

EMBEDDINGS AND BRANCHED COVERING SPACES FOR THREE AND FOUR DIMENSIONAL MANIFOLDS

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1. **Introduction.** The main purpose in writing this paper is to point out a connection between embeddings of manifolds and branched covering spaces of manifolds. The following theorem is a corollary to Theorems 3, 4, and 5, and can be regarded as the main result of this paper.

THEOREM. *Let $p: M^n \rightarrow S^n$, $n = 3$ or 4 , be a 3-fold dihedral branched covering space branched over a polyhedral knot or link if $n = 3$, or a closed orientable polyhedral surface, if $n = 4$.*

Then there is a locally flat embedding $e: M^n \rightarrow S^n \times S^2$ such that the following diagram commutes.

$$\begin{array}{ccc}
 & & S^n \times S^2 \\
 & \nearrow e & \downarrow \\
 M^n & \xrightarrow{p} & S^n
 \end{array}$$

It is a result of the author and José M. Montesinos ([2], [5]) that every closed orientable 3-manifold is a three fold dihedral covering of S^3 branched over a knot or link. Indeed, this can be done in a wide variety of ways satisfying various side conditions ([3]).

This result, together with the above theorem can be viewed as saying that every closed orientable 3-manifold and certain closed orientable 4-manifolds are topologically like Riemann surfaces.

Indeed, given such an $M^{3 \text{ or } 4}$ there is an S^2 multivalued function f (see §4) defined on $S^{3 \text{ or } 4}$ such that $M^{3 \text{ or } 4}$ is the graph of f . Moreover, locally the singularities of f look like $(x, z) \rightarrow \sqrt{z}$ or $(x_1, x_2, z) \rightarrow \sqrt{z}$.

It is unknown which closed orientable 4-manifolds can be 3-fold dihedral covering spaces of S^4 branched over orientable surfaces. But Montesinos ([7]) has recently shown that a large and important class of four manifolds with boundary are three fold dihedral coverings of D^4 , branched over locally flat, but not necessarily orientable, properly embedded surfaces. On the other hand, it is a result of Edmonds and Berstein that $S^1 \times S^1 \times S^1 \times S^1$ and many other closed orientable four manifolds cannot be threefold branched covering

spaces of S^4 at all. Also, not every closed orientable four manifold W^4 is a 3-fold dihedral branched covering space of S^4 branched over a locally flat orientable surface because not every closed oriented W^4 embeds in $S^4 \times S^2$. (For example, if CP^2 embeds in $S^4 \times S^2$ then $\tau(CP^2) \oplus \nu(CP^2)$ is trivial because $S^4 \times S^2$ is almost parallelizable. This implies $p(\tau(CP^2))p(\nu(CP^2)) = 1$ where p stands for the total Pontrjagin class. But $p(\tau(CP^2)) = 1 + 3\alpha^2$, and $p(\nu(CP^2)) = \chi(\nu(CP^2))^2$ where χ means Euler class, and α generates $H^2(CP^2)$. (See [4].) This is impossible.

2. **Definitions, notations, and standing assumptions.** We shall work in the piecewise linear category throughout the paper. All manifolds and maps will be assumed *PL* without it being explicitly stated. Sometimes, such as when we refer to k -skeleta, we shall assume fixed triangulations and simplicial maps.

Given a nondegenerate simplicial map $p: X \rightarrow Y$ between n -manifolds, $p: X \rightarrow Y$ is said to be a branched covering space if the restriction to the complements of the $n - 2$ skeletons is a covering space in the usual sense. The set of points C in X at which p fails to be a local homeomorphism is called the singular set. The set $B = p(C)$ is called the branch set. The branch covering space is k -fold if the associated covering space map is k to one. Since our results are well known or trivial if $\dim X = \dim Y = 2$ we shall assume throughout this paper that $n \geq 3$.

A k -fold branched covering space $p: X \rightarrow Y$ determines an equivalence class of representations $\rho: \pi_1(Y - B) \rightarrow \Sigma_k$, where Σ_k is the permutation group on k letters. The representation ρ corresponds to the permutation induced on the left cosets of $p_*\pi_1(X - p^{-1}(B))$ by left multiplication.

