PACIFIC JOURNAL OF MATHEMATICS
Vol. 78, No. 1, 1978

THE CARATHÉODORY METRIC AND HOLOMORPHIC MAPS ON A CLASS OF WEAKLY PSEUDOCONVEX DOMAINS

R. MICHAEL RANGE

The boundary behavior of proper holomorphic maps between two smoothly bounded pseudoconvex domains in C^n is studied by means of the Carathéodory metric. The Hölder continuity of such maps is proved in case the image domain satisfies some technical conditions; these are satisfied, for example, by strictly pseudoconvex domains and convex domains with real analytic boundary.

In recent years it has become clear that pseudoconvex domains with smooth boundary may exhibit rather pathological behavior in the absence of strict pseudoconvexity (cf. the examples of Kohn and L. Nirenberg [13] and Diederich and Fornaess [4]). Therefore it might be of interest to consider conditions weaker than strict pseudoconvexity and to extend classical results to more general settings.

Investigations related to Hölder estimates for solutions of the $\bar{\partial}$-equation (cf. Range [18]) have led the author to introduce a technical refinement of the following classical condition (cf. Behnke and Thullen [1], p. 29): The domain D is called *totally pseudoconvex* at $P \in \partial D$ if there is an analytic hypersurface M_P in a neighborhood U of P, such that $M_G \cap \bar{D} = \{P\}$. The refinement involves two parts. First, there should be supporting analytic hypersurfaces M_ζ for all points $\zeta \in \partial D$ near P, and M_ζ should depend smoothly on ζ. Next, in order to obtain estimates of some sort, one needs finite order contact between M_ζ and ∂D at ζ. The resulting condition is called *uniform total pseudoconvexity of finite order* (cf. Definition 1.8 for the precise formulation). Simple examples of domains which satisfy this condition at every boundary point are strictly pseudoconvex domains and convex domains with real analytic boundary.

In this paper we prove the following generalization of a classical result.

MAIN THEOREM. *Let D_1 and D_2 be bounded domains in $\boldsymbol{C}^n$ with smooth boundary. Assume that D_2 is uniformly totally pseudoconvex of finite order at every point $P \in \partial D_2$, and that $\bar{D}_2$ has a Stein neighborhood basis.*[1] *Then there is $\alpha > 0$, such that every proper*

[1] Theorem 2.2 and, as a consequence, the Main Theorem, are valid without assuming the existence of a Stein neighborhood basis, provided one assumes high differentiability

holomorphic map $F: D_1 \to D_2$ *is Hölder continuous of order* α; *in particular,* F *extends continuously to* $\bar{D}_1$.

ADDED IN PROOF. The Main Theorem holds without requiring the existence of a Stein neighborhood basis for D_2. The necessary modifications for the proof are sketched in footnote 1.

For D_2 strictly pseudoconvex (plus a mild restriction for D_1) the Main Theorem was proved by Henkin [9] and, independently, by Pinchuk [16]; a somewhat weaker result was obtained by Vormoor [20]. Based on these results, the author [17] proved the Main Theorem for biholomorphic maps between domains with *piecewise* smooth strictly pseudoconvex boundary. By a different method, Fefferman [5] proved that a biholomorphic map between strictly pseudoconvex domains with C^∞ boundary extends as a C^∞ diffeomorphism $\bar{D}_1 \to \bar{D}_2$.

The proof of the Main Theorem uses the argument of Henkin and Pinchuk. The main analytic tool is an estimate for the Carathéodory metric of the image domain D_2. For strictly pseudoconvex domains such an estimate was obtained by Henkin and Pinchuk, and also by Graham [6], by approximating the domain by balls and using supremum norm estimates for $\bar{\partial}$. The proof given here is based on an explicit local construction, and the passage from local to global is handled by Hörmander's L^2 estimates for $\bar{\partial}$; in particular, one obtains a new proof for the strictly pseudoconvex case.

Briefly, this paper is organized as follows. In § 1 we introduce the various notions of total pseudoconvexity and prove some basic results; in particular, we discuss the relationship with (Euclidean) convexity and the existence of peaking functions. The estimate for the Carathéodory metric is proved in § 2. In § 3 we combine a result of Diederich and Fornaess [3] and the estimates of § 2 with the techniques of Henkin and Pinchuk to prove the Hölder continuity of proper holomorphic maps.

The results in § 1 were, essentially, obtained in 1975, but they have not been published in detail before. The author has lectured on several occasions about different versions of these results in the

---

of the boundary. In order to see this, one observes that in the proof of Theorem 2.2 the $\bar{\partial}$-closed $(0,1)$ forms α_j, $1 \leqq j \leqq n$, which are defined on D_ε, satisfy, for each $k \in N$, an estimate $\|\alpha_j\|_{C^k_{0,1}(\bar{D})} \leqq \gamma_k \|\alpha_j\|_{L^\infty_{0,1}(L_\varepsilon)}$, where $0 < \gamma_k < \infty$. By Kohn's global regularity result ([**12**], Theorem 3.19) and Sobolev's Lemma, for sufficiently large k there is a bounded solution operator $T_k: C^k_{0.1}\bar{D} \cap \ker \bar{\partial} \to C^1(\bar{D})$ for $\bar{\partial}$. So $u_j = T_k(\alpha_j)$ satisfies $\bar{\partial} u_j = \alpha_j$ and $\|u_j\|_{C^1(\bar{D})} \leqq B_k \|_{L^\infty_{0,1}(D_\varepsilon)}$ for some constant B_k; this estimate is sufficient to complete the proof of (2.2).

context of Hölder estimates for $\bar{\partial}$, notably at the 1975 Seminar on Spaces of Analytic Functions in Kristiansand, Norway, and at the 1975 AMS Summer Institute on Several Complex Variables in Williamstown, Massachusetts (cf. [18]). Initially, uniform total pseudoconvexity of finite order was formulated in terms of special coordinate systems which are now relegated to a technical device; the version adopted here, which emphasizes the supporting analytic hypersurface, seems the more natural one. The problem of finding a tractable characterization of total pseudoconvexity in terms of local invariants of the boundary remains open; its solution should contribute to a better understanding of pseudoconvexity.

NOTATIONS. For $x \in \boldsymbol{R}^n$ and $E \subset \boldsymbol{R}^n$, $d(x, E)$ denotes the Euclidean distance from x to E. For $a \in \boldsymbol{C}^n$, the components of a are denoted by $a_1, \cdots, a_n$; we sometimes write $a = (a', a_n)$, where $a' = (a_1, \cdots, a_{n-1}) \in \boldsymbol{C}^{n-1}$; the Euclidean norm $(\sum_{i=1}^n a_i\bar{a}_i)^{1/2}$ is denoted by $|a|$. For $P \in \boldsymbol{C}^n$ and $c > 0$, $B(P, c)$ denotes the open ball in $\boldsymbol{C}^n$ with center P and radius c; Δ denotes the open unit disc in $\boldsymbol{C}$ with center 0.

