\} + J ,$$

say.

We have

$$(3.4)\qquad J_{m,n} = \lim_{T\to\infty} J_{m,n}(T) ,$$

where

$$J_{m,n}(T) = \frac{-1}{4\pi^2}\int_{(1/x,T)}\int_{(1/x,T)} \exp(-\lambda_m s - \lambda_n \bar{t})\frac{dsdt}{s\bar{t}}\int_0^x \exp(z(s+\bar{t}))dz$$

$$(3.5)\qquad = \int_0^x dz \frac{1}{2\pi i}\int_{(1/x,T)} \exp((z-\lambda_m)s)\frac{ds}{s}\,\frac{1}{2\pi i}\int_{(1/x,T)} \exp((z-\lambda_n)\bar{t})\frac{dt}{\bar{t}}$$

$$= \int_0^x dz \frac{1}{2\pi i}\int_{(1/x,T)} \exp((z-\lambda_m)s)\frac{ds}{s}\,\frac{(-1)}{2\pi i}\overline{\int_{(1/x,T)} \exp((z-\lambda_n)t)\frac{dt}{t}} .$$

Now (see [14, p. 346]), as $T\to\infty$,

$$(3.6)\qquad \left|\int_{(a,T)} e^{ws}\frac{ds}{s} - 2\pi i\right| \leqq \frac{2e^{wa}}{Ta}$$

if $w > 0$,

$$(3.7)\qquad \left|\int_{(a,T)} ds/s - \pi i\right| \leqq 2/T$$

for $w = 0$ and

$$\left|\int_{(a,T)} (e^{ws}/s)ds\right| \leqq 2e^{wa}/(T|w|) \tag{3.8}$$

if $w < 0$.

For fixed m and n let $A = \{z \in [0, x]: |z - \lambda_m| \geqq T - 1/2$ and $|z - \lambda_n| \geqq T^{-1/2}\}$ and $B = [0, x] - A$. Then

$$\int_0^x = \int_A + \int_B . \tag{3.9}$$

If $a = 1/x$, $w = z - \lambda_n$ or $z - \lambda_m$ and $z \in A$, then

$$e^{wa}/T|w| \leqq e/\sqrt{T} . \tag{3.10}$$

Thus, by (3.6), (3.8), and (3.10), we have

$$\begin{aligned}
&\int_A dz \frac{1}{2\pi i}\int_{(1/x,T)} \exp((z-\lambda_m)s)\frac{ds}{s}\frac{(-1)}{2\pi i}\overline{\int_{(1/x,T)} \exp((z-\lambda_n)t)\frac{dt}{t}} \\
&= \int_{A\cap(\max(\lambda_n,\lambda_m),x]} \frac{1}{2\pi i}\int_{(1/x,T)} \exp((z-\lambda_m)s)\frac{ds}{s}\frac{(-1)}{2\pi i} \\
&\quad \times \overline{\int_{(1/x,T)} \exp((z-\lambda_n)t)\frac{dt}{t}}dz \\
&\quad + \int_{A\cap([0,\lambda_n]\cup[0,\lambda_m])} \frac{1}{2\pi i}\int_{(1/x,T)} \exp((z-\lambda_m)s)\frac{ds}{s} \\
&\quad \times \frac{-1}{2\pi i}\overline{\int_{(1/x,T)} \exp((z-\lambda_n)t)\frac{ds}{s}}dz \qquad (3.11)\\
&= \int_{A\cap(\max(\lambda_n,\lambda_m),x]} (1+O(T^{-1/2}))(-1+O(T^{-1/2}))dz \\
&\quad + \int_{A\cap([0,\lambda_n]\cup[0,\lambda_m])} O(T^{-1/2})O(T^{-1/2})dz \\
&= -\int_{A\cap(\max(\lambda_n,\lambda_m),x]} dz + O(T^{-1/2}) \\
&= -(x-\max(\lambda_n,\lambda_m))^+ + O(T^{-1/2}) ,
\end{aligned}$$

as $T \to \infty$, since x is fixed, where + indicates the positive part.

As $T \to \infty$ we have

$$\int_{(a,T)} (e^{ws}/s)ds \ll e^{wa}\int_{-T}^{+T}\frac{du}{a+|u|} \ll e^{wa}\log(1+T/a) .$$

This gives, with $w \leqq x$ and $a = 1/x$, as $T \to \infty$,

$$\int_{(a,T)} (e^{ws}/s)ds \ll \log T . \tag{3.12}$$

Thus, by (3.12), as $T \to \infty$,

$$\begin{aligned}(3.13)\quad &\int_B dz\frac{1}{2\pi i}\int_{(1/x,T)}\exp((z-\lambda_m)s)\frac{ds}{s}\frac{(-1)}{2\pi i}\int_{(1/x,T)}\exp((z-\lambda_n)t)\frac{dt}{t}\\ &\quad\ll\int_B \log^2 T dz\\ &\quad\ll T^{-1/2}\log^2 T\,,\end{aligned}$$

since the length of B is at most $4T^{-1/2}$.

Combining (3.5), (3.9), (3.11), and (3.13) we see that

$$(3.14)\qquad J_{m,n}(T) = -(x-\max(\lambda_n,\lambda_m))^+ + O(T^{-1/2}\log^2 T)\,,$$

as $T\to\infty$. Then, by (3.4) and (3.14), we have

$$(3.15)\qquad J_{m,n} = -(x-\max(\lambda_n,\lambda_m))^+\,.$$

Since $\hat{Q}(t)$ is the Laplace transform of $Q(x)$ we have [8, p. 227]

$$(3.16)\qquad \frac{1}{2\pi i}\int_{(a)} e^{zt}\hat{Q}(t)dt = Q(z)$$

if $a>0$. Thus

$$(3.17)\quad \frac{1}{2\pi i}\int_{(a,T)} e^{zt}\hat{Q}(t)dt = Q(z)-\frac{1}{2\pi i}\int_{a+iT}^{a+i\infty} e^{zt}\hat{Q}(t)dt-\frac{1}{2\pi i}\int_{a-i\infty}^{a-iT} e^{zt}\hat{Q}(t)dt\,.$$

Now, as $T\to\infty$, we have, by (2.6),

$$\begin{aligned}\int_{a\pm iT}^{a\pm i\infty} e^{zt}\hat{Q}(t)dt &= \int_{a\pm iT}^{a\pm i\infty} e^{zt}t\hat{Q}(t)\frac{dt}{t}\\ &\ll e^{za}\int_T^\infty u^{-\beta}\log^{\rho-1}u\frac{du}{u}\\ &\ll e^{za}T^{-\beta}\log^{\rho-1}T\,,\end{aligned}$$

since $\beta>0$. This estimate, combined with (3.17), gives, for $a=1/x$ and $z\leqq x$,

$$(3.18)\qquad Q(z) = \frac{1}{2\pi i}\int_{(a,T)} e^{zt}\hat{Q}(t)dt + O(T^{-\beta}\log^{\rho-1}T)\,,$$

as $T\to\infty$.

Let

$$k_m(x) = \begin{cases}0 & \text{if}\quad x\leqq\lambda_m\\ 1 & \text{if}\quad x>\lambda_m\,.\end{cases}$$

As above we have

$$(3.19)\qquad J_m = \lim_{T\to\infty} J_m(T)\,,$$

where

$$(3.20)\quad \begin{aligned} J_m(T) &= \frac{-1}{4\pi^2}\int_0^x \int_{(1/x,T)} \exp((z-\lambda_m)s)\frac{ds}{s}\int_{(1/x,T)} \overline{e^{z\bar{t}}t\hat{Q}(t)\frac{dt}{t}}dz \\ &= \int_0^x dz\frac{1}{2\pi i}\int_{(1/x,T)} \exp((z-\lambda_m)s)\frac{ds}{s}\,\overline{\frac{-1}{2\pi i}\int_{(1/x,T)} e^{zt}\hat{Q}(t)\frac{dt}{t}}\,. \end{aligned}$$

In (3.20) we estimate the innermost integral by (3.18) and the middle integral as in (3.11) and (3.13), by the use of (3.6)-(3.10). This gives, as $T \to \infty$,

$$(3.21)\quad J_m(T) = -k_m(x)\int_{\lambda_m}^x \hat{Q}(z)dz + o(1)\,.$$

Thus, by (3.19) and (3.21), we have

$$(3.22)\quad J_m = -k_m(x)\int_{\lambda_m}^x \hat{Q}(z)dz\,.$$

Finally,

$$(3.23)\quad J = \lim_{T\to\infty} J(T)\,,$$

where

$$(3.24)\quad \begin{aligned} J(T) &= \frac{-1}{4\pi^2}\int_{(1/x,T)}\int_{(1/x,T)} \hat{Q}(s)\overline{\hat{Q}(t)}\int_0^x e^{z(s+\bar{t})}dzdsdt \\ &= \int_0^x dz\frac{1}{2\pi i}\int_{(1/x,T)} e^{zs}\hat{Q}(s)ds\,\overline{\frac{-1}{2\pi i}\int_{(1/x,T)} e^{zt}\hat{Q}(t)dt}\,. \end{aligned}$$

In (3.24) we estimate the inner two integrals by (3.18). This gives, as $T \to \infty$,

$$(3.25)\quad J(T) = -\int_0^x |Q(z)|^2 dz + o(1)\,.$$

Thus, by (3.23) and (3.25), we have

$$(3.26)\quad J = -\int_0^x |Q(z)|^2 dz\,.$$

The result, (3.1), follows from (3.2), (3.3), (3.15), (3.21), and (3.26), if we note that the integral I in (3.3) in minus the integral on the right hand side of (3.1).

LEMMA 3.2. *If $s = 1/x + ui$ and $0 \leqq u \leqq x^{-1/2}$, then as $x \to \infty$*

$$(3.27)\quad \{F(s) - s\hat{Q}(s)\}/s \ll x^{(m+1)/2}\,,$$

where m is given by (2.7).

Proof. We have, by (2.3), (2.4), and (2.7),

$$
\begin{aligned}
|F(s) - s\hat{Q}(s)| &= \left|\sum b(n)\mu_n^{1-r}\int_{-\infty}^{+\infty} e^{-sx}I(\mu_n x)dx\right| \\
&\leqq c_3|s|^{-m}\sum |e(n)\exp(-k(\mu_n/s)^\alpha)| \\
&\leqq c_4|s|^{-m}\exp(-c_5\operatorname{Re}(1/s)^\alpha) ,
\end{aligned} \tag{3.28}
$$

since the series on the right hand side of (2.7) converges absolutely for $\operatorname{Re}(s) > 0$. Since $0 \leqq u \leqq x^{-1/2}$ we have, for x sufficiently large,

$$
\begin{aligned}
\operatorname{Re}(1/s)^\alpha &= \operatorname{Re}(x/(1 + xui))^\alpha \\
&= (x/(1 + x^2u^2))^\alpha \operatorname{Re}(1 - xui)^\alpha \\
&\geqq c_6(x/(1 + x^2u^2))^\alpha .
\end{aligned} \tag{3.29}
$$

Thus, by (3.28) and (3.29), we have

$$
\begin{aligned}
|(F(s) - s\hat{Q}(s))/s| &\leqq c_4|s|^{-m-1}\exp\{-c_7(x/(1 + x^2u^2))^\alpha\} \\
&\leqq c_4\frac{x^{m+1}}{(1 + x^2u^2)^{(m+1)/2}}\exp\{-c_7(x/(1 + x^2u^2))^\alpha\} \\
&= c_4x^{(m+1)/2}(x/(1 + x^2u^2))^{(m+1)/2}\exp\{-c_7(x/(1 + x^2u^2))^\alpha\} \\
&\ll x^{(m+1)/2} ,
\end{aligned}
$$

as $x \to \infty$, since $x^a \exp(-bx^\alpha)$ is a decreasing function of x for $b > 0$ and x sufficiently large. This completes the proof of the lemma.

REMARK. Here there is no change in the result in the case that $f(s)$ is an entire function since $s\hat{Q}(s) \equiv 0$, by (2.4), in that case.

LEMMA 3.3. *If $s = 1/x + 2\pi ui$, then as $x \to \infty$*

$$
\int_{B(h,k)} |\hat{Q}(s)|du \ll (1/hk)x^{\beta/2}\log^{\rho-1}x . \tag{3.30}
$$

Proof. By (2.6) and (2.11), we have

$$
\begin{aligned}
\int_{B(h,k)} |\hat{Q}(s)|du &\ll \int_{B(h,k)} |s|^{-\beta-1}\log^{\rho-1}|s|du \\
&\ll (1/k\sqrt{x})(k/h)^{\beta+1}\log^{\rho-1}x \\
&\ll (1/hk)x^{\beta/2}\log^{\rho-1}x ,
\end{aligned}
$$

as $x \to \infty$, since $k \leqq \sqrt{x}$ by Definition 2.1. This completes the proof of the lemma.

REMARK. If $f(s)$ is an entire function, then $\hat{Q}(s) = f(0)/s$. In this case the integral in (3.30) is $\ll 1/hk$, as $x \to \infty$.