Conversely, a representation $\rho: \pi_1(Y - B) \rightarrow \Sigma_k$ determines a branched covering space $p: X \rightarrow Y$. The subgroup of the associated covering space is the inverse image of the group of permutations fixing some particular letter. If $k \geq 3$ and ρ is surjective, the covering is not regular because the subgroup is not normal. (Details on branched covering spaces and the representations that induce them are given in [1] and [8].)

Let D be a 2-disc in Y that intersects B exactly once, transversally, in its interior. Then D is called a meridian disc and any element of $\pi_1(Y - B)$ conjugate to the path that travels once around the boundary of D is called a meridian.

A threefold dihedral branched covering space is one in which meridians are mapped into transpositions by the representation ρ .

We shall be concerned in the sequel with a particular three dimensional manifold. Let $R(\hat{R})$ be the set of unordered (ordered)

triples of points in S^2 that lie on a great circle and at the vertices of an equilateral triangle. Then $q: \hat{R} \rightarrow R$ is a regular 6-fold covering space. Note that \hat{R} is homeomorphic to $SO(3)$. (The group $SO(3)$ acts transitively on \hat{R} and the isotropy subgroup of a point is trivial.)

Let $\{A, B, C\}((A, B, C))$ be a fixed base point in $R(\hat{R})$. Let $R_A = \{(D, E, F) \in R \mid \text{One of } D, E, \text{ or } F \text{ equals } A.\}$ and let R_B and R_C be defined similarly. Then $R_A, R_B,$ and R_C are all homeomorphic to S^1 and $R_A \cap R_B \cap R_C = \{A, B, C\}$.

We shall denote the natural maps $Z/4Z \rightarrow Z/2Z$, and $\Sigma_3 \rightarrow Z/2Z$ by α and β respectively and we shall denote the pullback $\{(x, y) \in Z/4Z \oplus \Sigma_3 \mid \alpha(x) = \beta(y)\}$ by G . Given a commutative diagram of groups indicated by the solid lines below, the homomorphism indicated by the dotted lines exists, is unique, and the diagram is still commutative. The natural maps γ and δ are indicated in the diagram

$$(1) \quad \begin{array}{ccc} G^1 & \longrightarrow & \Sigma_3 \\ \downarrow & \searrow & \nearrow \gamma \\ & G & \\ \downarrow \delta & & \downarrow \beta \\ Z/4Z & \xrightarrow{\alpha} & Z/2Z \end{array}$$

All homology and cohomology groups are assumed to have Z coefficients unless otherwise stated.

3. The homotopy groups of R .

LEMMA 1. *We have the following computations for the homotopy groups of R :*

$$\pi_1(R) = G, \quad \pi_2(R) = 0, \quad \pi_3(R) = Z.$$

Proof. The last two statements follow from the fact, mentioned in §2, that R is covered by $\hat{R} = SO(3)$ which is covered by S^3 .

There is a natural homomorphism $\theta: \pi_1(R; \{A, B, C\}) \rightarrow \Sigma_3 =$ permutations of the letters $A, B,$ and C . The homomorphism $\beta\theta$ maps $\pi_1(R; \{A, B, C\})$ onto $Z/2Z$. It is clear from the covering spaces that order $\pi_1(R)$ is twelve. If we let x equal continuous rotation through 120° about an axis perpendicular to the plane through $\{A, B, C\}$ then x belongs to kernel $\beta\theta$ and has order six. (Note that x^3 is the nontrivial element of $\pi_1(SO(3))$.) Thus we have the exact sequence:

$$(2) \quad 0 \longrightarrow Z/6Z \longrightarrow \pi_1(R) \longrightarrow Z/2Z \longrightarrow 0.$$

There is, up to isomorphism, only one noncommutative group satisfying such an exact sequence. To see this, let x generate $Z/6Z$ and let b be an element mapped into the nontrivial element of $Z/2Z$. Then $bx b^{-1} = x^r$ and the group structure is completely determined by the value of r . Since $bx b^{-1}$ also generates $Z/6Z$, $r = \pm 1$. Since the group is noncommutative $r = -1$. Thus $\pi_1(R) = G$ since G also satisfies this exact sequence. (The element $(2, (A, B, C))$ generates the kernel.)

4. **The R -value function f .** In this section we shall begin the construction of a function f defined on the complement of a tubular neighborhood of the branch set and taking values in R . It is useful to think of f as a multivalued function such as, for example, an algebraic function on the Riemann sphere.