For a C^1 function f in a neighborhood of $P \in \boldsymbol{C}^n$, $\partial f(P)$ denotes the (1, 0) form $\partial f = \sum_{i=1}^n \partial f/\partial z_i(P)dz_i$. The natural pairing between a cotangent vector α at P and a tangent vector v at P is denoted by $\langle \alpha, v\rangle$; in particular, $\langle \partial f(P), v\rangle = \sum_{i=1}^n \partial f/\partial z_i(P)v_i$.

A domain $D \subset \boldsymbol{C}^n$ has a C^k boundary at $P \in \partial D$, $1 \leqq k \leqq \infty$, if there is a C^k function $r: U \to \boldsymbol{R}$ defined on a neighborhood U of P, such that $dr(P) \neq 0$ and $D \cap U = \{z \in U: r(z) < 0\}$; a function r satisfying these conditions is called a *defining function for* D at P. The real tangent space of ∂D at P is denoted by $T_P(\partial D)$, and the complex tangent space $T_P(\partial D) \cap \sqrt{-1}\, T_P(\partial D)$ is denoted by $H_P(\partial D)$; for any defining function r, $H_p(\partial D) = \{v \in \boldsymbol{C}^n: < \partial r(P), v > = 0\}$.

In order to avoid the use of many constants, we adopt the following convention: If $A(x)$ and $B(x)$ denote expressions which depend on a variable $x \in \boldsymbol{R}^l$, $A(x) \lesssim B(x)$ means that there is a constant $K, 0 < K < \infty$, such that $|A(x)| \leqq K|B(x)|$ for all x under consideration; $A(x) \sim B(x)$ is equivalent to $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$.

1. Total pseudoconvexity.

We first discuss some properties of the simple point version of total pseudoconvexity which may be of independent interest.

DEFINITION 1.1. *A domain D in $\boldsymbol{C}^n$ is called totally pseudoconvex at the point $P \in \partial D$ if there is a nonsingular analytic hypersurface M in a neighborhood U of P, such that $M \cap \bar{D} \cap U = \{P\}$. M is called a supporting analytic hypersurface for D at P.*

From now on we will assume that D has a C^1 boundary at P; the coordinates of $\boldsymbol{C}^n$ and the defining function r for D are chosen so that $P=0$, $H_0(\partial D)=\{z\in\boldsymbol{C}^n: z_n=0\}$ and $\partial r(0)=dz_n$. If M is a supporting analytic hypersurface for D at 0, the tangent space of M at 0 coincides with $H_0(\partial D)$; so, near $0\in\boldsymbol{C}^n$, M can be described as the graph of a holomorphic function g defined in a neighborhood U' of $0\in\boldsymbol{C}^{n-1}$: $M\cap(U'\times\boldsymbol{C})=\{(z', z_n)\in U'\times\boldsymbol{C}: z_n=g(z')\}$. To say that D is totally pseudoconvex at $0\in\partial D$ is therefore equivalent to the following: *There is a holomorphic function g on U', with $g(0)=0$, such that $r(z', g(z'))>0$ for $z'\neq 0$.*

It will be convenient to linearize M by a suitable holomorphic change of coordinates. Choose a holomorphic function ϕ on a neighborhood U of 0, such that $M\cap U=\{z\in U: \phi(z)=0\}$ and $d\phi(0)=dz_n$. Define $F: U\to\boldsymbol{C}^n$ by $w=F(z)=(z', \phi(z))$; the Jacobian matrix F_{*0} of F at 0 is the identity matrix; hence, after shrinking U, $F: U\to F(U)$ is biholomorphic, and

$$F(U\cap M)=F(U)\cap\{w\in\boldsymbol{C}^n: w_n=0\}=F(U)\cap H_0(\partial F(D\cap U)) .$$

we thus have the following definition equivalent to Definition 1.1.

1.2. *D is totally pseudoconvex at $P\in\partial D$ if there is a holomorphic change of coordinates $w=w(z)$ in a neighborhood U of P, such that*

$$w(\bar{D}\cap U)\cap H_{w(P)}(\partial w(D\cap U))=\{w(P)\} .$$

So, geometrically, total pseudoconvexity is just the biholomorphic image of convexity in the directions of the *complex* tangent vectors. The following result shows that by relaxing the regularity of the coordinate change at P, one can achieve convexity also in the remaining tangential direction.

PROPOSITION 1.3. *Let D be totally pseudoconvex at $P\in\partial D$. Then there are a neighborhood U of P, a neighborhood Ω of $\bar{D}\cap U-\{P\}$, and a biholomorphic map $G:\Omega\to G(\Omega)\subset\boldsymbol{C}^n$ with the following properties*:

(a) *G extends continuously to P and $G(P)=0$;*

(b) *$G(\bar{D}\cap U-\{P\})\subset\{w\in\boldsymbol{C}^n: \operatorname{Re} w_n>0\}$.*

Proof. Choose the coordinates of $\boldsymbol{C}^n$, r and F as before; $\rho=r\circ F^{-1}$ is a defining function for $F(D\cap U)$ at 0, and $\partial_w\rho(0)=dw_n$. By assumption, there is $c>0$ such that $\rho(w', 0)>0$ for $0<|w'|<c$. Therefore, by Taylor's theorem, if $|w|<c$ and $w'\neq 0$,

$$\begin{aligned}\rho(w)=\rho(w', w_n)&=\rho(w', 0)+2\operatorname{Re}\frac{\partial\rho}{\partial w_n}(w', 0)\cdot w_n+o(|w_n|)\\&>2\operatorname{Re}1\cdot w_n+o(1)\cdot|w_n| .\end{aligned}$$

Choose U so small that $o(1)\cdot|w_n| \leqq |\operatorname{Re} w_n| + |\operatorname{Im} w_n|$ for $w \in F(U)$; if $\rho(w) \leqq 0$, one obtains

$$0 > 2 \operatorname{Re} w_n - |\operatorname{Re} w_n| - |\operatorname{Im} w_n| ,$$

or

$$(1.4.) \qquad -\operatorname{Re} w_n > -|\operatorname{Im} w_n| \text{ for } w \in F(\bar{D} \cap U) \text{ with } w' \neq 0.$$

This shows that, on $F(\bar{D} \cap U - \{P\})$, $-w_n$ omits the nonpositive real axis $\boldsymbol{R}^-$. If one chooses that branch of the square root defined on $\boldsymbol{C} - \boldsymbol{R}^-$ which satisfies $\sqrt{1} = 1$, $\sqrt{-w_n}$ is holomorphic on $\boldsymbol{C} - \boldsymbol{R}^-$ and, by 1.4, satisfies $\operatorname{Re}\sqrt{-w_n} > 0$ for $w \in F(\bar{D} \cap U - \{P\})$, and hence for $w \in F(\Omega)$, where Ω is some neighborhood of $\bar{D} \cap U - \{P\}$. If $S(w) = (w', \sqrt{-w_n})$, the map $G = S \circ F$ is biholomorphic on Ω and satisfies (a) and (b).