LEMMA 3.4. *If $s = 1/x + 2\pi ui$, then as $x \to \infty$*

$$\int_{B(h,k)} |F(s)/s|\, du \ll (1/h)x^{\delta-1/2} \log^{(\rho-1)\eta} x \; . \tag{3.31}$$

Proof. By (2.3), (2.4), (2.6), and (2.7), we have, for $\operatorname{Re}(t) > 0$,

$$\begin{aligned} F(t + 2hi/k) &= \sum a(n) \exp(-\lambda_n(t + 2hi/k)) \\ &= (t + 2hi/k)\hat{Q}(t + 2hi/k) \\ &\quad + \sum b(n)\mu_n^{1-r} \int_0^\infty e^{-(t+2hi/k)w} I(\mu_n w) dw \\ &\leqq c_8(|t + 2hi/k|^{-\beta} \log^{\rho-1} |t2hi/k| + |t + 2hi/k|^{-m}) \\ &\leqq c_9 |t + 2hi/k|^{-\delta} \log^{(\rho-1)\eta} |t + 2hi/k| \; . \end{aligned} \tag{3.32}$$

Now let $t = 1/x + 2\pi(u - h/k)i$ in (3.32). Then $F(t + 2hi/k) = F(1/x + 2\pi ui) = F(s)$. By (2.11), we see that if $u \in B(h, k)$, then $u = O(h/k)$ as $x \to \infty$. Then, by (3.32), we have, as $x \to \infty$,

$$\begin{aligned} \int_{B(h,k)} \left| \frac{F(1/x + 2\pi ui)}{1/x + 2\pi ui} \right| du &\leqq c_{10} \int_{B(h,k)} |F(1/x + 2\pi ui)| \frac{du}{u} \\ &\ll \int_{B(h,k)} (k/h)\{1/x^2 + (\pi u - h/k)^2\}^{-\delta/2} \log^{(\rho-1)\eta} x du \\ &\ll (k/h) \log^{(\rho-1)\eta} x \int_{-1/k\sqrt{x}}^{1/k\sqrt{x}} (1/x^2 + \pi^2 u^2)^{-\delta/2} du \\ &\ll (1/h)x^{\delta-1/2} \log^{(\rho-1)\eta} x \; . \end{aligned}$$

This completes the proof of the lemma.

REMARKS. (1) If $f(s)$ is entire, then we take $\eta = 0$ and $\delta = m$.

(2) In [21] Walfisz is able to get an asymptotic result for the integral (1.5), in his special case, in place of our Theorem 1. There he considers the error term $P_m(x)$, which is associated with the problem of lattice points in m dimentional ellipsoids. He is working with quadratic forms, which have special properties that allow him to get his better result. The most important of these properties is the homogeneity property. This allows him to get a better estimate for Lemma 3.4 by getting positive powers of k in the denominator on the right hand side of (3.31), which, when he later sums on k, reduces the power of x he finally obtains. We conjecture that (3.31) can be improved to

$$\int_{B(h,k)} |F(s)/s|\, du \ll (1/h)(x/k)^{\delta-1} \log^{(\rho-1)\eta} x \; ,$$

but we are not able to prove this. The previous Lemmas 3.1, 3.2, and 3.3, are exact generalizations of his resaults are so it is Lemma 3.4 that should be improved the obtain better results.

We use the result of Lemma 3.1 to rewrite the mean square integral (1.5) as a sum of four semi-infinite double integrals. By making a change of variables we write these latter double integrals as double integrals over the semi-infinite segments $(1/x, 1/x + i\infty)$. This allows us to use the covering property of the intervals $B(h, k)$ to rewrite these integrals as sums of integrals over the integrals $B(h, k)$. We can then use the results of Lemma 3.2, 3.3, and 3.4 to estimate these finite integrals and so derive Theorem 1.

By Lemma 3.1, we have

$$\begin{aligned}(3.33)\qquad \int_0^x |E(y)|^2 dy &= \frac{1}{4\pi^2}\left\{\int_{1/x}^{1/x+i\infty}\int_{1/x}^{1/x+i\infty} + \int_{1/x}^{1/x+i\infty}\int_{1/x-i\infty}^{1/x}\right.\\ &\quad \left. + \int_{1/x-i\infty}^{1/x}\int_{1/x}^{1/x+i\infty} \int_{1/x-i\infty}^{1/x}\int_{1/x-i\infty}^{1/x}\right\} G(s, t)dsdt\\ &= P_1 + P_2 + P_3 + P_4 ,\end{aligned}$$

say, where $G(s, t)$ is the integrand of the integral on the right hand side of (3.1). In P_2 replace t by $\bar{t}$, in P_3 replace s by $\bar{s}$, and in P_4 replace s and t by $\bar{s}$ and $\bar{t}$. This gives

$$P_1 = \frac{1}{4\pi^2}\int_{1/x}^{1/x+i\infty}\int_{1/x}^{1/x+i\infty}\{F(s) - s\hat{Q}(s)\}\overline{\{F(t) - t\hat{Q}(t)\}}\frac{\exp(x(s + \bar{t})) - 1}{s\bar{t}(s + \bar{t})}dsdt ,$$

$$P_2 = \frac{1}{4\pi^2}\int_{1/x}^{1/x+i\infty}\int_{1/x}^{1/x+i\infty}\{F(s) - s\hat{Q}(s)\}\overline{\{F(\bar{t}) - \bar{t}\hat{Q}(\bar{t})\}}\frac{\exp(x(s + t)) - 1}{st(s + t)}dsdt ,$$

$$P_3 = \frac{1}{4\pi^2}\int_{1/x}^{1/x+i\infty}\int_{1/x}^{1/x+i\infty}\{F(\bar{s}) - \bar{s}\hat{Q}(\bar{s})\}\overline{\{F(t) - t\hat{Q}(t)\}}\frac{\exp(x(\bar{s} + \bar{t})) - 1}{\bar{s}\bar{t}(\bar{s} + \bar{t})}dsdt$$

and

$$P_4 = \frac{1}{4\pi^2}\int_{1/x}^{1/x+i\infty}\int_{1/x}^{1/x+i\infty}\{F(\bar{s}) - \bar{s}\hat{Q}(\bar{s})\}\overline{\{F(\bar{t}) - \bar{t}\hat{Q}(\bar{t})\}}\frac{\exp(x(\bar{s} + t)) - 1}{\bar{s}t(\bar{s} + t)}dsdt .$$

Let $B(h, k)$ be as in Definition 2.1. In the remainder of §3 we will denote by

$$\sum_{h,k} \quad \text{the sum} \quad \sum_{h=1}^{\infty}\sum_{0<k\leq\sqrt{x}} ,$$

when working with sums of integrals over the intervals $B(h, k)$.

LEMMA 3.5. *For $s = 1/x + 2\pi ui$ and $t = 1/x - 2\pi vi$, where u and v are real, we have, as $x \to \infty$,*

$$\begin{aligned}(3.35)\qquad P_1 = &-\sum_{h,k}\int_{B(h,k)}\int_{B(h,k)}\{F(s) - s\hat{Q}(s)\}\overline{\{F(t) - t\hat{Q}(t)\}}\\ &\times \frac{\exp(x(s + \bar{t})) - 1}{s\bar{t}(s + \bar{t})}dsdt + O(M(x))\end{aligned}$$

and

$$(3.36)\qquad \begin{aligned} P_4 = -\sum_{h,k}\int_{B(h,k)}\int_{B(h,k)} &\{F(\bar{s})-\bar{s}\hat{Q}(\bar{s})\}\overline{\{F(\bar{t})-\bar{t}\hat{Q}(\bar{t})\}} \\ &\times \frac{\exp(x(\bar{s}+t))-1}{\bar{s}t(\bar{s}+t)}dsdt + O(M(x)) , \end{aligned}$$

where $M(x)$ *is defined by* (2.9).

Proof. From (3.34) we see that the difference between P_1 and P_4 is the replacement of s and t by $\bar{s}$ and $\bar{t}$. For this reason we give the details for (3.35) only, since the estimates for P_4 go exactly in the same manner.

By (3.34) and the definitions of s and t, we have

$$P_1 = -\int_0^\infty\int_0^\infty G(s, t)dudv ,$$

where $G(s, t)$, as above, is the integrand of the integral on the right hand side of (3.1). Let $B(h, k)$ and $B(p, q)$ be Farey intervals as defined in Definition 2.1. By Theorem 36 of [10], we have

$$\bigcup_{h=1}^{\infty}\bigcup_{1\leqq k\leqq\sqrt{x}} B(h, k) = \bigcup_{p=1}^{\infty}\bigcup_{1\leqq q\leqq\sqrt{x}} B(p, q) = [([\sqrt{x}]+1)^{-1}, \infty) .$$

Let $B_0 = [0, ([\sqrt{x}]+1)^{-1}]$. Then we have

$$B_0 \cup \bigcup_{h=1}^{\infty}\bigcup_{1\leqq k\leqq\sqrt{x}} B(h, k) = B_0 \cup \bigcup_{p=1}^{\infty}\bigcup_{1\leqq q\leqq\sqrt{x}} B(p, q) = [0, \infty) .$$

Thus

$$\begin{aligned} P_1 &= -\Big\{\int_{B_0}\int_{B_0} + \sum_{h,k}\int_{B(h,k)}\int_{B_0} + \sum_{p,q}\int_{B_0}\int_{B(p,q)} + \sum_{h,k}\sum_{\substack{p,q \\ hq\neq pk}}\int_{B(h,k)}\int_{B(p,q)} \\ &\qquad + \sum_{h,k}\int_{B(h,k)}\int_{B(h,k)}\Big\}G(s, t)dudv \\ &= -\Big\{\int_{B_0}\int_{B_0} + 2\sum_{h,k}\int_{B_0}\int_{B(h,k)} + \sum_{h,k}\sum_{\substack{p,q \\ hq\neq pk}}\int_{B(h,k)}\int_{B(p,q)} \\ &\qquad + \sum_{h,k}\int_{B(h,k)}\int_{B(h,k)}\Big\}G(s, t)dudv . \end{aligned}$$

Now if $a, b \geqq 0$, then $\sqrt{2(a+b)} \geqq \sqrt{a} + \sqrt{b}$. Thus

$$(3.38)\qquad \begin{aligned} |s+\bar{t}| &= |1/x + 2\pi ui + 1/x - 2\pi vi| \\ &= 2|1/x + \pi(u-v)i| \\ &= 2(1/x^2 + \pi^2(u-v)^2)^{1/2} \\ &\geqq \sqrt{2}(1/x + \pi|u-v|) . \end{aligned}$$

Thus, by Lemma 3.2, we have, as $x \to \infty$

$$\left|\int_{B_0}\int_{B_0} G(s, t)dudv\right| \leqq c_{10}\int_{B_0}\int_{B_0} |F(s) - s\hat{Q}(s)||F(t) - t\hat{Q}(t)|\frac{dudv}{|s\bar{t}||s + \bar{t}|}$$
$$\ll x^{m+1}\int_{B_0}\int_{B_0}|s + \bar{t}|^{-1}dudv$$
$$\ll x^{m+1}\int_0^{1/\sqrt{x}}\int_0^{1/\sqrt{x}}(1/x + \pi|u - v|)^{-1}dudv$$
$$\ll x^{m+1}\int_0^{1/\sqrt{x}} du\int_0^u (1/x + \pi(u - v))^{-1}dudv .$$

In the last integral we let $w = \pi(u - v)x$. Then, as $\to \infty$,

$$\left|\int_{B_0}\int_{B_0} G(s, t)dudv\right| \ll x^{m+1}\int_0^{1/\sqrt{x}} du\int_0^{ux}(1 + w)^{-1}dw \ll x^{m+1/2}\log x . \tag{3.39}$$

Next

$$\sum_{h,k}\int_{B(h,k)}\int_{B_0}|G(s, t)|\,dudv = \Bigg\{\sum_{\substack{h,k\\ R(h,k)\leqq 4/\sqrt{x}}}\int_{B(h,k)}\int_{B_0} + \sum_{\substack{h,k\\ R(h,k)>4/\sqrt{x}}}\int_{B(h,k)}\int_{B_0}\Bigg\}|G(s, t)|\,dudv , \tag{3.40}$$

where $R(h, k)$ is defined as in Definition 2.1.

By Lemma 3.2, we have, as $x \to \infty$,

$$\sum_{\substack{h,k\\ R(h,k)\leqq 4/\sqrt{x}}}\int_{B(h,k)}\int_{B_0}|G(s, t)|\,dudv \ll \int_0^{4/\sqrt{x}}\int_0^{1/\sqrt{x}}|F(s) - s\hat{Q}(s)||F(t) - t\hat{Q}(t)|\frac{dudv}{|st|\,|s + \bar{t}|} \ll x^{m+1}\int_0^{4/\sqrt{x}}\int_0^{1/\sqrt{x}}|s + \bar{t}|^{-1}dudv \ll x^{m+1/2}\log x , \tag{3.41}$$

where the last estimate is made as above for (3.39).