We note that if B is a codimension two, locally flat, properly embedded submanifold of the manifold Y , then a regular neighborhood of B is a closed tube neighborhood of B . This is because a regular neighborhood of B is the total space of a 2-dimensional block bundle over B and a 2-dimensional block bundle over B is a linear disk bundle (see [9] for details and definitions).

We shall need the following.

LEMMA 2. *Let $p: X \rightarrow Y$ be a threefold dihedral branched covering space of compact orientable n -manifolds. Suppose the branch set B is a properly embedded, locally flat, $n - 2$ dimensional orientable manifold. Assume $H_1(Y) = H_2(Y) = 0$. Let T be a closed tubular neighborhood of B and let $\rho: \pi_1(Y - \text{int } T; y_0) \rightarrow \Sigma_3$ be the representation induced by the covering space.*

Then there is a map $\sigma: \pi_1(Y - \text{int } T; y_0) \rightarrow G = \pi_1(R, \{A, B, C\})$ such that $\gamma\sigma = \rho$ and there is a map $f: (2\text{-skeleton}(Y - \text{int } T)) \cup \partial T \rightarrow R$ such that $f_*: \pi_1(Y - \text{int } T; y_0) \rightarrow \pi_1(R; \{A, B, C\})$ equals σ and such that f restricted to a component of ∂T takes values in one of the submanifolds R_A, R_B, R_C . Also, f takes any meridian circle of ∂T into a generator of the homology of R_A, R_B , or R_C .

Proof. It follows from Alexander duality that $H^{n-2}(B, \partial B) = H_2(Y - \partial B, Y - B)$ and it follows from the exact homology sequence of the pair $(Y - \partial B, Y - B)$ and the assumption $H_1(Y) = H_2(Y) = 0$ that $H_2(Y - \partial B, Y - B) = H_1(Y - B)$. (Note $Y - \partial B \cong Y$.) Thus $H_1(Y - B)$ is a free abelian group generated by meridians. It has as many generators as there are components of B .

Now let T be a tubular neighborhood of B in Y . We have the representation $\beta\rho: \pi_1(Y - \text{int } T) \rightarrow Z/2Z$ which factors through

$H_1(Y - \text{int } T)$. Let $\varphi: H_1(Y - \text{int } T) \rightarrow Z$ be defined by sending each meridinal generator into ± 1 . Making a choice of $+1$ or -1 for each generator is equivalent to choosing an orientation for Y and each component of B . Let $\varepsilon: Z \rightarrow Z/4Z$ and $\kappa: \pi_1 \rightarrow H_1$ be the natural maps. We then have the commutative diagram given by the solid lines below. It is the assumption that meridians are sent to transpositions that makes this diagram commutative. The homomorphism σ exists because, as explained in § 2, G is the pullback of $Z/4Z$ and Σ_3 .

$$(3) \quad \begin{array}{ccc} \pi_1(Y - \text{int } T; y_0) & \xrightarrow{\rho} & \Sigma_3 \\ \varepsilon\varphi\kappa \downarrow & \sigma \nearrow & \downarrow \beta \\ & G & \\ \delta \swarrow & & \searrow \gamma \\ Z/4Z & \xrightarrow{\alpha} & Z/2Z \end{array}$$

Now we must construct the function f . We begin by choosing disjoint arcs from y_0 to each component of ∂T and letting Arcs be union of these arcs. Let B_1 be one of the components of B , let T_1 and ∂T_1 be the corresponding components of T and ∂T and let $y_1 = \text{Arcs} \cap \partial T_1$. We obtain by restriction a representation (which we also by σ) from $\pi_1(\partial T_1; y_1) \rightarrow G = \pi_1(R; \{A, B, C\})$.

Since T_1 is a tubular neighborhood of B_1 , it follows that ∂T_1 is an S^1 bundle over B_1 and the top row of the following diagram is a segment of the exact homotopy sequence of this fibration. The other maps are natural or are induced by inclusion

$$(4) \quad \begin{array}{ccccc} \pi_1(S^1) & \xrightarrow{e} & \pi_1(\partial T_1) & \longrightarrow & \pi_1(B_1) \longrightarrow 0 \\ \downarrow d & & \downarrow & & \\ H_1(S^1) & \xrightarrow{a} & H_1(\partial T_1) & & \\ & \searrow c & \downarrow b & & \\ & & H_1(Y - \text{int } T) & & \end{array}$$

The fiber S^1 is a meridian, so c is injective and therefore a is injective. Since d is an isomorphism, it follows that e is injective.