Remark. As the proof shows, the singularity of G at P is quite simple. It is not known whether one can choose G holomorphic at P, so that 1.3(b) still holds.[2]

Corollary 1.5. *Let D be totally pseudoconvex at $P \in \partial D$. Then there is a neighborhood U of P such that P is a peak point for the uniform algebra $A(D \cap U)$.*

Proof. The function $h(z) = \exp[-G_n(z)]$ is in $A(D \cap U)$ and peaks at P.

By using Kohn's global regularity result for $\bar{\partial}$ [12] and standard techniques (cf. Pflug [15], or Hakim and Sibony [8]), one obtains the following global version of 1.5.

Corollary 1.6. *Suppose D is a bounded pseudoconvex domain with smooth boundary. If D is totally pseudoconvex at P, then P is a peak point for the uniform algebra $A(D)$.*

Furthermore, by a variation of the proof of a result of Rossi ([19], Theorem 5.13), Corollary 1.5 implies:

Corollary 1.7. *Suppose D has a C^2 boundary near P and D is totally pseudoconvex at P. Then P is a limit point of strictly pseudoconvex boundary points of D.*

For the details of the simple modification required, see, for example, [8].

---

[2] T. Bloom recently found an example for which there is no such map G holomorphic at P. (cf. Duke Math. J. **45** (1978), 133–148.)

We now come to the parametrized version of total pseudoconvexity.

DEFINITION 1.8. *Let D have a C^1 boundary at $P \in \partial D$, and let r be a defining function for D. D is uniformly totally pseudoconvex at P if there are positive constants δ, c and a C^1-map $\phi: \partial D \cap B(P, \delta) \times B(P, 2\delta) \to \boldsymbol{C}$, such that, for all $\zeta \in \partial D \cap B(P, \delta)$ the following are satisfied:*

(i) *$\phi(\zeta, \cdot)$ is holomorphic on $B(\zeta, \delta)$;*

(ii) *$\phi(\zeta, \zeta) = 0$ and $d_z\phi|_{z=\zeta} \neq 0$;*

(iii) *$r(z) > 0$ for all z with $\phi(\zeta, z) = 0$ and $0 < |z - \zeta| < c$.*

Clearly Definition 1.8 implies that D is totally pseudoconvex at all points $\zeta \in \partial D$ near P; the supporting analytic hypersurface for D at ζ is given by $M_\zeta = \{z: \phi(\zeta, z) = 0\}$.

For $\zeta \in \partial D$, we denote by π_ζ the orthogonal projection $T_\zeta \boldsymbol{C}^n \to H_\zeta(\partial D)$.

1.8 (Continued). *D is uniformly totally pseudoconvex of finite order at P if, in addition to* (i), (ii), (iii), *there are $m \in \boldsymbol{N}$ and $\gamma > 0$, such that*

(iv) *$r(z) \geqq \gamma |\pi_\zeta(z - \zeta)|^m$ for all $z \in B(\zeta, c)$ with $\phi(\zeta, z) = 0$.*

REMARK 1.9. Definition 1.8 is independent of the choice of holomorphic coordinates in a neighborhood of P and of the particular defining function r which appears in (iii) and (iv). The smallest integer m for which (iv) holds with some constants γ and c for all $\zeta \in \partial D$ in a neighborhood of P is called the order of ∂D at P. Note that if $D \subset \boldsymbol{C}^n$ with $n > 1$, one must have $m \geqq 2$.

By multiplying r and ϕ by suitable nonzero functions of ζ, one may further assume

(v) $\quad |\partial r(\zeta)| = 1$ and $\partial r(\zeta) = d_z\phi|_{z=\zeta}$ for $\zeta \in \partial D \cap B(P, \delta)$.

EXAMPLE 1.10. A bounded domain $D \subset \boldsymbol{R}^d$ with a C^1 boundary is called *totally convex* if for each $P \in \partial D$ the tangent space $T_p(\partial D)$ intersects $\bar{D}$ only at P. If $D \subset \boldsymbol{C}^n$ is totally convex, then D is uniformly totally pseudoconvex at all points $P \in \partial D$; in fact, if r is a defining function for D in some neighborhood U of ∂D, the function $\phi(\zeta, z) = \sum_{i=1}^n \partial r/\partial\zeta_i(\zeta)(z_i - \zeta_i)$ satisfies 1.8(i)-(iii).

EXAMPLE 1.11. Let D be strictly Levi pseudoconvex at $P \in \partial D$, i.e., if r is a C^2 defining function for D near P, the Levi form

$$L(r; P, v) = \sum_{i,j=1}^n \frac{\partial^2 r}{\partial\zeta_i \partial\bar{\zeta}_j}(P) v_i \bar{v}_j$$

satisfies, with some $\gamma > 0$,

$$L(r; P, v) \geqq \gamma |v|^2 \text{ for } v \in H_P(\partial D) .$$

By continuity, it follows that $L(r; \zeta, v) \geqq \gamma/2|v|^2$ for $v \in H_\zeta(\partial D)$ and $\zeta \in \partial D$ near P. It is classical that in this case D is uniformly totally pseudoconvex of finite order 2 at P; the function ϕ is given by

$$\phi(\zeta, z) = \sum_{i=1}^{n} \frac{\partial r}{\partial \zeta_i}(\zeta)(z_i - \zeta_i) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 r}{\partial \zeta_i \partial \zeta_j}(\zeta)(z_i - \zeta_i)(z_j - \zeta_j) .$$

Typically, this is proved by choosing r strictly plurisubharmonic (cf. Gunning and Rossi [7], Chapter IXB); however, it is easy to obtain this result by just using any defining function r, as follows. By Taylor's expansion,

$$r(z) = r(\zeta) + 2 \operatorname{Re} \phi(\zeta, z) + L(r; \zeta, z - \zeta) + o(|z - \zeta|^2);$$

fix $\zeta \in \partial D$; for z with $\phi(\zeta, z) = 0$ one has $z - \zeta = \pi_\zeta(z - \zeta) + o(|z - \zeta|)$; therefore, for some $c > 0$, one obtains

$$\begin{aligned} r(z) &= L(r; \zeta, z - \zeta) + o(|z - \zeta|^2) \geqq 1/2\, L(r; \zeta, \pi_\zeta(z - \zeta)) \\ &\geqq \gamma/4 |\pi_\zeta(z - \zeta)|^2 \text{ for all } z \in B(\zeta, c) \text{ with } \phi(\zeta, z) = 0 . \end{aligned}$$