If $R(h, k) > 4/\sqrt{x}$, then $h/k \geqq R(h, k) - 1/\sqrt{x} \geqq 3/\sqrt{x}$ by (2.11). Then $2h/3k \geqq 2/\sqrt{x}$ and $h/k - 2/\sqrt{x} \geqq h/3k$. Thus, by (2.11),

$$L(h, k) - 1/\sqrt{x} \geqq h/k - 2/\sqrt{x} \geqq h/3k , \tag{3.42}$$

where $L(h, k)$ is defined in Definition 2.1. Thus, if $R(h, k) > 4/\sqrt{x}$, then, by (3.42), we have, as $x \to \infty$,

$$\int_{B(h,k)}\int_{B_0}|G(s, t)|\,dudv \leqq c_{11}\int_{B(h,k)}\int_{B_0}(|F(s)| + |s\hat{Q}(s)|)|F(t) - t\hat{Q}(t)|\frac{dudv}{|st|\,|u - v|} \tag{3.43}$$

$$\ll \int_{B(h,k)}\int_{B_0} (|F(s)| + |s\hat{Q}(s)|)|F(t) - t\hat{Q}(t)|\,|L(h, k) - 1/\sqrt{x}\,|^{-1}|st|^{-1}dudv$$

$$\ll \int_{B(h,k)}\int_{B_0} (|F(s)| + |s\hat{Q}(s)|)|F(t) - t\hat{Q}(t)|(k/h)|st|^{-1}dudv .$$

We use Lemmas 3.2, 3.3 and 3.4, and the definition of B_0 to estimate the integrand in (3.43). Thus, if $R(h, k) > 4/\sqrt{x}$, then, as $x \to \infty$,

$$\int_{B(h,k)}\int_{B_0} |G(s, t)|\,dudv$$
$$\ll ((1/h)x^{\delta-1/2}\log^{(\rho-1)\eta} x + (1/hk)x^{\beta/2}\log^{\rho-1} x)x^{(m+1)/2}(k/h)\int_0^{1/\sqrt{x}} dv$$
$$\ll (k/h^2)x^{m/2}(x^{\delta-1/2}\log^{(\rho-1)\eta} x + (1/k)x^{\beta/2}\log^{\rho-1} x) .$$

This estimate gives

$$\sum_{\substack{h,k \\ R(h,k)>4/\sqrt{x}}} \int_{B(h,k)}\int_{B_0} |G(s, t)|\,dudv$$
$$(3.44)\qquad \ll x^{m/2}\sum_{h=1}^{\infty} h^{-2} \sum_{k\leqq\sqrt{x}} k(x^{\delta-1/2}\log^{(\rho-1)\eta} x + (1/k)x^{\beta/2}\log^{\rho-1} x)$$
$$\ll x^{m/2}(x^{\delta+1/2}\log^{(\rho-1)\eta} x + (1/k)x^{(\beta+1)/2}\log^{\rho-1} x) ,$$

as $x \to \infty$.

Combining the results of (3.40), (3.41), and (3.44) we have, since $m \geqq 0$ by (2.7) and $\beta > 0$ by hypothesis,

$$\sum_{h,k}\int_{B(h,k)}\int_{B_0} G(s, t)dudv \ll x^{m+1/2}\log x$$
$$(3.45)\qquad + x^{m/2}(x^{\delta+1/2}\log^{(\rho-1)\eta} x + x^{(\beta+1)/2}\log^{\rho-1} x)$$
$$= x^{m+1/2}\log x + x^{(m+1)/2\delta}\log^{(\rho-1)\eta} x$$
$$+ x^{(m+\beta+1)/2}\log^{\rho-1} x ,$$

as $x \to \infty$.

We have

$$\left|\int_{B(h,k)}\int_{B(p,q)} G(s, t)dudv\right|$$
$$\leqq c_{12}\int_{B(h,k)}\int_{B(p,q)} (|F(s)| + |s\hat{Q}(s)|)(|F(t)| + |t\hat{Q}(t)|)\frac{dudv}{|st|\,|s + \bar{t}|}$$
$$(3.46)\qquad = c_{12}\left\{\int_{B(h,k)}\int_{B(p,q)}\left|\frac{F(s)F(t)}{st}\right|\frac{dudv}{|s+\bar{t}|} + \int_{B(h,k)}\int_{B(p,q)}\left|\frac{F(s)}{s}\hat{Q}(t)\right|\frac{dudv}{|s+\bar{t}|}\right.$$
$$\left. + \int_{B(h,k)}\int_{B(p,q)}\left|\frac{F(t)}{t}\hat{Q}(s)\frac{dudv}{|s+\bar{t}|} + \int_{B(h,k)}\int_{B(p,q)} |\hat{Q}(s)\hat{Q}(t)|\frac{dudv}{|s+\bar{t}|}\right\} .$$

By (2.6), the estimate on $|s + \bar{t}|$, (3.38), and the definitions of $B(h, k)$ and $B(p, q)$, we have, as $x \to \infty$,

$$\sum_{\substack{h,k\\hq\neq pk}}\sum_{p,q}\int_{B(h,k)}\int_{B(p,q)}|\hat{Q}(s)\hat{Q}(t)|\frac{dudv}{|s+\bar{t}|}$$

$$\ll \int_{([\sqrt{x}]+1)^{-1}}^{\infty}\int_{([\sqrt{x}]+1)^{-1}}^{\infty}\frac{u^{-\beta-1}v^{-\beta-1}\log^{\rho-1}(xu)\log^{\rho-1}(xv)dudv}{1/x+\pi|u-v|}$$

$$\ll x\int_{1/2\sqrt{x}}^{\infty}\frac{\log^{\rho-1}(xu)du}{u^{\beta+1}}\int_{1/2\sqrt{x}}^{u}\frac{\log^{\rho-1}(xv)dv}{v^{\beta+1}(1+\pi(u-v)x)} \tag{3.47}$$

$$\ll x\int_{1/2\sqrt{x}}^{\infty}\frac{\log^{\rho-1}(xu)du}{u^{\varepsilon+1}}\left\{\int_{1/2\sqrt{x}}^{u/2}+\int_{u/2}^{u-1/x}+\int_{u-1/x}^{u}\right\}\frac{\log^{\rho-1}(xv)dv}{v^{\beta+1}(1+\pi(u-v)x)}$$

$$\ll x\int_{1/2\sqrt{x}}^{\infty}\frac{\log^{2\rho-2}(xu)}{u^{\beta+2}}x^{\beta/2-1}du+\int_{1/2\sqrt{x}}^{\infty}\frac{\log^{2\rho-1}(xu)}{u^{2\beta+2}}du$$

$$\ll z^{\beta+1/2}\log^{2\rho-1}x\ .$$

Let $M(h, k, p, q)=\min\{|u-v|: u\in B(h, k),\ v\in B(p, q)\}$. Then, by (2.11), we have

$$M(h, k, p, q)=\begin{cases}L(p, q)-R(h, k) & \text{if}\quad h/k<p/q\\ L(h, k)-R(p, q) & \text{if}\quad h/k>p/q\ .\end{cases} \tag{3.48}$$

Let

$$D(h, k, p, q)=\begin{cases}x & \text{if}\quad M(h, k, p, q)=0\\ M(h, k, p, q)^{-1} & \text{if}\quad M(h, k, p, q)\neq 0\ .\end{cases} \tag{3.49}$$

Then, for $s=1/x+2\pi ui$ and $t=1/x+2\pi vi$, $u\in B(h, k)$ and $v\in B(p, q)$, we have, by (3.38),

$$|s+\bar{t}|^{-1}\leqq 2^{-1/2}\{1/x+\pi|u-v|\}^{-1}\leqq c_{13}D(h, k, p, q)\ . \tag{3.50}$$

By Lemmas 3.3 and 3.4 and (3.50), we have, as $x\to\infty$,

$$\int_{B(h,k)}\int_{B(p,q)}\left|\frac{F(s)F(t)}{st}\right|\frac{dudv}{|s+\bar{t}|}\ll\frac{D(h, k, p, q)}{hp}x^{2\delta-1}\log^{2(\rho-1)\eta}x\ ,$$

$$\int_{B(h,k)}\int_{B(p,q)}\left|\frac{F(s)}{s}\hat{Q}(t)\right|\frac{dudv}{|s+\bar{t}|}\ll\frac{D(h, k, p, q)}{hpq}x^{\delta+(\beta-1)/2}\log^{(\rho-1)(1+\eta)}x$$

and

$$\int_{B(h,k)}\int_{B(p,q)}\left|\hat{Q}(s)\frac{F(t)}{t}\right|\frac{dudv}{s+\bar{t}}\ll\frac{D(h, k, p, q)}{hkp}x^{\delta+(\beta-1)/2}\log^{(\rho-1)(1+\eta)}x\ .$$

Combining these estimates with (3.46) and (3.47) gives, as $x\to\infty$,

$$\begin{aligned}\sum_{\substack{h,k\\hq\neq pk}}\sum_{p,q}\int_{B(h,k)}\int_{B(p,q)}|G(s, t)|dudv &\ll \sum_{\substack{h,k\\hq\neq pk}}\sum_{\substack{p,q\\k\leqq q}}\frac{D(h, k, p, q)}{hp}\{x^{2\delta-1}\log^{2(\rho-1)\eta}x\\ &\quad+(1/q+1/k)x^{\delta+(\beta-1)/2}\log^{(\rho-1)(1+\eta)}x\}\\ &\quad+x^{\beta+1/2}\log^{2\rho-1}x\end{aligned} \tag{3.51}$$

$$= \left\{\sum_1 + \sum_2 + \sum_3\right\} \frac{D(h,k,p,q)}{hp} \{x^{2\delta-1} \log^{2(\rho-1)} x$$
$$+ (1/q + 1/k) x^{\delta+(\beta-1/2)} \log^{(\rho-1)(1+\eta)} x\}$$
$$+ x^{\beta+1/2} \log^{2\rho-1} x$$
$$= S_1 + S_2 + S_3 + x^{\beta+1/2} \log^{2\rho-1} x \, ,$$

say, where in (3.51)

(3.52) $\sum_1$ is the sum over h, k, p, q such that $M(h, k, p, q) \leqq |hq - pk|/2kq$,

(3.53) $\sum_2$ is the sum over h, k, p, q such that $hq - pk > 0$ and $M(h, k, p, q) > (hq - pk)/2kq$

and

(3.54) $\sum_3$ is the sum over h, k, p, q such that $hq - pk < 0$ and $M(h, k, p, q) > (pk - hq)/2kq$.

Suppose (h, k, p, q) is a quadruple being summed over in (3.52). Then, by (2.11), (3.48), and (3.49), we have $D(h, k, p, q) \leqq c_{14}x$. Thus

$$S_1 \leqq c_{15} x \sum_1 (1/hp)(x^{2\delta-1} \log^{2(\rho-1)\eta} x + (1/q + 1/k) x^{\delta+(\beta-1)/2} \log^{(\rho-1)(1+\eta)} x) \, . \tag{3.55}$$

If $M(h, k, p, q) = |u_0 - v_0|$, then, by (2.11) and (3.52),

$$|hq - pk|/kq = |h/k - p/q| \leqq |h/k - u_0| + |p/q - v_0| + |u_0 - v_0|$$
$$\leqq 1/k\sqrt{x} + 1/q\sqrt{x} + |hq - pk|/kq \, .$$

Thus

$$|hq - pk|/2kq \leqq (1/k + 1/q)/\sqrt{x} \, .$$

Since h, k, p and q, are integers $hq \neq pk$, $k \leqq q$ by (3.51) and $q \leqq \sqrt{x}$ by Definition 2.1, this gives

$$1 \leqq |hq - pk| \leqq 2(k + q)/\sqrt{x} \leqq 4q/\sqrt{x} \leqq 4 \, .$$

Thus

$$|hq - pk| \leqq 4 \quad \text{and} \quad q \geqq \sqrt{x}/4 \, . \tag{3.56}$$

If h and k are given, then q belongs to at most 8 residue classes modulo k, since $hq \equiv a \pmod k$ and $|a| \leqq 4$ by (3.56). Thus, by (2.11), there are at most $c_{16}\sqrt{x}/k$ values of q being summed over in (3.52). If h, k, and q are given, then by (3.56) there are at most c_{17} values of p. Finally, for x sufficiently large, we have, by (3.56),

$$p \geqq (hq - 4)/k \geqq (h\sqrt{x} - 16)/4k \geqq h\sqrt{x}/5k \, . \tag{3.57}$$

By (3.55)-(3.57), we have, as $x \to \infty$,

$$S_1 \ll x \sum_{h=1}^{\infty} (1/h) \sum_{k \leqq \sqrt{x}} \sum_{q \leqq c_{17}\sqrt{x}/k} \sum_{p \leqq c_{18}} (k/h\sqrt{x})(x^{2\delta-1} \log^{2(\rho-1)\eta} x + (1/\sqrt{x} + 1/k)x^{\delta+(\beta-1)/2} \log^{(\rho-1)(1+\eta)} x)$$

$$\ll \sqrt{x} \sum_{h=1}^{\infty} h^{-2} \sum_{k \leqq \sqrt{x}} k(\sqrt{x}/k)(x^{2\delta-1} \log^{2(\rho-1)\eta} x + (1/\sqrt{x} + 1/k)x^{\delta+(\beta-1)/2} \log^{(\rho-1)(1+\eta)} x) \tag{3.58}$$

$$\ll x^{2\delta+1/2} \log^{2(\rho-1)\eta} x + x^{\delta+(\beta+1)/2} \log^{(\rho-1)(1+\eta)} x\{1 + \sum_{k \leqq \sqrt{x}} (1/k)\}$$

$$\ll x^{2\delta+1/2} \log^{2(\rho-1)\eta} x + x^{\delta+(\beta+1)/2} \log^{(\rho-1)(1+\eta)+1} x .$$

Suppose (h, k, p, q) is a quadruple being summed over in (3.53). Then, by (3.49) and (3.53), we have

$$D(h, k, p, q) \leqq 2kq/(hq - pk) . \tag{3.59}$$

Define integers $m = m(h, k, p, q)$ and $n = n(h, k, p, q)$ by

$$hq - pk = m + nk , \quad n \geqq 0 \quad \text{and} \quad 0 < m \leqq k . \tag{3.60}$$

If h, k, and m are given, then q belongs to a definite residue class modulo q, since $hq \equiv m \pmod{k}$. If h, k, m, and q are given then n must satisfy $n \geqq 0$ and $hq - m - nk \geqq k$, since $p = (hq - m - nk)/k \geqq 1$. Finally, if h, k, m, q, and n are given, then there is exactly one value for p.