Let x generate $\pi_1(S; y_1)$ and let y be any element of $\pi_1(\partial T_1; y_1)$ then y must commute with x because $yx y^{-1} \in \pi_1(S^1)$ and $d(x) = d(yx y^{-1})$. Thus $\sigma(x)$ commutes with $\sigma(y)$ and $\rho(x)$ commutes with $\rho(y)$. Since $\rho(x)$ is a transposition we see that $\rho(\pi_1(\partial T_1; y_1))$ is a cyclic group of order two and $\sigma(\pi_1(\partial T_1; y_1))$ is a cyclic group of order four. This cyclic group of order 4 is mapped isomorphically onto $Z/4Z$ by δ . The representation σ factors through $H_1(\partial T_1) \rightarrow H_1(Y - \text{int } T) \rightarrow Z$. Suppose $\rho(\pi_1(\partial T_1; y_1)) = \{1, (B, C)\}$. Then since $R_A = S^1$ and $\pi_1(S^1) = 0$

for $i \geq 2$ we can find a map $f: \partial T_1 \rightarrow R_A \subset R$ such that $f_*: (\partial T_1; y_1) \rightarrow \pi_1(R; \{A, B, C\})$ equals σ . Note that f takes any meridian in ∂T_1 into a generator of $H_1(R_A)$.

We define the map f in a similar way on the other components of ∂T and we map the set Arcs into the point $\{A, B, C\}$.

We can now just use the definition of σ to extend the map f to the rest of the 1-skeleton and to the 2-skeleton so the proof of this lemma is complete.

To extend the function f from the 2-skeleton to all of $Y - \text{int } T$ will require some additional assumptions about dimension. We do this in the next two theorems.

THEOREM 3. *Given $p: X \rightarrow Y$ as in Lemma 2, if $n = 3$, then the map f extends to all of $Y - \text{int } T$.*

Proof. By Lemma 1, $\pi_2(R) = 0$ so we can extend to the 3-skeleton.

THEOREM 4. *Given $p: X \rightarrow Y$ as in Lemma 2, if $n = 4$, then the map f extends to all of $Y - \text{int } T$.*

Proof. Let H be the subgroup of $\pi_1(Y - \text{int } T)$ equal to the kernel of $\sigma: \pi_1(Y - \text{int } T) \rightarrow G = \pi_1(R)$ and let \hat{Y} be the covering space. We have the following commutative diagram of twelve fold covering spaces

$$(5) \quad \begin{array}{ccc} \hat{Y} & \xrightarrow{f} & S^3 \\ \downarrow \pi & & \downarrow \pi \\ Y - \text{int } T & \xrightarrow{f} & R. \end{array}$$

The map $f(\hat{f})$, the lift of f is only defined on $\partial T(\pi^{-1}\partial T)$ and the 2-skeleton of $Y - \text{int } T$ (of \hat{Y}).

As in Theorem 3, the map $f(\hat{f})$ can be extended to the 3-skeleton but there is an obstruction $O(\hat{O})$ belonging to $H^4(Y - \text{int } T, \partial T; \pi_3(R))(H^4(\hat{Y}), \pi^{-1}\partial T; \pi_3(S^3))$ to the extension of $f(\hat{f})$. Since $\pi: \hat{Y} \rightarrow (Y - \text{int } T)$ is just a covering space map, the induced map on the fourth cohomology is groups multiplication by twelve and O vanishes if and only if \hat{O} does.

The obstruction \hat{O} is a primary obstruction. To compute it we only need to know \hat{f} restricted to $\pi^{-1}(\partial T)$. According to Theorem 17 of [4] p. 431, \hat{f} can be extended if and only if $\delta \hat{f}^*(i) = 0$ in $H^4(\hat{Y}, \pi^{-1}(\partial T))$ where i is a generator of $\pi_3(S^3) = \mathbb{Z}$.