If D is not strictly pseudoconvex at P, it is usually condition 1.8 (iv) which is hardest to verify. Even though one may be able to obtain for each $\zeta \in \partial D$ near P an estimate $r(z) \geqq \gamma_\zeta |\pi_\zeta(z - \zeta)|^{m_\zeta}$ for $z \in B(\zeta, c_\zeta)$ with $\phi(\zeta, z) = 0$, there remains the nontrivial problem of choosing γ_ζ, c_ζ, m_ζ independently of ζ. As an example, consider, for $m > 2$, even, the domain $B_m = \{|z_1|^2 + |z_2|^m < 1\}$ with defining function $r_m(z) = |z_1|^2 + |z_2|^m - 1$. B_m is totally convex, and at points $\zeta = (\zeta_1, \zeta_2) \in \partial B_m$ with $\zeta_2 \neq 0$ it is strictly pseudoconvex. If $\phi(\zeta, z) = \langle \partial r_m(\zeta), z - \zeta \rangle$, one obtains the following estimates for $\zeta \in \partial B_m$ and $z \in B(\zeta, 1)$ with $\phi(\zeta, z) = 0$:

$$r_m(z) \geqq \begin{cases} \gamma_\zeta |\pi_\zeta(z - \zeta)|^2 \geqq \gamma_\zeta |\pi_\zeta(z - \zeta)|^m , & \text{if } \zeta_2 \neq 0; \\ 1 \cdot |\pi_\zeta(z - \zeta)|^m , & \text{if } \zeta_2 = 0 . \end{cases}$$

Here, for $\zeta_2 \neq 0$, $\gamma_\zeta > 0$ is, essentially, the eigenvalue of the Levi form; since $\gamma_\zeta \to 0$ as $\zeta_2 \to 0$, the constants one obtains by the "obvious" point estimates do not depend continuously on ζ at non-strictly pseudoconvex boundary points. Nevertheless, as pointed out in [18], one can show that B_m is uniformly totally pseudoconvex of finite order m.

More generally, one has the following result.

PROPOSITION 1.12. *Let D be a bounded domain in C^n with real analytic boundary. Suppose D is uniformly totally pseudoconvex* *at every point $P \in \partial D$, and that the function $\phi(\zeta, z)$ given by* 1.8 *can be chosen real analytic in (ζ, z). Then D is uniformly totally pseudoconvex of finite order at every $P \in \partial D$.*

COROLLARY 1.13. *Let D be a bounded convex domain in C^n with real analytic boundary. Then D is uniformly totally pseudoconvex of finite order at every $P \in \partial D$.*

To prove the corollary, observe that the hypotheses imply that D is totally convex; the conclusion then follows by 1.10 and the proposition.

In order to prove 1.12 we first introduce the parametrized version of the coordinate system given by 1.2; this will be used in § 2 as well.

Thus, suppose D is uniformly totally pseudoconvex at P, and let $U = B(P, \delta)$, c, $r(z)$, and $\phi(\zeta, z)$ be as in 1.8, so that (i), (ii), (iii), and (v) are satisfied. Choose smooth orthonormal sections $E^{(1)}(\zeta), \cdots, E^{(n-1)}(\zeta)$ of the holomorphic tangent bundle $H(\partial D)$ over $\partial D \cap U$. For $\zeta \in \partial D \cap U$ define the holomorphic map $z \to w = F_\zeta(z)$ by

$$w_\nu = \sum_{i=1}^{n} \bar{E}_i^{(\nu)}(\zeta)(z_i - \zeta_i),\ \nu = 1, \cdots, n-1, \text{ and } w_n = \phi(\zeta, z)\ .$$

The Jacobian matrix $(F_\zeta)_{*\zeta}$ has rows $\bar{E}^{(1)}(\zeta), \cdots, \bar{E}^{(n-1)}(\zeta)$, $(\partial\phi/\partial z_1(\zeta, \zeta), \cdots, \partial\phi/\partial z_n(\zeta, \zeta))$, and so it is unitary, by (v). After shrinking the neighborhood U of P, one may choose $c > 0$, $d > 0$ so small that F_ζ maps $B(\zeta, c)$ biholomorphically onto the neighborhood $F_\zeta(B(\zeta, c)) \supset B(0, d)$ of 0 in C^n for all $\zeta \in \partial D \cap U$. Also, one may assume that F_ζ and F_ζ^{-1} have uniformly bounded Jacobian matrices; hence there are positive constants A_1, A_2 such that

$$A_1|z - z^*| \leqq |F_\zeta(z) - F_\zeta(z^*)| \leqq A_2|z - z^*| \quad \text{for } z, z^* \in B(\zeta, c)\ . \tag{1.14}$$

The analytic hypersurface $\{z \in B(\zeta, c): \phi(\zeta, z) = 0\}$ is mapped by F_ζ biholomorphically into $\{w \in C^n: w_n = 0\}$. The function $\rho_\zeta = r \circ F_\zeta^{-1}$ is a defining function for $F_\zeta(D \cap B(\zeta, c))$; a calculation shows

$$\partial_w \rho_\zeta(0) = dw_n\ . \tag{1.15}$$

The conditions (iii) and (iv) are, respectively, equivalent to

(iii bis) $\quad \rho_\zeta(w', o) > 0$ for $0 < |w'| < d$;

(iv bis) $\quad \rho_\zeta(w', o) \geqq \gamma\,|w'|^m$ for $0 \leqq |w'| < d$.

Proof of Proposition 1.12. By assumption, the functions r and ϕ may be chosen real analytic, which implies that the map $F_\zeta(z)$

constructed above may be chosen real analytic in (ζ, z). One thus obtains a nonegative real analytic function $R(\zeta, w') = \rho_\zeta(w', o)$ defined on $\Omega = (\partial D \cap U) \times \{w' \in \boldsymbol{C}^{n-1}: |w'| < d\}$. Let $Z = \{(\zeta, w') \in \Omega: R(\zeta, w') = 0\}$; by (iii bis), $Z = \{(\zeta, w') \in \Omega: w' = 0\}$, and $d((\zeta, w'), Z) = |w'|$. By a theorem of Lojasiewicz ([14], p. 124), given $U' \subset\subset U$ and $0 < d' < d$, one can find a constant $\gamma > 0$ and a positive integer m, such that

$$R(\zeta, w') \geqq \gamma d((\zeta, w'), Z)^m = \gamma |w'|^m \text{ for } \zeta \in \partial D \cap U' \text{ and } |w'| \leqq d' .$$

Thus (iv bis) holds.