By (3.51), (3.59), and (3.60), we have

$$S_2 \leqq c_{18} \sum_{h=1}^{\infty} h^{-1} \sum_{k \leqq \sqrt{x}} k^2 \sum_{m=1}^{k} \sum_{\substack{h \leqq q \leqq \sqrt{x} \\ hq = m(k)}} q \sum_{\substack{n \geqq 0 \\ hq - m - nk \ k}} (x^{2\delta-1} \log^{2(\rho-1)\eta} x + (1/q + 1/k)x^{\delta+(\beta-1)/2} \log^{(\rho-1)(1+\eta)} x)(hq - m - nk)^{-1}(m + nk)^{-1} .$$

We then proceed as in [21, pp. 26-27] to estimate the inner sums on q and n. This gives, as $x \to \infty$,

$$S_2 \ll x^{2\delta+1/2} \log^{2(\rho-1)\eta+1} x + x^{\delta+(\beta+1)/2} \log^{(\rho-1)(1+\eta)+2} x . \tag{3.61}$$

We estimate S_3 in a way similar to S_2 and get, as $x \to \infty$,

$$S_3 \ll x^{2\delta+1/2} \log^{2(\rho-1)\eta+1} x + x^{\delta+(\beta+1)/2} \log^{(\rho-1)(1+\eta)+2} x . \tag{3.62}$$

Thus, by (3.51), (3.58), (3.61), and (3.62), we have, as $x \to \infty$,

$$\sum_{h,k} \sum_{\substack{p,q \\ hq \neq pk}} \int_{B(h,k)} \int_{B(p,q)} |G(s, t)| \, du dv$$

$$\ll S_1 + S_2 + S_3 + x^{\beta+1/2} \log^{2\rho-1} x \tag{3.63}$$

$$\ll x^{2\delta+1/2} \log^{2(\rho-1)\eta+1} x + x^{\delta+(\beta+1)/2} \log^{(\rho-1)(1+\eta)+2} x + x^{\beta+1/2} \log^{2\rho-1} x .$$

Finally, by (3.37), (3.39), (3.45), and (3.63), we have, as $x \to \infty$,

$$\begin{aligned} P_1 = & -\sum_{h,k} \int_{B(h,k)} \int_{B(h,k)} G(s, t) du dv + O(x^{m+1/2} \log x) + O(x^{(m+1)/2+\delta} \log^{(\rho-1)\eta} x) \\ & + O(x^{(m+\beta+1)/2} \log^{\rho-1} x) + O(x^{2\delta+1/2} \log^{2(\rho-1)\eta+1} x) \\ & + O(x^{\delta+(\beta+1)/2} \log^{(\rho-1)(1+\eta)+2} x) + O(x^{\beta+1/2} \log^{2\rho-1} x) \\ = & -\sum_{h,k} \int_{B(h,k)} \int_{B(h,k)} G(s, t) du dv + O(M(x)) , \end{aligned}$$

by (2.9). This completes the proof.

REMARK. If $f(s)$ is an entire function, then the error term is $O(x^{2m+1/2} \log x + x^{3m/2+1})$.

LEMMA 3.6. *If* $s = 1/x + 2\pi ui$ *and* $t = 1/x + 2\pi vi$, *where* u *and* v *are real, then, as* $x \to \infty$,

$$P_2 \ll M(x) \tag{3.64}$$

and

$$P_3 \ll M(x) , \tag{3.65}$$

where P_2 *and* P_3 *are defined by* (3.34) *and* $M(x)$ *by* (2.9).

Proof. As in the proof of Lemma 3.5 we prove only (3.64) since the only difference between P_2 and P_3 is the replacement of s and t by $\bar{s}$ and $\bar{t}$.

By (3.34) and the definition of s and t, we have

$$P_2 = -\int_0^\infty \int_0^\infty G(s, \bar{t}) du dv ,$$

where $G(s, t)$ is the integrand of the integral on the right hand side of (3.1). Let B_0, $B(h, k)$ and $B(p, q)$ be as in the proof of Lemma 3.5. Then

$$P_2 = -\left\{ \int_{B_0} \int_{B_0} + 2 \sum_{h,k} \int_{B(h,k)} \int_{B_0} + \sum_{h,k} \sum_{p,q} \int_{B(h,k)} \int_{B(p,q)} \right\} G(s, \bar{t}) du dv . \tag{3.66}$$

By Lemma 3.2 and the arithmetic-geometric mean inequality, we have, as $x \to \infty$,

$$\begin{aligned} & \left| \int_{B_0} \int_{B_0} G(s, \bar{t}) du dv \right| \\ & \ll c_{24} \int_{B_0} \int_{B_0} |F(s) - s\hat{Q}(s)| \, |F(t) - t\hat{Q}(t)| \frac{du dv}{|st| \, |s + t|} \end{aligned}$$

$$\begin{aligned}(3.67)\qquad &\ll x^{m+1}\int_{B_0}\int_{B_0}|s+t|^{-1}dudv\\ &\ll x^{m+1}\int_0^{1/\sqrt{x}}\int_0^{1/\sqrt{x}}(u+v)^{-1}dudv\\ &\ll x^{m+1}\int_0^{1/\sqrt{x}}\int_0^{1/\sqrt{x}}(uv)^{-1/2}dudv\\ &\ll x^{m+1/2}\,.\end{aligned}$$

Again, by the arithmetic-geometric mean inequality, we have

$$\begin{aligned}&\left|\int_{B(h,k)}\int_{B_0}G(s,\bar{t})dudv\right|\\ (3.68)\qquad &\leqq c_{25}\int_{B(h,k)}\int_{B_0}(|F(s)|+|s\hat{Q}(s)|)|F(\bar{t})-\bar{t}\hat{Q}(\bar{t})|\frac{dudv}{|st||s+t|}\\ &\leqq c_{26}\int_{B(h,k)}(|F(s)|+|s\hat{Q}(s)|)\frac{du}{|s|\sqrt{u}}\int_{B_0}|F(\bar{t})-\bar{t}\hat{Q}(\bar{t})|\frac{dv}{|t|\sqrt{v}}\,.\end{aligned}$$

We estimate the integral in (3.68) over B_0 by Lemma 3.2 and the integral in (3.68) over $B(h,k)$ by Lemmas 3.3 and 3.4. Thus, by (2.11), we have

$$\begin{aligned}&\int_{B(h,k)}\int_{B_0}G(s,\bar{t})dudv\\ &\quad\ll x^{(m+1)/2}(k/h)^{1/2}\int_{B(h,k)}(|F(s)|+|s\hat{Q}(s)|)|s|^{-1}du\int_{B_0}v^{-1/2}dv\\ &\quad\ll x^{(m+1)/2}(k/h)^{1/2}((1/h)x^{\delta-1/2}\log^{(\rho-1)\eta}x+(1/hk)x^{\beta/2}\log^{\rho-1}x)\int_0^{1/\sqrt{x}}v^{-1/2}dv\\ &\quad\ll x^{(m+1)/2}k^{1/2}h^{-3/2}(x^{\delta-1/2}\log^{(\rho-1)\eta}x+(1/k)x^{\beta/2}\log^{\rho-1}x)x^{1/4}\,,\end{aligned}$$

as $x\to\infty$. Thus, as $x\to\infty$,

$$\begin{aligned}&\sum_{h,k}\int_{B(h,k)}\int_{B_0}G(s,\bar{t})dudv\\ (3.69)\quad &\ll x^{(m+1)/2+1/4}\sum_{h=1}^{\infty}h^{-3/2}(x^{\delta-1/2}\log^{(\rho-1)\eta}x\sum_{k\leqq\sqrt{x}}k^{1/2}+x^{\beta/2}\log^{\rho-1}x\sum_{k\leqq\sqrt{x}}k^{-1/2})\\ &\ll x^{(m+1)/2+1/4}(x^{\delta+1/4}\log^{(\rho-1)\eta}x+x^{\beta/2+1/4}\log^{\rho-1}x)\,.\end{aligned}$$

By (2.11), the arithmetic-geometric mean inequality and Lemmas 3.2 and 3.4, we have, as $x\to\infty$,

$$\begin{aligned}&\int_{B(h,k)}\int_{B(p,q)}G(s,\bar{t})dudv\\ &\quad\ll c_{27}\int_{B(h,k)}(|F(s)|+|s\hat{Q}(s)|)|s|^{-1}u^{-1/2}du\int_{B(p,q)}(|F(\bar{t})|+|\bar{t}\hat{Q}(\bar{t})|)|t|^{-1}v^{-1/2}dv\\ &\quad\leqq c_{28}(kq/hp)^{1/2}\int_{B(h,k)}(|F(s)|+|s\hat{Q}(s)|)\frac{du}{|s|}\int_{B(p,q)}(|F(\bar{t})|+|\bar{t}\hat{Q}(\bar{t})|)\frac{dv}{|t|}\\ &\quad\ll (kq/hp)^{1/2}((1/h)x^{\delta-1/2}\log^{(\rho-1)\eta}x+(1/hk)x^{\beta/2}\log^{\rho-1}x)\end{aligned}$$

$$\times ((1/p)x^{\delta-1/2}\log^{(\rho-1)\eta} x + (1/pq)x^{\beta/2}\log^{\rho-1} x)$$
$$= (kq/hp)^{1/2}((1/hp)x^{2\delta-1}\log^{2(\rho-1)\eta} x$$
$$+ (1/hpq + 1/hkp)x^{\delta+(\beta-1)/2}\log^{(\rho-1)(1+\eta)} x$$
$$+ (1/hpkq)x^{\beta}\log^{2(\rho-1)} x) \,.$$

Thus, as $x \to \infty$,

$$\sum_{h,k}\sum_{p,q}\int_{B(h,k)}\int_{B(p,q)} G(s, \bar{t})dudv \ll x^{2\delta+1/2}\log^{2(\rho-1)\eta} x + x^{\delta+(\beta+1)/2}\log^{(\rho-1)(1+\eta)} x + x^{\beta+1/2}\log^{2\rho-2} x \,, \tag{3.70}$$

by the definition of

$$\sum_{h,k} \text{ and } \sum_{p,q} .$$

Thus, by (3.66), (3.67), (3.69), and (3.70), we have, as $x \to \infty$,

$$P_2 \ll x^{m+1/2} + x^{m/2+1+\delta}\log^{(\rho-1)\eta} x + x^{(m+\beta)/2+1}\log^{\rho-1} x$$
$$+ x^{2\delta+1/2}\log^{2(\rho-1)\eta} x + x^{\delta+(\beta+1)/2}\log^{(\rho-1)(1+\eta)} x$$
$$+ x^{\beta+1/2}\log^{2(\rho-1)} x$$
$$\ll M(x) \,,$$

by (2.9). This completes the proof.

REMARK. If $f(s)$ is an entire function, then the error term is $O(x^{2m+1/2}\log x + x^{3m/2+1})$.

LEMMA 3.7. *If $s = 1/x + 2\pi ui$ and $t = 1/x + 2\pi vi$, where u and v are real, then, as $x \to \infty$,*

$$\int_0^x |E(y)|^2 dy = -\sum_{h,k}\int_{B(h,k)}\int_{B(h,k)} \Big\{F(s)\overline{F(t)}\frac{\exp(s(s+\bar{t}))-1}{s\bar{t}(s+\bar{t})}$$
$$+ F(\bar{s})\overline{F(\bar{t})}\frac{\exp(x(\bar{s}+t))-1}{\bar{s}t(\bar{s}+t)}\Big\}dudv$$
$$+ O(M(x)) \,,$$

where $M(x)$ is defined by (2.9).