Finally, by modifying the proof of Proposition 1.3, one obtains a more precise estimate for $\phi(\zeta, z)$.

PROPOSITION 1.16. *Suppose D is uniformly totally pseudoconvex of finite order m at the point $P \in \partial D$. Let $\phi(\zeta, z)$ satisfy* (i)—(v) *in* (1.8) *for $\zeta \in \partial D \cap B(P, \delta)$. Then there are constants $A, c^\# > 0$, such that*

(vi) $|\phi(\zeta, z)| \geqq A[d(z, \partial D) + |\operatorname{Im} \phi(\zeta, z)| + |\zeta - z|^m]$
for $\zeta \in \partial D \cap B(P, \delta)$ and $z \in \bar{D} \cap B(\zeta, c^\#)$.

Proof. Fix $\zeta \in \partial D \cap B(P, \delta)$ and introduce $\rho_\zeta = r \circ F_\zeta^{-1}$ as above. Taylor's theorem, 1.15 and (iv bis) imply, for $|w| < d$,

$$\begin{aligned} \rho_\zeta(w', w_n) &= \rho_\zeta(w', 0) + 2 \operatorname{Re}\left(\frac{\partial \rho_\zeta}{\partial w_n}(w', 0) \cdot w_n\right) + o(|w_n|) \\ &\geqq 2 \operatorname{Re} w_n + \gamma |w'|^m + o(1) \cdot |w_n| , \end{aligned}$$

where $o(1) \to 0$ as $w \to 0$, the convergence being uniform in ζ, as $o(1)$ depends only on the modulus of continuity of the first order partial derivatives of $r \circ F_\zeta^{-1}$.

Since $m \geqq 2$, $|w'|^m = |w|^m + o(1) \cdot |w_n|$; thus, if $0 < d^\# < d$ is chosen so small that the combined terms $o(1) \cdot |w_n|$ satisfy $o(1) \cdot |w_n| \leqq |\operatorname{Re} w_n| + |\operatorname{Im} w_n|$ for $|w| \leqq d^\#$, one obtains

$$(1.17) \quad -2 \operatorname{Re} w_n + |\operatorname{Re} w_n| \geqq -\rho_\zeta(w) - |\operatorname{Im} w_n| + \gamma |w|^m \text{ for } |w| \leqq d^\# ,$$

and hence

$$3 |\operatorname{Re} w_n| \geqq -\rho_\zeta(w) - |\operatorname{Im} w_n| + \gamma |w|^m .$$

Choose $c^\#$ so small that $F_\zeta(B(\zeta, c^\#)) \subset B(o, d^\#)$. If $z \in B(\zeta, c^\#)$, the last inequality and (1.14) imply

$$3 |\operatorname{Re} \phi(\zeta, z)| \geqq -r(z) - |\operatorname{Im} \phi(\zeta, z)| + \gamma A_1 |z - \zeta|^m .$$

Since $|r(z)| \gtrsim d(z, \partial D)$ for $z \in \bar{D}$ and $5|\phi| > 3 |\operatorname{Re} \phi| + 2 |\operatorname{Im} \phi|$, it follows that

$$|\phi(\zeta, z)| \gtrsim d(z, \partial D) + |\operatorname{Im} \phi(\zeta, z)| + \gamma A_1 |z - \zeta|^m$$

for $z \in \bar{D} \cap B(\zeta, c^\#)$.

2. The Carathéodory metric. The infinitesimal form C_D of the Carathéodory metric on a domain D in C^n is defined as follows: for $z \in D$ and $v \in C^n$,

$$C_D(z, v) = \sup \{| \langle \partial f(z), v \rangle |: f: D \longrightarrow \Delta, \text{ holomorphic}\} .$$

For a holomorphic map $F: D_1 \to D_2$ one trivially obtains

$$C_{D_2}(F(z), F_{*_z} v) \leqq C_{D_1}(z, v) ,$$

where F_{*_z} denotes the Jacobian matrix of F at z. Furthermore, by restricting $f: D \to \Delta$ to the ball $B(z, d(z, \partial D)) \subset D$ and applying Cauchy's derivative estimates, one obtains

LEMMA 2.1. *The Carathéodory metric satisfies*

$$C_D(z, v) \leqq |v| \, d(z, \partial D)^{-1}$$

for all $z \in D$ *and* $v \in C^n$.

The main result of this section is the following estimate from below for the Carathéodory metric.

THEOREM 2.2. *Let D be a bounded domain in C^n with C^1 boundary, and let r be a defining function for D defined on a neighborhood of $\bar{D}$. Suppose that D is uniformly totally pseudoconvex of finite order m at every point $P \in \partial D$, and that $\bar{D}$ has a Stein neighborhood basis*[1]. *Then*

$$C_D(z, v) \gtrsim |v| \, d(z, \partial D)^{-1/m} + |\langle \partial r(z), v \rangle| \, d(z, \partial D)^{-1}$$

for $z \in D$ *and* $v \in C^n$.

REMARK 2.3. $C_D(z, v)$ may grow faster than $d(z, \partial D)^{-1/m}$ for certain tangential vectors v, but in general, no better estimate is possible for *all* v. As an example, consider $D = \{z \in C^3: |z_1|^2 + |z_2|^2 + |z_3|^4 < 1\}$ and $P = (1, 0, 0) \in \partial D$; one can show that D is uniformly totally pseudoconvex of order 4 at P; for $v = (0, 1, 0)$ and $v^* = (0, 0, 1) \in H_P(\partial D)$ one obtains $C_D(z, v) \sim d(z, \partial D)^{-1/2}$ and $C_D(z, v^*) \sim d(z, \partial D)^{-1/4}$ as $z \to P$ along the inner normal to ∂D at P.

The proof of Theorem 2.2 involves a technical local result which we state separately. First we define, for $\zeta \in \partial D$, $\delta > 0$ and $\varepsilon > 0$,

$$\Omega(\zeta, \delta, \varepsilon) = (D \cap B(\zeta, \delta)) \cup \{z: \delta/2 < |z - \zeta| < \delta; r(z) < \varepsilon\} .$$

MAIN LEMMA 2.4. *Let D be uniformly totally pseudoconvex of order m at $P \in \partial D$. Then there are positive real numbers δ, ε, a, M, such that the following holds.*

For each $x \in D \cap B(P, \delta)$, if $\zeta_x \in \partial D \cap B(P, 2\delta)$ is chosen so that $|x - \zeta_x| = d(x, \partial D)$, there are

(i) *functions $h_1^x, \cdots, h_n^x$ defined and holomorphic on $\Omega(\zeta_x, \delta, \varepsilon)$,*

(ii) *an orthonormal basis $v_1^x, \cdots, v_n^x$ of C^n with v_n^x perpendicular to $H_{\zeta_x}(\partial D)$,*

which satisfy the following conditions:

(iii) $|h_j^x(z)| \leqq M$ *for* $z \in \Omega(\zeta_x, \delta, \varepsilon)$, $j = 1, \cdots, n$;

(iv) $|\langle \partial h_j^x(x), v_j^x \rangle| \gtrsim d(x, \partial D)^{-1/m}$ *for* $d(x, \partial D) < a$ *and* $f = 1, \cdots, n - 1$;

(v) $|\langle \partial h_n^x(x), v_n^x \rangle| \gtrsim d(x, \partial D)^{-1}$ *for* $d(x, \partial D) < a$.

We first show how the Main Lemma implies the theorem.

Proof of 2.2. Fix $P \in \partial D$; for $x \in D \cap B(P, \delta)$ let $h_1^x, \cdots, h_n^x$ be the functions given by the Main Lemma. The essential part of the proof involves replacing these functions by functions H_j^x, $j = 1, \cdots, n$, which are holomorphic on D and still satisfy properties (iii), (iv), and (v) above.

Choose $\chi \in C^\infty(\boldsymbol{R})$ such that $0 \leqq \chi \leqq 1$ and

$$\chi(t) = \begin{cases} 1 \text{ for } t \leqq 5\delta/8 \\ 0 \text{ for } t \geqq 7\delta/8 \,; \end{cases}$$

define, for $\zeta \in \boldsymbol{C}^n$, the function $\chi_\zeta \in C_0^\infty(\boldsymbol{C}^n)$ by

$$\chi_\zeta(z) = \chi(|z - \zeta|) \,.$$

Now fix $x \in D \cap B(P, \delta)$ with $d(x, \partial D) < \delta/2$; to simplify notation, we will omit the superscript x in h_j^x and v_j^x, and we set $\zeta = \zeta_x$; unless otherwise noted, the index j runs from 1 to n.

Set $\alpha_j = \bar\partial(\chi_\zeta h_j)$ on $\Omega(\zeta, \delta, \varepsilon)$; α_j extends trivially as a $\bar\partial$-closed $C_{0,1}^\infty$ – form to the domain $D_\varepsilon = D \cup \{z : r(z) < \varepsilon\}$. Choose a Stein domain G, such that $D \subset\subset G \subset D_\varepsilon$. By Hörmander [10], there are functions $u_j \in C^\infty(G)$, such that $\bar\partial u_j = \alpha_j$ and

$$\|u_j\|_{L^2(G)} \lesssim \|\alpha_j\|_{L^2(G)} \lesssim \sup_{z \in \Omega(\zeta, \delta, \varepsilon)} |h_j(z)| \leqq M \,.$$

By interior elliptic estimates for $\bar\partial$,

$$\sup_{z \in D} |u_j(z)| \lesssim \|u_j\|_{L^1(G)} + \|\bar\partial u_j\|_{L^\infty(G)} \leqq K_1 \cdot M \,. \tag{2.5}$$

Define $H_j = \chi_\zeta h_j - u_j$; H_j is holomorphic on D, and by 2.4 (iii) and (2.5)

$$|H_j(z)| \leqq (1 + K_1)\cdot M = M' \text{ for } z \in D .$$

Furthermore, observe that u_j is holomorphic on $G \cap B(\zeta, 5\delta/8)$; since for $z \in D \cap B(\zeta, \delta/2)$ and $w \notin G \cap B(\zeta, 5\delta/8)$, $|z - w| \geqq \min(\delta/8, \text{dist}(D, C^n - G)) > 0$, it follows that for some constant K_2,

$$|\partial u_j(z)| \lesssim ||u_j||_{L^2(G)} \leqq K_2 \cdot M \text{ for } z \in D \cap B(\zeta, \delta/2) .$$

From $\partial H_j(x) = \partial h_j(x) - \partial u_j(x)$ one thus obtains

$$|\langle \partial H_j(x), v_j \rangle| \geqq |\langle \partial h_j(x), v_j \rangle| - K_2 \cdot M ,$$

which implies that (iv) and (v) in 2.4 still hold with H_j instead of h_j, provided $d(x, \partial D) < a'$, where $0 < a' \leqq a$ is suitably chosen. Since $H_j/M' \colon D \to \Delta$, one has $C_D(x, v_j) \geqq 1/M' |\langle \partial H_j(x), v_j \rangle|$; it follows that

$$C_D(x, v_j) \gtrsim d(x, \partial D)^{-1/m} , \quad j = 1, \cdots, n-1 ,$$

and

$$C_D(x, v_n) \gtrsim d(x, \partial D)^{-1} .$$

By 2.4 (ii), this implies

$$C_D(x, v) \gtrsim |v| \, d(x, \partial D)^{-1/m} + |\langle \partial r(x), v \rangle| \, d(x, \partial D)^{-1}$$

for all $v \in C^n$; here x is any point in $D \cap B(P, \delta)$ with $d(x, \partial D) < a'$, and the constant implicit in $\gtrsim$ is independent of x. A standard compactness argument now shows that the above estimate holds for all $x \in D$.

Proof of the Main Lemma. The plan of the proof is as follows: one first constructs the required functions and vectors with respect to the coordinate system $w = F_{\zeta_x}(z)$ given in § 1, and then one pulls back everything to the domain D.

We use the notation developed in § 1. $\delta > 0$ is chosen so small that for all $\zeta \in \partial D \cap B(P, 2\delta)$ the biholomorphic map F_ζ is defined on $B(\zeta, \delta)$ and $\rho_\zeta = r \circ F_\zeta^{-1}$ satisfies (1.17) for $|w| < d^\#$, i.e.,

$$-2 \operatorname{Re} w_n + |\operatorname{Re} w_n| + |\operatorname{Im} w_n| \geqq -\rho_\zeta(w) + \gamma |w|^m . \tag{2.6}$$

By (1.14), if δ is chosen sufficiently small, there is $b > 0$, such that

$$F_\zeta(\{z \colon \delta/2 < |z - \zeta| < \delta\}) \subset \{w \colon b < |w| < d^\#\} .$$

Let $\varepsilon = \gamma b^m/2$ and define

$$R(\zeta) = \{w \colon |w| < d^\#, \ \rho_\zeta(w) < 0\} \cup \{w \colon b < |w| < d^\#, \ \rho_\zeta(w) < \varepsilon\} ;$$

observe that $F_\zeta(\Omega(\zeta, \delta, \varepsilon)) \subset R(\zeta)$.