Proof. By (3.35), we have

$$\Big|P_1 - \Big\{-\sum_{h,k}\int_{B(h,k)}\int_{B(h,k)} F(s)\overline{F(t)}\frac{\exp(x(s+\bar{t}))-1}{s\bar{t}(s+\bar{t})}dudv\Big\}\Big|$$
$$\le c_{30}\Big\{\sum_{h,k}\int_{B(h,k)}\int_{B(h,k)} |F(s)||\hat{Q}(t)|\frac{dudv}{|s||s+\bar{t}|} \tag{3.71}$$
$$+ \sum_{h,k}\int_{B(h,k)}\int_{B(h,k)} |F(t)||\hat{Q}(s)|\frac{dudv}{|t||s+\bar{t}|}$$

$$+ \sum_{h,k} \int_{B(h,k)} \int_{B(h,k)} |\hat{Q}(s)\hat{Q}(t)| \frac{dudv}{|s + \bar{t}|}\Big\} + O(M(x)) .$$

By (2.6), (2.11), and Lemma 3.3, we have, as $x \to \infty$,

$$\begin{aligned} &\sum_{h,k} \int_{B(h,k)} \int_{B(h,k)} |F(s)\hat{Q}(t)| \frac{dudv}{|s||s + \bar{t}|} \\ &\quad \ll \sum_{h,k} \int_{B(h,k)} |F(s)/s|\, du \int_{B(h,k)} v^{-\beta-1} \log^{\rho-1}(xv) \frac{dv}{1/x} \\ &\quad \ll x \sum_{h=1} \sum_{k=\sqrt{x}}^{\infty} (h^{-1}x^{\delta-1/2} \log^{(\rho-1)\eta} x)(1/k\sqrt{x})(k/h)^{\beta+1} \log^{\rho-1}(hx/k) \qquad (3.72) \\ &\quad \ll x^{\delta} \log^{(\rho-1)\eta} x \sum_{h=1}^{\infty} h^{-\beta-2} \log^{\rho-1}(hx) \sum_{k \leqq \sqrt{x}} (1/k)k^{\beta+1} \log^{\rho-1} k \\ &\quad \ll x^{\delta+(\beta+1)/2} \log^{(\rho-1)(2+\eta)} x , \end{aligned}$$

since $\beta > 0$.

Similarly, as $x \to \infty$,

$$\sum_{h,k} \int_{B(h,k)} \int_{B(h,k)} |F(t)\hat{Q}(t)| \frac{dudv}{|t||s + \bar{t}|} \ll x^{\delta+(\beta+1)/2} \log^{(\rho-1)(2+\eta)} x . \tag{3.73}$$

Finally, by (2.6), we have, as $x \to \infty$,

$$\begin{aligned} &\sum_{h,k} \int_{B(h,k)} \int_{B(h,k)} |\hat{Q}(s)\hat{Q}(t)| \frac{dudv}{|s + \bar{t}|} \\ &\quad \ll \int_{(1+[\sqrt{x}])^{-1}}^{\infty} \int_{(1+[\sqrt{x}])^{-1}}^{\infty} \frac{\log^{\rho-1}(xu) \log^{\rho-1}(xv)}{(uv)^{\beta+1}(1/x + \pi|u - v|)} dudv \qquad (3.74) \\ &\quad \ll x^{\rho+1/2} \log^{2\rho-1} x , \end{aligned}$$

where the last estimate is obtained in the same way as was the estimate (3.47).

Thus, by (3.71)-(3.74), we have, as $x \to \infty$,

$$\begin{aligned} &P_1 - \Big\{-\sum_{h,k} \int_{B(h,k)} \int_{B(h,k)} F(s)\overline{F(t)} \frac{\exp(x(s + \bar{t})) - 1}{s\bar{t}(s + \bar{t})} dudv\Big\} \\ &\quad \ll x^{\delta+(\beta+1)/2} \log^{(\rho-1)(2+\eta)} x + x^{\beta+1/2} \log^{\rho-1} x + M(x) \qquad (3.75) \\ &\quad \ll M(x) , \end{aligned}$$

by (2.9).

A similar argument, using the estimate (3.36), gives, as $x \to \infty$,

$$P_4 - \Big\{-\sum_{h,k} \int_{B(h,k)} \int_{B(h,k)} F(\bar{s})\overline{F(\bar{t})} \frac{\exp(x(\bar{s} + t)) - 1}{\bar{s}t(\bar{s} + t)} dudv\Big\} \ll M(x) . \tag{3.76}$$

By (3.33), (3.75), (3.76), and Lemma 3.6, we have, as $x \to \infty$,

$$\begin{aligned}\int_0^x |E(y)|^2 dy &- \left\{-\sum_{h,k} \int_{B(h,k)} \int_{B(h,k)} \left(F(s)\overline{F(t)}\frac{\exp(x(s+\bar{t}))-1}{s\bar{t}(s+\bar{t})}\right.\right. \\ &\left.\left. + F(\bar{s})\overline{F(\bar{t})}\frac{\exp(x(\bar{s}+t))-1}{\bar{s}t(\bar{s}+t)}\right)dudv\right\} \\ &= P_1 - \left\{-\sum_{h,k} \int_{B(h,k)} \int_{B(h,k)} F(s)\overline{F(t)}\frac{\exp(x(s+\bar{t}))-1}{s\bar{t}(s+\bar{t})}dudv\right\} \\ &+ P_4 - \left\{-\sum_{h,k} \int_{B(h,k)} \int_{B(h,k)} F(\bar{s})\overline{F(\bar{t})}\frac{\exp(x(\bar{s}+t))-1}{\bar{s}t(\bar{s}+t)}dudv\right\} \\ &+ P_2 + P_3 \\ &\ll M(x) .\end{aligned}$$

This completes the proof of the lemma.

REMARK. If $f(s)$ is an entire function, then the error term is $O(x^{2m+1/2} + x^{3m/2+1})$.

3.2. *Proof of Theorem* 1 *and Corollary* 1. We give the details only for the case when $f(s)$ is not entire. The proof when $f(s)$ is entire is similar and is obtained by using the estimates given in the remarks to the lemmas.

Proof of Theorem 1. For $s = 1/x + 2\pi ui$ and $t = 1/x + 2\pi vi$ let

$$I(h, k) = \int_{B(h,k)} \int_{B(h,k)} F(s)\overline{F(t)}\frac{\exp(x(s+\bar{t}))-1}{s\bar{t}(s+\bar{t})}dudv . \tag{3.77}$$

By (2.3), (2.4), and (2.7), we have, as $x \to \infty$,

$$\begin{aligned}|F(s) - s\hat{Q}(s)| &= \left|\sum b(n)\mu_n^{1-r} \int_0^\infty I(\mu_n x)e^{-sx}dx\right| \\ &\leqq c_{31}|s|^{-m} \sum |e(n)| \exp(-k(\mu_n \operatorname{Re}(1/s))^\alpha) \\ &\leqq c_{32}|s|^{-m} \exp(-c_{33} \operatorname{Re}(1/s)^\alpha) \\ &\ll x^{m/2} ,\end{aligned} \tag{3.78}$$

where the last estimate is obtained in the same way as the estimate in Lemma 3.2. Similarly, as $x \to \infty$,

$$|F(t) - t\hat{Q}(t)| \ll x^{m/2} . \tag{3.79}$$

Let

$$F(u, v) = s\hat{Q}(s)t\hat{Q}(t)\frac{\exp(2 + 2\pi x(u-v)i) - 1}{(1/x + 2\pi ui)(1/x - 2\pi vi)(2/x + 2\pi(u-v)i)} \tag{3.80}$$

and

$$I'(h, k) = \int_{B(h,k)}\int_{B(h,k)} F(u, v)dudv . \tag{3.81}$$

If $u, v \in B(h, k)$, then, by (3.80) and (2.11), we have

$$\begin{aligned}\left|\frac{F(u, v)}{s\hat{Q}(s)t\hat{Q}(t)}\right| &= \left|\frac{\exp(2 + 2\pi x(u - v)i) - 1}{(1/x + 2\pi ui)(1/x - 2\pi vi)(2/x + 2\pi(u - v)i)}\right| \\ &\leqq c_{34}(uvx^{-1})^{-1} \\ &\leqq c_{35}((h/k)^2x^{-1})^{-1} .\end{aligned} \tag{3.82}$$

By (3.77)-(3.80), we have

$$I(h, k) = \int_{B(h,k)}\int_{B(h,k)} (s\hat{Q}(s) + O(x^{m/2}))(t\hat{Q}(t) + O(x^{m/2}))\frac{F(u, v)}{s\hat{Q}(s)t\hat{Q}(t)}dudv .$$

Thus, by (3.81), (3.82), and (2.11), we have, as $x \to \infty$,

$$|I(h, k) - I'(h, k)| \ll x^{(m+1)/2}(k/h^2)\int_{B(h,k)} |s\hat{Q}(s)|du + x^m/h^2 . \tag{3.83}$$

By (2.6), we have, as in the proof of Lemma 3.4,

$$\begin{aligned}\int_{B(h,k)} |s\hat{Q}(s)|du &\ll \int_{B(h,k)} |s|^{-\beta}\log^{\rho-1}|s|du \\ &\ll (1/k\sqrt{x})(k/h)^\beta \log^{\rho-1} x \\ &\ll (1/k)x^{(\beta-1)/2}\log^{\rho-1} x .\end{aligned} \tag{3.84}$$

Thus, by (3.83) and (3.84), we have, as $x \to \infty$,

$$I(h, k) - I'(h, k) \ll x^{(m+\beta)/2}h^{-2}\log^{\rho-1} + x^m h^{-2} . \tag{3.85}$$

In a similar way we let

$$J(h, k) = \int_{B(h,k)}\int_{B(h,k)} F(\bar{s})\overline{F(\bar{t})}\frac{\exp(x(\bar{s} + t)) - 1}{\bar{s}t(\bar{s} + t)}dudv \tag{3.86}$$

and

$$\begin{aligned}&J'(h, k) \\ &= \int_{B(h,k)}\int_{B(h,k)} \bar{s}\hat{Q}(\bar{s})\overline{\bar{t}\hat{Q}(\bar{t})}\frac{\exp(2 + 2\pi(v - u)xi) - 1}{(1/x-2\pi ui)(1/x+2\pi vi)(2/x+2\pi(v-u)i)}dudv ,\end{aligned}$$

in analogy to (3.77) and (3.81), and obtain in a similar way

$$J(h, k) - J'(h, k) \ll x^{(m+\beta)/2}h^{-2}\log^{\rho-1} x + x^m h^{-2} . \tag{3.87}$$

By Lemma 3.7, (3.77) and (3.86), have, as $x \to \infty$,

$$
\begin{aligned}
\int_0^x |E(y)|^2 dy &= -\sum_{h,k} I(h, k) - \sum_{h,k} J(h, k) + O(M(x)) \\
&= -\sum_{h,k} (I(h, k) - I'(h, k)) - \sum_{h,k} I'(h, k) \\
&\quad - \sum_{h,k} (J(h, k) - J'(h, k)) - \sum_{h,k} J'(h, k) + O(M(x)) .
\end{aligned} \tag{3.88}
$$

As in the estimate (3.47), we have, as $x \to \infty$,

$$\sum_{h,k} I'(h, k) \ll x^{\beta+1/2} \log^{2\rho-1} x \tag{3.89}$$

and

$$\sum_{h,k} J'(h, k) \ll x^{\beta+1/2} \log^{2\rho-1} x . \tag{3.90}$$

By (3.85), we have, as $x \to \infty$,

$$
\begin{aligned}
\sum_{h,k} (I(h, k) - I'(h, k)) &\ll (x^{(m+\beta)/2} \log^{\rho-1} x + x^m) \sum_{h=1}^{\infty} h^{-2} \sum_{k \leqq \sqrt{x}} 1 \\
&\ll x^{(m+\beta+1)/2} \log^{\rho-1} x + x^{m+1/2} .
\end{aligned} \tag{3.91}
$$

In a similar way, by (3.87), we have, as $x \to \infty$,

$$\sum_{h,k} (J(h, k) - J'(h, k)) \ll x^{(m+\beta+1)/2} \log^{\rho-1} x + x^{m+1/2} . \tag{3.92}$$

Thus, by (3.88)-(3.92), we have

$$
\begin{aligned}
\int_0^x |E(y)|^2 dy &\ll x^{\beta+1/2} \log^{2\rho-1} x + x^{(m+\beta+1)/2} \log^{\rho-1} x + x^{m+1/2} + M(x) \\
&\ll M(x) ,
\end{aligned}
$$

as $x \to \infty$, by (2.9). This completes the proof of the theorem.

Proof of Corollary 1. By [3, p. 152], we can take $m = r$. If $f(s)$ is entire, then the result follows immediately from Theorem 1. If $f(s)$ is not entire, then we have, by the hypotheses on the poles of $f(s)$, that $0 < \beta \leqq r$. Thus, by (2.8), $\delta = \beta \leqq r = m$. The result then follows from Theorem 1 and (2.9). This completes the proof of the corollary.

4. The sums of the squares of the coefficients.

4.1. *Preliminary lemmas.* In this section we assume that $\lambda_n = n$ for all n.

By [14, p. 121] we have $|a(n)| \leqq c_{36} n^{c_{37}}$, for all n. Thus, the sum defined in (2.2) converges absolutely for $\operatorname{Re}(s) > 0$.