(2.6) shows that for $w \in R(\zeta)$ the function w_n omits the nonnegative real axis; hence one can define a holomorphic branch of $w_n^{1/m}$ on $R(\zeta)$. For $j = 1, \cdots, n-1$ we define holomorphic functions g_j on $R(\zeta)$ by $g_j(w) = w_j \cdot w_n^{-1/m}$; then

$$(2.7) \qquad \left(\frac{\partial}{\partial w_j} g_j\right)(w) = w_n^{-1/m}\,, \quad j = 1, \cdots, n-1\,.$$

From (2.6) one obtains

$$4|w_n| \geqq \varepsilon \quad \text{if} \quad b < |w| < d^* \quad \text{and} \quad \rho_\zeta(w) < \varepsilon\,;$$
$$4|w_n| \geqq \gamma |w|^m \geqq \gamma |w_j|^m \quad \text{if} \quad |w| < d^* \quad \text{and} \quad \rho_\zeta(w) < 0\,;$$

this implies that there is a constant M such that

$$(2.8) \qquad |g_j(w)| \leqq M \text{ for } w \in R(\zeta)\,, \quad j = 1, \cdots, n-1\,.$$

In order to define g_n we modify the function $f(w) = \exp(-\sqrt{-w_n})$ which was used in the proof of Corollary 1.5; f is well defined and holomorphic on $R(\zeta)$, $|f(w)| < 1$ for $w \in R(\zeta)$ and $f(w) \to 1$ for $w \to 0$. Fix $y \in R(\zeta)$ and let φ_q be the holomorphic automorphism of Δ which sends $q = f(y)$ to 0 and 1 to 1. Since $|\varphi_q'(q)| = (1 - |q|^2)^{-1}$ and

$$1 - |q| = 1 - |f(y)| = 1 - \exp(-\operatorname{Re}\sqrt{-y_n}) \lesssim |\sqrt{y_n}|\,,$$

it follows that

$$|\varphi_q'(q)| \gtrsim |y_n|^{-1/2}\,.$$

Therefore, if one defines $g_n^y = \varphi_{f(y)} \circ f$, one obtains, by the chain rule,

$$(2.9) \qquad \left|\frac{\partial g_n^y}{\partial w_n}(y)\right| \gtrsim |y_n|^{-1}\,,$$

and

$$(2.10) \qquad |g_n^y(w)| < 1 \text{ for } w \in R(\zeta)\,.$$

Now fix $x \in D \cap B(P, \delta)$ and choose $\zeta_x \in \partial D \cap B(P, 2\delta)$ such that $|x - \zeta_x| = d(x, \partial D)$. Let $y = F_{\zeta_x}(x)$, and set $h_j^x = g_j \circ F_{\zeta_x}$ for $j = 1, \cdots, n-1$, $h_n^x = g_n^y \circ F_{\zeta_x}$. Then, by (2.8) and (2.10), conditions 2.4 (i) and (iii) are satisfied. From the explicit form of the matrix $(F_\zeta)_{*\zeta}$ (cf. the definition of F_ζ in § 1), it follows that the vectors $v_j^x = (F_{\zeta_x}^{-1})_{*0}(\partial/\partial w_j)$, $j = 1, \cdots, n$, satisfy 2.4 (ii). Let $t_j^x = (F_{\zeta_x}^{-1})_{*y}(\partial/\partial w_j)$; then

$$v_j^x = t_j^x + o(|x - \zeta_x|) = t_j^x + o(d(x, \partial D))\,,$$

and therefore there is $K < \infty$, such that

$$|\langle \partial h_j^x(x), v_j^x - t_j^x \rangle| \leqq K$$

whenever $d(x, \partial D)$ is sufficiently small. Also, $y_n = \phi(x, \zeta_x)$ implies $|y_n| \lesssim |x - \zeta_x| = d(x, \partial D)$. Hence, there is $a > 0$, such that for all $x \in D \cap B(P, \delta)$ with $d(x, \partial D) < a$ one obtains, by (2.7),

$$|\langle \partial h_j^x(x), v_j^x \rangle| \geqq |\langle \partial h_j^x(x), t_j^x \rangle| - K = \left| \frac{\partial g_j}{\partial w_j}(y) \right| - K$$
$$\gtrsim d(x, \partial D)^{-1/m} \text{ for } j = 1, \cdots, n-1 ,$$

and, by (2.9),

$$|\langle \partial h_n^x(x), v_n^x \rangle| \geqq \left| \frac{\partial g_n^y}{\partial w_n}(y) \right| - K \gtrsim d(x, \partial D)^{-1} .$$

This completes the proof of the Main Lemma.

3. Proper holomorphic maps.

LEMMA 3.1. *Let D_1 and D_2 be bounded pseudoconvex domains in $\boldsymbol{C}^n$ with smooth boundary. Then there is a positive integer l such that every proper holomorphic map $F: D_1 \to D_2$ satisfies*

$$d(z, \partial D_1)^l \lesssim d(F(z), \partial D_2) \lesssim d(z, \partial D_1)^{1/l}$$

for all $z \in D_1$.

Proof. By a theorem of Diederich and Fornaess [3], there are continuous functions $\phi_\nu: \bar{D}_\nu \to \boldsymbol{R}$, $\nu = 1, 2$, with the following properties:

(i) $\varphi_\nu | D_\nu$ is smooth and plurisubharmonic;
(ii) $\varphi_\nu | D_\nu < 0$ and $\varphi_\nu | \partial D_\nu = 0$;
(iii) for some $l \in N$, $(\varphi_\nu)^l$ is smooth on $\bar{D}_\nu$.

iii implies

$$|\varphi_\nu(x)| \lesssim d(x, \partial D_\nu)^{1/l} \text{ for } x \in D_\nu . \tag{3.2}$$

Let $\psi_1 = \varphi_2 \circ F$. Since F is proper, ψ_1 is continuous on $\bar{D}_1$, and it satisfies (i) and (ii) with respect to D_1.

In order to push forward φ_1, observe that $F: D_1 \to D_2$ represents D_1 as a λ-sheeted branched analytic covering over D_2. Define ψ_2 on D_2 by

$$\psi_2(w) = \max \{\varphi_1(z_1), \cdots, \varphi_1(z_\lambda)\} ,$$

where $\{z_1, \cdots, z_\lambda\} = F^{-1}(w)$, counted with multiplicities. ψ_2 is continuous on $\bar{D}_2$ and plurisubharmonic on D_2 (cf. [16], p. 646); also, $\psi_2 | D_2 < 0$ and $\psi_2 | \partial D_2 = 0$.

The classical normal derivative lemma [11], also known as Hopf lemma (cf. [2]), implies

$$|\psi_\nu(x)| \gtrsim d(x, \partial D_\nu) \text{ for } x \in D_\nu, \nu = 1, 2 . \tag{3.3}$$

By combining (3.2) and (3.3) one obtains

$$d(z, \partial D_1) \lesssim |\psi_1(z)| = |\varphi_2(F(z))| \lesssim d(F(z), \partial D_2)^{1/l}$$

and

$$d(F(z), \partial D_2) \lesssim |\psi_2(F(z))| \leqq |\varphi_1(z)| \lesssim d(z, \partial D_1)^{1/l} .$$

THEOREM 3.4. *Let D_1 and D_2 be bounded domains in C^n with smooth boundary. Suppose there is $\delta>0$ such that the Carathéodory metric of D_2 satisfies*

$$(*) \qquad C_{D_2}(w, v) \gtrsim |v|\, d(w, \partial D_2)^{-\delta}$$

for all $w \in D_2$ and $v \in C^n$. Then there is $\alpha > 0$, such that every proper holomorphic map $F\colon D_1 \to D_2$ is Hölder continuous of order α, i.e., *there is $K < \infty$, such that*

$$|F(z) - F(z^*)| \leqq K|z - z^*|^\alpha \text{ for all } z, z^* \in D_1 .$$

Proof. The hypothesis (*) implies that D_2 is a domain of holomorphy, hence pseudoconvex. If there is a proper holomorphic map $F\colon D_1 \to D_2$, then D_1 must be pseudoconvex also, and hence 3.1 holds. So one has all the ingredients which are required to apply the classical argument of Henkin and Pinchuk. As the argument is very short, we include it here for the convenience of the reader.

By applying (*) to $F(z)$ and $F_{*_z}v$, and by Lemma 2.1, one obtains

$$|F_{*_z}v|\, d(F(z), \partial D_2)^{-\delta} \lesssim C_{D_2}(F(z), F_{*_z}v) \leqq C_{D_1}(z, v) \leqq |v|\, d(z, \partial D_1)^{-1} ;$$

by multiplying with $d(F(z), \partial D_2)^\delta$ and 3.1,

$$|F_{*_z}v| \lesssim |v|\, d(z, \partial D_1)^{-1+\delta/l} ,$$

i.e.,

$$\|F_{*_z}\| \lesssim d(z, \partial D_1)^{-1+\alpha}, \text{ with } \alpha = \delta/l > 0 .$$

The analogue of a classical result of Hardy and Littlewood now implies that F is Hölder continuous of order α.

Theorem 2.2. and Theorem 3.4 clearly imply the Main Theorem stated in the introduction. From Corollary 1.13 one obtains the following special case of the Main Theorem.

COROLLARY 3.5. *Let D_1 and D_2 be bounded convex domains in C^n with real analytic boundary. Then every biholomorphic map $F\colon D_1 \to D_2$ extends to a homeomorphism $\hat{F}\colon \bar{D}_1 \to \bar{D}_2$.*

Open Problems 3.6. Some natural questions arise at this point. First, one would expect that the extension $\hat{F}$ in 3.5 is differentiable, or even real analytic, up to the boundary. Next, one may ask whether Corollary 3.5 remains true if one only assumes that D_1 and D_2 are bounded pseudoconvex domains with smooth boundary. Finally, one may consider similar questions for proper holomorphic maps; specifically, can the Main Theorem be strengthened to yield a differentiable extension to the boundary? It appears that methods quite different from those used in this paper would be needed to attack any of these problems.

REFERENCES

1. H. Behnke and P. Thullen, *Theorie der Funktionen mehrerer komplexer Veränderlichen*, 2nd rev. ed., Springer-Verlag, Berlin-Heidelberg-New York, 1970.
2. L. Bers, F. John and M. Schechter, *Partial Differential Equations*. Interscience, New York, 1966.
3. K. Diederich, and J. E. Fornaess, *Exhaustion functions and Stein neighborhoods for smooth pseudoconvex domains*. Proc. Nat. Acad. Sci. U.S.A., **72** (1975), 3279-3280.
4. ———, *A strange bounded smooth domain of holomorphy*, Bull. Amer. Math. Soc., **82** (1976), 74-76.
5. C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math., **26** (1974), 1-65.
6. I. Graham, *Boundary behavior of the Carathéodory and Kobayashi metrics on strongly pseudoconvex domains in C^n with smooth boundary*, Trans. Amer. Math. Soc., **207** (1975), 219-240.
7. R. C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, N.J., 1965.
8. M. Hakim and N. Sibony, *Frontière de Silov et spectre de $A(D)$ pour des domaines faiblement pseudoconvexes*, C. R. Acad. Sci. Paris, **281** (1975), A959-A962.
9. G. M. Henkin, *An analytic polyhedron is not holomorphically equivalent to a strictly pseudoconvex domain*, Dokl. Akad. Nauk SSSR, **210** (1973), 1026-1029=Soviet Math. Dokl., **14** (1973), 858-862.
10. L. Hörmander, *L^2 estimates and existence theorems for the $\bar{\partial}$ operator*, Acta Math., **113** (1965), 89-152.
11. M. Keldyš and M. Lavrent'ev, *Sur l'unicité de la solution du problème de Neumann*, C. R. (Dokl.) Acad. Sci. URSS, **26** (1937), 141-142.
12. J. J. Kohn, *Global regularity for $\bar{\partial}$ on weakly pseudoconvex manifolds*, Trans. Amer. Math. Soc., **181** (1973), 273-292.
13. J. J. Kohn and L. Nirenberg, *A pseudoconvex domain not admitting a holomorphic support function*, Math. Ann., **201** (1973), 265-268.
14. S. Lojasiewicz, *Sur le problème de la division*, Studia Math., **18** (1959), 87-136.
15. P. Pflug, *Über polynomiale Funktionen auf Holomorphiegebieten*, Math. Z., **139** (1974), 133-139.
16. S. J. Pinchuk, *On proper holomorphic mappings of strictly pseudoconvex domains*, Siberian Math. J., **15** (1975), 644-649.
17. R. M. Range, *On the topological extension to the boundary of biholomorphic maps in C^n*, Trans. Amer. Math. Soc., **216** (1976), 203-216.
18. ———, *Hölder estimates for $\bar{\partial}$ on convex domains in C^2 with real analytic boundary*, Proc. Symp. Pure Math., **30**, part 2, 31-33. Amer. Math. Soc., Providence, R. I., 1977.

19. H. Rossi, *Holomorphically convex sets in several complex variables*, Ann. Math., **74** (1961), 470–493.
20. N. Vormoor, *Topologische Fortsetzung biholomorpher Funktionen auf dem Rande bei beschränkten streng pseudokonvexen Gebieten im C^n mit C^∞-Rand*, Math. Ann., **204** (1973), 239-261.

Received September 29, 1977. Research supported by NSF Grant MCS75-07062A01.

STATE UNIVERSITY OF NEW YORK
ALBANY, NY 12222