LEMMA 4.1. *If $s = 1/x + 2\pi ui$ and $t = 1/x + 2\pi vi$, where u and v are real, then for each positive integer x, we have*

$$\sum_{n=1}^{x-1} |a(n)|^2 = \int_0^1 \int_0^1 F(s)\overline{F(t)} \frac{\exp(x(s+\bar{t}))-1}{\exp(s+\bar{t})-1} dudv . \tag{4.1}$$

Proof. We have, for $n \geqq 0$,

$$\begin{aligned}\int_0^1 F(s)e^{ns}du &= \int_0^1 \sum a(m)e^{-ms+ns}du \\ &= \sum a(m) \int_0^1 e^{(n-m)s}du \\ &= a(n) .\end{aligned}$$

Thus

$$\begin{aligned}|a(n)|^2 &= \int_0^1 F(s)e^{ns}du \int_0^1 \overline{F(t)}e^{n\bar{t}}dv \\ &= \int_0^1 \int_0^1 F(s)\overline{F(t)}e^{n(s\bar{t})}dudv .\end{aligned}$$

If we sum on n, we have

$$\begin{aligned}\sum_{n=1}^{x-1} |a(n)|^2 &= \sum_{n=1}^{x-1} |a(n)|^? \\ &= \int_0^1 \int_0^1 F(s)\overline{F(t)} \sum_{n=0}^{x-1} e^{n(s+\bar{t})}dudv \\ &= \int_0^1 \int_0^1 F(s)\overline{F(t)} \frac{\exp(x(s+\bar{t}))-1}{\exp(s+\bar{t})-1} dudv .\end{aligned}$$

This completes the proof of the lemma.

The integrand of the double integral on the right hand side of (4.1) is periodic in u and v of period 1. Thus we may integrate over any interval E of length 1. Thus, by Lemma 3.1,

$$\sum_{n=1}^{x-1} |a(n)|^2 = \int_E \int_E F(s)\overline{F(t)} \frac{\exp(x(s+\bar{t}))-1}{\exp(s+\bar{t})-1} dudv . \tag{4.2}$$

Let $E = [1/[\sqrt{x}], 1 + 1/[\sqrt{x}]]$. Let $B(h, k)$ be as defined in Definition 2.1. If we note that $B(1, 1) = [1 - 1/[\sqrt{x}], 1 + 1/[\sqrt{x}]]$, then we see that

$$E = \bigcup_{1 \leqq k \leqq \sqrt{x}} \bigcup_{1 \leqq h \leqq k} B(h, k) .$$

In the remainder of §4 we will denote by

$$\sum_{h,k} \text{ the sum } \sum_{1 \leqq k \leqq \sqrt{x}} \sum_{1 \leqq h \leqq k} .$$

Then, from (4.2), we have

$$\sum_{n=1}^{x-1} |a(n)|^2 = \sum_{h,k} \sum_{p,q} \int_{B(h,k)} \int_{B(p,q)} F(s)\overline{F(t)} \frac{\exp(x(s+\bar{t}))-1}{\exp(s+\bar{t})-1} dudv . \tag{4.3}$$

LEMMA 4.2. *For $s = 1/x + 2\pi ui$ and $t = 1/x + 2\pi vi$, where u and v are real, we have, as $x \to \infty$,*

$$\sum_{n \leqq x-1} |a(n)|^2 = \sum_{h,k} \int_{B(h,k)} \int_{B(h,k)} F(s)\overline{F(t)} \frac{\exp(x(s+\bar{t})) - 1}{\exp(s+\bar{t}) - 1} dudv + O(x^{2\delta} \log^{2(\rho-1)\eta+1} x) .$$

Proof. If $|z| < 3\pi/2$, then $|e^z - 1| \geqq c_{38}|z|$. Thus, for $0 < |z| < 3\pi/2$, we have

$$|e^z - 1|^{-1} \leqq c_{39}|z|^{-1} . \tag{4.4}$$

Suppose $u \in B(h, k)$ and $v \in B(p, q)$ with $hq \neq pk$. Then, with s and t as defined above, we have $s + \bar{t} = 2/x + 2\pi(u - v)i$, where $-1 \leqq u - v \leqq 1$, since the integration in (4.2) is over and interval of length 1.

In (4.4) we take $z = s + \bar{t} + 2\pi i$ if $-1 \leqq u - v \leqq -1/2$, $z = s + \bar{t}$ if $-1/2 < u - v < 1/2$ and $z = s + \bar{t} - 2\pi i$ if $1/2 \leqq u - v \leqq 1$. Then we have, by (3.8) and (3.50),

$$\begin{aligned} |e^{s+\bar{t}} - 1|^{-1} &\leqq c_{40}(|s + \bar{t}|^{-1} + |s + \bar{t} + 2\pi i|^{-1} + |s + \bar{t} - 2\pi i|^{-1}) \\ &\leqq c_{41}((1/x + \pi|u - v|)^{-1} + (1/x + \pi|u - v + 1|)^{-1} \\ &\quad + (1/x + \pi|u - v - 1|)^{-1}) \qquad (4.5) \\ &\leqq c_{42}(D(h, k, p, q) + D(h + k, k, p, q) + D(h, k, p + q, q)) . \qquad (4.6) \end{aligned}$$

By Lemma 3.3 and (2.11), we have, as $x \to \infty$,

$$\begin{aligned} \int_{B(h,k)} \int_{B(p,q)} |F(s)F(t)| dudv &\leqq c_{43} \int_{B(h,k)} u|F(s)/s| du \int_{B(p,q)} |F(t)/t| vdv \\ &\ll (1/h)(x^{\delta-1/2} \log^{(\rho-1)\eta} x)(h/k)(1/p)(x^{\delta-1/2} \log^{(\rho-1)\eta} x)(p/q) \\ &= (x^{2\delta-1} \log^{2(\rho-1)\eta} x)/kq . \qquad (4.7) \end{aligned}$$

Thus, by (4.3) and (4.7), we have, as $x \to \infty$,

$$\begin{aligned} &\left| \sum_{n=1}^{x-1} |a(n)|^2 - \sum_{h,k} \int_{B(h,k)} \int_{B(h,k)} F(s)\overline{F(t)} \frac{\exp(x(s+\bar{t})) - 1}{\exp(s+\bar{t}) - 1} dudv \right| \\ &\leqq c_{44} \sum_{h,k} \sum_{\substack{p,q \\ hq \neq pk}} \int_{B(h,k)} \int_{B(p,q)} |F(s)F(t)| |e^{s+\bar{t}} - 1|^{-1} dudv \\ &\ll x^{2\delta-1} \log^{2(\rho-1)\eta} x \sum_{h,k} \sum_{\substack{p,q \\ hq \neq pk}} (1/kq)(D(h, k, p, q) \\ &\quad + D(h + k, k, p, q) + D(h, k, p + q, q)) \\ &\leqq c_{45} x^{2\delta-1} \log^{2(\rho-1)\eta} x \sum_{h,k}{}' \sum_{\substack{p,q \\ hq \neq pk}}{}' D(h, k, p, q)/kq \qquad (4.8) \\ &\leqq c_{46} x^{2\delta-1} \log^{2(\rho-1)\eta} x \left(\sum_1 + \sum_2 + \sum_3\right) D(h, k, p, q)/kq , \qquad (4.9) \end{aligned}$$

where in (4.8) the dash indicates that the sums are over h, k, p, q

such that $k \leqq \sqrt{x}$, $h \leqq 2k$, $(h, k) = 1$ and $q \leqq \sqrt{x}$, $p \leqq 2q$, $(p, g) = 1$, respectively, and in (4.9) the sums are over the regions (3.52), (3.53), and (3.54), respectively

We estimate the sums

$$\sum_1, \quad \sum_2, \quad \text{and} \quad \sum_3$$

as in [21, pp. 43-47] and obtain

$$\sum_1 + \sum_2 + \sum_3 \ll x \log x \,. \tag{4.10}$$

Thus, by (4.9) and (4.10), we have, as $x \to \infty$,

$$\sum_{n \leqq x-1} |a(n)|^2 - \sum_{h,k} \int_{B(h,k)} \int_{B(h,k)} F(s)\overline{F(t)} \frac{\exp(x(s+\bar{t})) - 1}{\exp(s+\bar{t}) - 1} du dv \ll x^{2\delta} \log^{2(\rho-1)\gamma+1} x \,.$$

This completes the proof of the lemma.

REMARK. If $f(s)$ is entire, then the error term is $O(x^{2m} \log x)$.

4.2. *Proof of Theorem* 2 *and Corollary* 2.

Proof of Theorem 2. We give the details only for the case that $f(s)$ is not entire. The details when $f(s)$ is entire are similar except that we use the estimate given in the remark to Lemma 4.2.

Let $s = 1/x + 2\pi ui$. Then we have, by (2.3), (2.4), and (2.7),

$$\begin{aligned} |F(s) - s\hat{Q}(s)| &= \left| \sum b(n) \mu_n^{1-r} \int_0^\infty I(\mu_n x) e^{-sx} dx \right| \\ &\leqq c_{62} |s|^{-m} \sum |e(n)| \exp(-k(\mu_n \operatorname{Re}(1/s)^\alpha)) \\ &\leqq c_{63} |s|^{-m} \exp\{-c_{64}(x/(1 + 4\pi^2 x^2 u^2)^\alpha)\} \\ &\leqq c_{63} x^\alpha |s|^{2\alpha-m} (1/x^\alpha |s|^{2\alpha}) \exp(-c_{64}/x^\alpha |s|^{2\alpha}) \\ &\ll x^\alpha |s|^{2\alpha-m} \,, \end{aligned} \tag{4.11}$$

as $x \to \infty$.

Let s be as above and $t = 1/x + 2\pi vi$, where u and v are real, and

$$I(h, k) = \int_{B(h,k)} \int_{B(h,k)} F(s)\overline{F(s)} \frac{\exp(x(s+\bar{t})) - 1}{\exp(s+\bar{t}) - 1} du dv \,. \tag{4.12}$$

Then, by (4.11) and (4.12), we have, as $x \to \infty$,

$$I(h, k) = \int_{B(h,k)} \int_{B(h,k)} (s\hat{Q}(s) + O(x^\alpha |s|^{2\alpha-m}))(\overline{t\hat{Q}(t)} + O(x^\alpha |s|^{2\alpha-m})) \frac{e^{x(s+\bar{t})} - 1}{e^{s+t} - 1} du dv \,.$$

Let

$$F(u, v) = s\hat{Q}(s)\overline{t\hat{Q}(t)}(\exp(x(s + \bar{t})) - 1)/(\exp(s + \bar{t}) - 1) \tag{4.13}$$

and

$$I'(h, k) = \int_{B(h,k)} \int_{B(h,k)} F(u, v)dudv .$$

Then, by (4.5), we have

$$I(h, k) - I'(h, k) \ll x^{m/2}k^{m-2\alpha} \int_{B(h,k)} \int_{B(h,k)} |s\hat{Q}(s)| \min(x, |u - v|^{-1})dudv + x^m k^{2m-4\alpha-2} ,$$

as $x \to \infty$, by (2.11). We estimate the double integral as in the estimate (3.84). This gives

$$I(h, k) - I'(h, k) \ll k^{m-2\alpha-1}x^{(m+\beta-1)/2} \log^{\rho} x + x^m k^{2m-\alpha-2} , \tag{4.14}$$

as $x \to \infty$.

By Lemma 4.2, (4.12), and (4.14), we have

$$\begin{aligned}\sum_{n \leqq x-1} |a(n)|^2 &= \sum_{h,k} I(h, k) + O(x^{2\delta} \log^{2(\rho-1)\eta+1} x) \\ &= \sum_{h,k} I'(h, k) + \sum_{h,k} (I(h, k) - I'(h, k)) + O(x^{2\delta} \log^{2(\rho-1)\eta+1} x) \\ &= \sum_{h,k} I'(h, k) + O(x^{2\delta} \log^{2(\rho-1)\eta+1} x + h(x)) , \end{aligned} \tag{4.15}$$

as $x \to \infty$, where $h(x)$ is defined by (2.10).

Suppose $\beta > 1$. Let $C(h, k)$ be the union of those intervals whose points are either all $\geqq R(h, k)$ or $\leqq L(h, k)$, so that $B(h, k) \cup C(h, k) = (-\infty, +\infty)$. Let

$$I''(h, k) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(u, v)dudv ,$$

which converges for $\beta > 1$, by (2.6) and (4.13). Then, as $x \to \infty$, we have, by (4.4) and (4.13),

$$\begin{aligned} I'(h, k) - I''(h, k) &\ll \left\{\int_{B(h,k)} \int_{C(h,k)} + \int_{C(h,k)} \int_{B(h,k)}\right\} F(u, v)dudv \\ &\ll \int_{-\infty}^{+\infty} |s\hat{Q}(s)|\, du \int_{C(h,k)} \min(x, |u - v|^{-1}) |t\hat{Q}(t)|\, dv . \end{aligned} \tag{4.16}$$

By (2.6) and (2.11), we have, as $x \to \infty$,

$$\int_{C(h,k)} \min(x, |u - v|^{-1}) |t\hat{Q}(t)|\, dv \ll x \int_{\substack{C(h,k) \\ |u-v| \leqq x^{-1}}} t\hat{Q}|(t)|\, dv + \int_{\substack{C(h,k) \\ x^{-1} \leqq |u-v| \leqq 1}} |t\hat{Q}(t)| |u - v|^{-1} dv$$

$$+\int_{\substack{C(h,k)\\|u-v|\geqq 1}} |t\hat{Q}(t)||u-v|^{-1}dv \tag{4.17}$$

$$\ll x(k\sqrt{x})^{\beta}\log^{\rho-1}x(1/x) + (k\sqrt{x})^{\beta}\log^{\rho-1}x\int_{1/x}^{1} v^{-1}dv$$

$$+\int_{1/2k\sqrt{x}}^{+\infty}(\log^{\rho-1}v)v^{-\beta}dv$$

$$\ll k^{\beta}x^{\beta/2}\log^{\rho-1}x\ .$$

Thus, by (4.16), (4.17), and (3.84), we have, as $x \to \infty$,

$$\begin{aligned} I'(h, k) - I''(h, k) &\ll ((1/k)x^{(\beta-1)/2}\log^{\rho-1}x)(k^{\beta}x^{\beta/2}\log^{\rho-1}x) \\ &\ll k^{\beta-1}x^{\beta-1/2}\log^{2(\rho-1)}x. \end{aligned} \tag{4.18}$$

By (4.15) and (4.18), we have, as $x \to \infty$,

$$\begin{aligned} \sum_{n\leqq x-1}|a(n)|^2 &= \sum_{h,k} I''(h, k) \\ &\quad + O(x^{3\beta/2}\log^{2\rho-2}x + x^{2\delta}\log^{2(\rho-1)\eta+1}x + h(x))\ . \end{aligned} \tag{4.19}$$

By (4.13) and the definition of s and t,

$$\begin{aligned} I''(h, k) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s\hat{Q}(s)\overline{t\hat{Q}(t)}\sum_{n=0}^{x-1}\exp\{n(2/x + 2\pi(u-v)i)\}dudv \\ &= \sum_{n=0}^{x-1}\left|\int_{-\infty}^{+\infty} s\hat{Q}(s)\exp\{n(1/x + 2\pi ui)\}du\right|^2 \\ &= \sum_{n=0}^{x-1}\left|(2\pi i)^{-1}\int_{(1/x)} w\hat{Q}(w)e^{nw}dw\right|^2 , \end{aligned}$$

where we have made the change of variables $w = 1/x + 2\pi ui$ to obtain the last integral on the right hand side. Since $w\hat{Q}(w)$ is the Laplace transform of $Q'(x)$ we have [8, p. 227]

$$I''(h, k) = \sum_{n=0}^{x-1}|Q'(n)|^2\ .$$

By (2.5), we see that $Q'(x) \sim cx^{\beta-1}\log^{\rho-1}{}^{\rho-1}x$, as $x \to \infty$, where c is some complex constant. Thus, as $x \to \infty$,

$$I''(h, , k) \sim \sum_{n=0}^{x-1}|c|^2 n^{2\beta-2}\log^{2\rho-2}n \sim Ax^{2\beta-1}\log^{2\rho-2}x\ , \tag{4.20}$$

where A is some positive constant.

Thus, by (4.19) and (4.20), we have, as $x \to \infty$,

$$\sum_{n\leqq x-1}|a(n)|^2 \ll x^{2\beta}\log^{2\rho-2}x + h(x)\ .$$

Replacing x by $x+1$ and letting it be an arbitrary real number gives the second part of Theorem 2.

If $0 < \beta \leqq 1$, then the integral defining $I''(h, k)$ does not converge and we must estimate $I'(h, k)$ in another manner. By (3.84), we have

$$I'(h, k) \leqq c_{65} \int_{B(h,k)} \int_{B(h,k)} |s\hat{Q}(s)t\hat{Q}(t)| \min(x, |u - v|^{-1}) du dv$$
$$\leqq c_{65} x \int_{B(h,k)} |s\hat{Q}(s)| du \int_{B(h,k)} |t\hat{Q}(t)| dv$$
$$\ll (1/k^2) x^\beta \log^{2\rho - 2} x \,,$$

as $x \to \infty$. Then, as $x \to \infty$,

$$\sum_{h,k} I'(h, k) \ll \sum_{k \leqq \sqrt{x}} \sum_{h \leqq k} (x^\beta \log^{2\rho-2} x)/k^2$$
$$\ll x^\beta \log^{2\rho - 1} x \,. \tag{4.21}$$

Thus, by (4.15) and (4.21) we have, as $x \to \infty$,

$$\sum_{n \leqq x-1} |a(n)|^2 \ll x^\beta \log^{2\rho-1} x + x^{2\delta} \log^{2(\rho-1)\eta+1} x + h(x) \,.$$

Replacing x by $x + 1$ and letting x be an arbitrary real number gives the first part of Theorem 2.

This completes the proof of Theorem 2.

Proof of Corollary 2. Corollary 2 follows from Theorem 2 exactly as Corollary 1 follows from Theorem 1 and so the details will be omitted.

5. Application of a theorem of E. M. Wright. In this section we will prove a theorem that will ensure the validity of (2.7) for a class of Dirichlet series that satisfy the functional equation (1.2).

Let $\Delta(s)$ be as defined in (1.1).

Let

$$A = \sum_{k=1}^{N} \alpha_k \,, \qquad B = \sum_{k=1}^{N} \beta_k \,, \qquad \log D = \sum_{k=1}^{N} \alpha_k \log \alpha_k \,,$$
$$\mu = (1 - N)/2 + B \,, \qquad \theta = 2(\log D - A \log A) \quad \text{and} \tag{5.1}$$
$$h = 2 \exp(-\theta/2A) \,.$$

As in [4, pp. 100-102], we can show that, as $|u| \to \infty$,

$$I(u) = A_1 u^{r/2+1/2A-1} J_{2\mu+Ar-1}(hu^{1/2A}) + O(u^{r/2+1/8A-1}) \,, \tag{5.2}$$

where $I(u)$ is defined by (2.1), $A_1 = D^{1/A}$ and $J_\upsilon(x)$ is the ordinary Bessel function of order υ. Then, as $|s| \to \infty$,

$$F_n(s) = \int_0^\infty I(\mu_n x)e^{-sx}dx$$
$$\sim A_1 \int_0^\infty (x\mu_n)^{r/2+1/2A-1} J_{2\mu+Ar-1}(h(\mu_n x)^{1/2A})e^{-sx}dx$$
$$(5.3) \qquad = (A_1/\mu_n)\int_0^\infty x^{r/2+1/2A-1}J_{2\mu+Ar-1}(hx^{1/2A})\exp(-sx/\mu_n)dx\ ,$$

since $r/2 + 1/8A > 0$.

We state as a lemma the special case that we will need of a more general thearem of E. M. Wright [23, Theorem 1].

LEMMA 5.1. *Suppose b and d are complex numbers, a and c are positive real numbers and $\Gamma(b + at)$ has no poles if t is a non-negative integer. Let, for* $\mathrm{Re}(z) > 0$,

$$H(z) = \sum_{k=0}^\infty \frac{\Gamma(b + ak)}{\Gamma(d + ck)} \cdot \frac{(-z)^k}{k!}\ .$$

Let $\varphi = b - d$, $A_0 = (1 + c - a)^{1/2-b+d}c^{1/2-d}a^{b-1/2}$ and

$$W = (1 + c - a)(a^a c^{-c} z)^{1/(1+c-a)}\ .$$

Then, as $|z| \to \infty$,

$$(5.4) \qquad H(z) = A_0 W^{-\varphi} e^{-W}\left\{1 + \sum_{l=1}^{M-1} B_l W^{-l} + O(W^{-M})\right\}\ ,$$

where M is a positive integer and the B_l, $1 \leq l \leq M - 1$, are certain constants independent of z.

LEMMA 5.2. *Suppose* $\mathrm{Re}(\omega) > 0$, *h, $\gamma > 0$ and* $\mathrm{Re}(\gamma\upsilon + \lambda) > 0$. *Then, as $|\omega| \to \infty$,*

$$(5.5) \qquad \int_0^\infty x^{\lambda-1}J_\upsilon(hx^\gamma)e^{-x/\omega}dx \sim A_2(h/2)^\upsilon \omega^{(\lambda-\gamma)/(1-\gamma)}\exp(-k\omega^{\gamma/(1-\gamma)})\ ,$$

where $A_2 = 2^{\upsilon+1}(1 - \gamma)^{1/2}(h\gamma^\gamma)^{(\lambda-1)/(1-\gamma)-\upsilon}$ and $k = (1 - \gamma)(h\gamma^\gamma)^{1/(1-\gamma)}$.

Proof. We have, since $\mathrm{Re}(\gamma\upsilon + \lambda) > 0$,

$$\int_0^\infty x^{\lambda-1}J_\upsilon(hx^\gamma)e^{-x/\omega}dx = (h/2)^\upsilon \sum_{l=0}^\infty \frac{(-h^2/4)^l}{l!\,\Gamma(\upsilon + l + 1)} \int_0^\infty x^{\lambda+\gamma\upsilon+2\gamma l-1}e^{-x/\omega}dx$$
$$(5.6) \qquad = (h/2)^\upsilon \omega^{\lambda+\gamma\upsilon} \sum_{l=0}^\infty \frac{\Gamma(\lambda + \gamma\upsilon + 2\gamma l)}{l!\,\Gamma(\upsilon + l + 1)}(-h^2\omega^{2\gamma}/4)^l\ .$$

We apply Lemma 5.1 with $a = 2\gamma$, $b = \lambda + \gamma\upsilon$, $c = 1$ and $d = \upsilon + 1$. Thus $\varphi = \lambda - 1 + \upsilon(\gamma - 1)$, $A_0 = 2^{\upsilon+1}(1 - \gamma)^{3/2-\lambda+\nu(1-\gamma)}\gamma^{\lambda+\gamma\upsilon-1/2}$ and $W = 2(1 - \gamma)((2\gamma)^\gamma h\omega^\gamma/2)^{1/(1-\gamma)}$. Thus, by (5.4), we have, as $|\omega| \to \infty$,

$$\sum_{l=0}^{\infty} \frac{\Gamma(\lambda + \gamma\upsilon + 2\gamma l)}{l!\,\Gamma(\upsilon + 1 + l)}(-h^2\omega^{2\gamma}/4)^l \sim A_2\omega^{\gamma(\lambda-1)/(1-\gamma)} \exp(-k\omega^{\gamma/(1-\gamma)}) . \tag{5.7}$$

Combining (5.6) and (5.7) we get (5.5). This completes the proof of the lemma.

THEOREM 3. *If* $A > 1/2$ *and* $\operatorname{Re}(\mu + Ar) > 0$, *then in* (2.7) *we may take* $m = Ar/(2A - 1)$ *and* $\alpha = (2A - 1)^{-1}$.

Proof. By (5.3) and (2.3), we have, as $|s| \to \infty$,

$$\begin{aligned} &\sum b(n)\mu_n^{1-r}\int_0^\infty I(\mu_n x)e^{-sx}dx \\ &\quad = \sum b(n)\mu_n^{1-r}F_n(s) \\ &\quad \sim A_1 \sum b(n)\mu_n^{1-r}\int_0^\infty x^{r/2+1/2A-1}J_{2\mu+Ar-1}(hx^{1/2A})\exp(-sx/\mu_n)dx . \end{aligned} \tag{5.8}$$

In Lemma 5.2 we take $\gamma = 1/2A$, $\omega = \mu_n/s$, $\lambda = r/2 + 1/2A$ and $\upsilon = 2\mu + Ar - 1$. The condition $\operatorname{Re}(\gamma\upsilon + \lambda) > 0$, of Lemma 5.2, translates into $\operatorname{Re}(r + \mu/A) > 0$ or, since $A > 1/2$, $\operatorname{Re}(\mu + Ar) > 0$. Thus, as $n \to \infty$, we have

$$F_n(s) \sim A_3 s^{-Ar/(2A-1)}\mu_n^{Ar/(2A-1)-1}\exp(-k(\mu_n/s)^{1/(2A-1)}) , \tag{5.9}$$

where $A_3 = A_1A_2(h/2)^{2\mu+Ar-1}$ and $k = (2A - 1)(h/2A)^{2A/(2A-1)}$. Since $h > 0$ and $A > 1/2$ we see $k > 0$. Thus, by (5.8) and (5.9), we have, as $|s| \to \infty$,

$$\begin{aligned} &\sum b(n)\mu_n^{1-r}\int_0^\infty I(\mu_n x)e^{-sx}dx \\ &\quad \sim s^{-Ar/(2A-1)}\sum A_3 b(n)\mu_n^{(1-A)r/(2A-1)}\exp(-k(\mu_n/s)^{1/(2A-1)}) . \end{aligned} \tag{5.10}$$

Comparing (5.10) and (2.7) gives the result and completes the proof of the theorem.

As an application of Theorem 3 we give the following theorem which is an application of Theorem 1 to Dirichlet series with positive coefficients.

THEOREM 4. *Suppose that* $f(s) = \sum a(n)\lambda_n^{-s}$ *satisfies the functional equation* (1.2) *with* $A > 1/2$ *and* $\operatorname{Re}(\mu + Ar) > 0$, *where* A *and* μ *are defined by* (5.1). *Suppose that for all* n *we have* $a(n) \geqq 0$. *Then we may take* $m = Ar/(2A - 1)$ *in Theorem* 1. *Also, as* $x \to \infty$,

(1) *if* $A > 1$, *then we have*

$$\int_0^x |E(y)|^2 dy \ll x^{3m/\rho+1} + x^{2m+1/2}\log x + x^{m+1/2}\log^{\rho-1} x + x^{r+1/2}\log^{2\rho-1} x ;$$

(2) *if* $A = 1$, *then we have*

$$\int_0^x |E(y)|^2 dy \ll x^{3r/2+1} \log^{2\rho-2} x + x^{2r+1/2} \log^{2\rho-1} x ;$$

(3) *if* $1/2 < A < 1$, *then we have*

$$\int_0^x |E(y)|^2 dy \ll x^{m+1/2} \log x + x^{m/2+r+1} \log^{\rho-1} x + x^{2r+1/2} \log^{2\rho-1} x .$$

Proof. By Theorem 3 we have $m = Ar/(2A-1)$. By a theorem of Landau (see [14, p. 874]), if $a(n) \geqq 0$, then $f(s)$ has a pole at $s = r$. Thus $\beta = r$. If $A > 1$, then $m < r$. Thus $\delta = m < r = \beta$ and $\eta = 0$, by (2.8). If $1/2 < A \leqq 1$, then $m \geqq r$. Thus $\delta = \beta = r \leqq m$ and $\eta = 1$, by (2.8). The results then follow by comparing the exponents of the terms in $M(x)$, in (2.9), in each of the three cases. This completes the proof of the theorem.

6. Comparison of Theorem 1 to the theorem of Chandrasekharan and Narasimhan. In this section we make a comparison of our Theorem 1 to Chandrasekharan and Narasimhan's Theorem 1 of [6]. For reference we state their theorem in our notation.

THEOREM A. *Suppose the functional equation* (1.2) *is satisfied with* $r > 0$, $A \geqq 1$ *and* $\mu_n = c_{66}n$, $\lambda_n = c_{67}n$, *where* c_{66} *and* c_{67} *are positive constants. Suppose the only singularities of* $f(s)$ *are poles and that for some real numbers* a *and* b

$$\sum_{\mu_n \leqq x} |b(n)|^2 \ll x^{2a-1} \log^b x ,$$

as $x \to \infty$. *If* $2a - r - 1/A \leqq 0$, *then, as* $x \to \infty$,

$$\int_0^x |E(y)|^2 dy = c_{68} x^{2d+1} + O(x^{2d+1/2A} \log^{b+2} x) ,$$

where c_{68} *is a certain positive constant and* $d = r/2 - 1/4A$. *If* $2a - r - 1/A > 0$, *then on the basis of the further assumptions that*

$$\sum_{\lambda_n \leqq x} |a(n)|^2 \ll x^{2a-1} \log^b x$$

and $b \geqq 2(\rho - 1)$ *we have, as* $x \to \infty$,

$$\int_1^x |E(y)|^2 dy \ll x^{2d+1} + x^{2a+1-1/2A} \log^{b+1} x .$$

From the estimates (4.19) and (4.20) and Remark 2 after the proof of Lemma 3.4 it seems likely that the estimate

$$\sum_{n \leqq x} |a(n)|^2 \ll x^{2\beta-1} \log^b x ,$$

as $x \to \infty$, hold for some nonnegative integer b in the case when $f(s)$ is not entire. In many of the special cases that estimates for the sum (1.6) are known an estimate of this type is obtained. For example, for the coefficients of zeta functions of algebraic number fields [5, Theorem 3] and for the case $a(n) = d_k(n)$ [19, p. 199] such estimates are obtained. Thus in Theorem A we could take $\alpha = \beta$.

Since our Theorem 1 gives only an 0-estimate for the integral of the square of the error term, Theorem A is better when $2\beta - r - 1/A \leqq 0$. In a sense this says that the parameters for the estimate are relatively small.

Suppose $2\beta - r - 1/A > 0$. Then both Theorem 1 and Theorem A give only O-estimates. The estimate from Theorem A is

$$\int_0^x |E(y)|^2 dy \ll x^{2\beta+1-1/2A} \log^{b+1} x \, , \tag{6.1}$$

as $x \to \infty$. If we take each possible term in (2.9) in both cases ($\delta = m$ and $\delta = \beta$) and suppose it to be maximal, we see that the result of Theorem 1 is no worse than the estimate (6.1) if either $A \geqq 1$ or $\beta \geqq 1/A$. The first condition, $A \geqq 1$, is part of the hypotheses of Theorem A and the second, $\beta \geqq 1/A$, is again a statement that the parameters are not too small, since $\beta < 1/A$ and $2\beta - r - 1/A > 0$ imply $r < 1/A$. If we were given $\beta \geqq 1$, then the second condition would also be fulfilled. An example of the latter would be a Dirichlet series with nonnegative coefficients satisfying the functional equation (1.2) with $r \geqq 1$.

We can then say that Theorem A gives better results if the parameters are small, while Theorem 1 gives better results if the parameters are large, as is the case in Examples 2 and 5 of §7 below. Moreover, Theorem 1 is applicable in those cases where one does not have estimates on

$$\sum_{\lambda_n \leqq x} |a(n)|^2 \, ,$$

whereas Theorem A is not.

7. Examples.

EXAMPLE 1. For $k > 0$, let $\sigma_k(n)$ be the sum of kth powers of the divisors of n and let $S_k(x)$ be the associated error term. Then, for $\mathrm{Re}\,(s) > k + 1$,

$$\sum \sigma_k(n) n^{-s} = \zeta(s)\zeta(s - k) \, .$$

Here we have $r = \beta = k + 1$, $A = 1$ and $\rho = 1$. By (2) of Theorem 4, we have, as $x \to \infty$,

$$\int_0^x |S_k(y)|^2 dy \ll x^{2k+5/2} \log x \ ,$$

which is the same result obtainable from the theorem of Chandrasekharan and Narasimhan, but here we did not need to refer to the size of the sum

$$\sum_{n \leqq x} \sigma_k^2(n) \ ,$$

as is required by their theorem.

EXAMPLE 2. Let $d_k(n)$ be the number of ways of writing n as a product of $k \geqq 2$ factors. Then, for $\mathrm{Re}\,(s) > 1$,

$$\sum d_k(n) n^{-s} = \zeta^k(s) \ .$$

Here we have $r = \beta = 1$, $A = k/2$ and $\rho = k$. Thus, by (1) of Theorem 4, we have, for $k \geqq 3$, as $x \to \infty$,

$$\int_0^x |\Delta_k(y)|^2 dy \ll x^{(4k-3)/(2k-2)} \log^{k-1} x \ ,$$

where $\Delta_k(x)$ is the error term associated with the coefficients $d_k(n)$. For $k = 2$ Chandrasekharan and Narasimhan get an asymptotic result. For $k \geqq 3$ this improves the result obtainable from their theorem. For $k \geqq 5$ our result improves the result in Titchmarsh [19, Theorem 12.3 and §12.5, p. 270].

EXAMPLE 3. Let K be an algebraic number field of degree n over the rationals, with $n \geqq 3$. Let $a_K(m)$ be the number of integral ideals with norm exactly equal to m. For $\mathrm{Re}\,(s) > 1$. Let

$$\zeta_K(s) = \sum a_K(m) m^{-s} \ .$$

Then, from [13, p. 27], we know that $\zeta_K(s)$ has a simple pole at $s = 1$ and is regular elsewhere. Also $\zeta_K(s)$ satisfies the functional equation

$$\Gamma^{r_1}(s/2)\Gamma^{r_2}(s)C^{-s}\zeta_K(s) = \Gamma^{r_1}((1-s)/2)\Gamma^{r_2}(1-s)C^{s-1}\zeta_K(1-s) \ ,$$

where r_1 is the number of real conjugates, $2r_2$ the number of imaginary conjugates of K, so that $n = r_1 + 2r_2$, and C is a positive constant depending only on the field K. Here we have $r = \beta = 1$, $A = n/2$ and $\rho = 1$. Thus, in Theorem 4, we take $m = n/(2n-2)$. If $E(x)$ is the error term associated with $\zeta_K(s)$, then by (1) of Theorem 4, we have, as $x \to \infty$,

$$\int_0^x |E(y)|^2 dy \ll x^{(4n-3)/(2n-2)} \ .$$

This result improves the result of Chandrasekharan and Narasimhan [6, Theorem 2] for $n \geqq 3$. For $n = 2$ they get an asymptotic result.

EXAMPLE 4. Let K, r_1, r_2, and C be as in Example 3. Let Λ be a nonprincipal Grössencharakter on the ideals of K. Let

$$e_q = \begin{cases} 1 & 1 \leqq q \leqq r_1 \\ 2 & r_1 + 1 \leqq q \leqq r_1 + r_2 \end{cases} .$$

For $\operatorname{Re}(s) > 1$, let

$$\varphi_\Lambda(s) = C^s \zeta(s, \Lambda) = C^s \sum{}' \Lambda(\mathfrak{A}) N(\mathfrak{A})^{-s} ,$$

where the sum is over all nonzero integral ideals $\mathfrak{A}$ and $N(\mathfrak{A})$ is the norm of $\mathfrak{A}$. If we let

$$c(m) = c_{.}(m) = \sum_{N(\mathfrak{A}) = m} \Lambda(\mathfrak{A}) ,$$

then we have, for $\operatorname{Re}(s) > 1$,

$$\varphi_\Lambda(s) = C^s \sum c(m) m^{-s} .$$

Then, from [11], we know that $\varphi_\Lambda(s)$ can be continued to an entire function and that there exist real numbers $\varepsilon_1, \cdots, \varepsilon_{r_1+r_2}$ and nonnegtive integers d_q, d'_q, $1 \leqq q \leqq r_1 + r_2$, such that $\varphi_\Lambda(s)$ satisfies the functional equation

$$\varphi_\Lambda(s) \Gamma_\Lambda(s) = L \Gamma_\Lambda(1 - s) \varphi_{\bar{\Lambda}}(1 - s) ,$$

where L is a constant depending on Λ and

$$\Gamma_\Lambda(s) = \prod_{p=1}^{r_1+r_2} \Gamma\{[e_p(s + (d_p + d'_p)/2 + i\varepsilon_p]/2\} .$$

Here we have $A = n/2$ and $r = 1$. Thus in Theorem 1 we may take $m = n/(2n - 2)$, by Theorem 3. This gives

$$\int_0^x |E(y)|^2 dy \ll x^{7/4 + 3/(4n-4)} ,$$

as $x \to \infty$, where $E(x)$ is the error term associated with the coefficients $c(m)$.

By Theorem 3, we have $\alpha = 1/(n - 1)$. Thus, by Theorem 2, we have, as $x \to \infty$,

$$\sum_{m \leqq x} |c(m)|^2 \ll x^{1 + 1/(n-1)} \log x .$$

EXAMPLE 5. Siegel's zeta function. Let Q be an indefinite quadratic form in $k \geqq 4$ variables with rational coefficients. Let

$\mu(Q, t)$ be the measure of representation of t by Q and let, for $\operatorname{Re}(s) > k/2$,

$$\zeta(Q, s) = \sum_{t>0} \mu(Q, t)t^{-s} .$$

This zeta function was first introduced by C. L. Siegel in [17]. From [17, p. 688] we know that $\zeta(Q, s)$ satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta(Q, s) = (-1)^{(k-n)/2}|Q|^{-1/2}\pi^{-(k/2-s)}\Gamma(k/2 - s)\zeta(Q^{-1}, k/2 - s) ,$$

where $|Q|$ is the determinant of Q, Q^{-1} is its inverse form and $(n, k - n)$ is its signature. From [17, p. 688] we know that $\zeta(Q, s)$ has a simple pole at $s = k/2$ and is a regular function elsewhere. Thus we have $r = k/2 = \beta$, $\rho = 1$, and $\eta = 1$. Let $E(x)$ be the error term associated with $\zeta(Q, s)$. Then, by Corollary 1, we have, as $x \to \infty$,

$$\int_0^x |E(y)|^2 dy \ll x^{k+1/2} \log x .$$

Now assume that the coefficients of Q are integers. By Theorem 3 (or [3, p. 152]) we may take $\alpha = 1$. Thus, by Corollary 2, we have, as $x \to \infty$,

$$\sum_{t \leq x} \mu(Q, t)^2 \ll \begin{cases} x^4 \log x & \text{if } k = 4 \\ x^k & \text{if } k \geqq 5 . \end{cases}$$

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Received September 28, 1977 and in revised form December 27, 1977. Some of the results in this paper appeared in the author's Ph. D thesis written under Professor Bruce C. Berndt at the University of Illinois in 1975.

SOUTHERN ILLINOIS UNIVERSITY
CARBONDALE IL 62901