The map \hat{f} restricted to $\pi^{-1}(\partial T)$ factors through $y: \pi^{-1}(R_A \cup R_B \cup R_C) \rightarrow S^3$. Since $\pi^{-1}(R_A \cup R_B \cup R_C)$ is a 1-complex we see that

$\hat{f}^*(i) = 0$ and \hat{f} extends. Thus \hat{O} vanishes, so does O , and we are done. We note that f may have to be redefined on the 3-skeleton before extending it.

5. The embedding theorem. In this section we prove the following

THEOREM 5. *Let $p: X \rightarrow Y$ be a threefold dihedral branched covering space of compact three or four dimensional manifolds. Let the branch set B be a properly embedded locally flat orientable submanifold of Y . Assume also $H_1(Y) = H_2(Y) = 0$.*

Then there is a locally flat embedding $e: X \rightarrow Y \times S^2$ such that the following diagram is commutative

$$(6) \quad \begin{array}{ccc} & Y \times S^2 & \\ e \nearrow & & \downarrow \pi \\ X & \xrightarrow{p} & Y \end{array}$$

Proof. Let T be a tubular neighborhood of B . By Theorems 3 and 4 there is a map $f: Y - \text{int } T \rightarrow R$ such that $\gamma f_*: \pi_1(Y - \text{int } T; y_0) \rightarrow \pi_1(R; \{A, B, C\}) \rightarrow \Sigma_3$ is the representation associated to the covering and f maps each component of ∂T into R_A, R_B , or R_C . Let $x_0 \in p^{-1}(y_0)$ be the base point defining the covering space. That is $p_*\pi_1(X - p^{-1}(\text{int } T); x_0) = \rho^{-1}\{1, (B, C)\}$.

We next define a function $h: X - p^{-1}(\text{int } T) \rightarrow S^2$. Define $h(x_0) = A$. If x is any point in $X - p^{-1}(\text{int } T)$ let $x(t)$ be a path from x_0 to x . Lift the path $f(x(t))$ which begins at $\{A, B, C\}$ in R to a path beginning at (A, B, C) in \hat{R} and define $h(x)$ to be the first coordinate of the endpoint of this path. By checking definitions we see that h is well defined and that if $f(x) = \{D, E, F\}$, then $h(p^{-1}\{x\}) = \{D, E, F\}$. It follows that the map $(p, h): X - p^{-1}(\text{int } T) \rightarrow Y - \text{int } T \times S^2$ is a locally flat embedding.

Next we must extend h to $p^{-1}(\text{int } T)$. Let T_1 be a connected component of T . Since T_1 is a linear disc bundle over B_1 we shall consider T_1 as the mapping cone on its boundary. That is $T_1 = (\partial T_1 \times I) \cup_q B_1$ where the map q on $\partial T_1 \times \{0\}$ is induced by the bundle projection. In this way each 2-disk fiber has a natural structure as the cone on its boundary. Assume T_1 and R have fixed metrics. Before we can extend h we must make the maps f and h nicer. Suppose $f(\partial T_1)$ is contained in, say, R_B . Let D be a fiber of T_1 . We know that the map f takes ∂D onto R_B in such a way that the image of ∂D generates the homology of R_B . We can homotopy f so that f maps ∂D homeomorphically onto R_B . To do this suppose

that $\tau: R_B \rightarrow \partial D$ is a locally length preserving homeomorphism. The map $g = f\tau: R_B \rightarrow R_B$ is naturally homotopic via g_t to a rotation. (Lift g to a map $\hat{g}: R^1 \rightarrow R^1$. Let $k: R^1 \rightarrow R^1$ be a straight line function of slope 1 such that the average area between the graphs of k and \hat{g} is zero. Homotopy \hat{g} to k through convex combinations and project the homotopy down to R_B .) It can be seen that the homotopy $g_i\tau^{-1}$ of f restricted to ∂D does not depend on the homeomorphism τ and can be carried out continuously and simultaneously on all the fibers of ∂T_1 . We homotopy f in similar fashion on the other components of ∂T , we extend the homotopy of f to $Y - p^{-1}(\text{int } T)$ and we homotopy the map h so that h still satisfies $h\{p^{-1}(x)\} = f(x)$ for all x in $Y - p^{-1}(\text{int } T)$.

Since $\rho([\partial D])$ is a transposition (consider $[\partial D]$ an element of $H_1(\partial D)$), $p^{-1}(D)$ consists of two components D_1 and D_2 . One of them, say D_1 , is mapped homeomorphically by p onto D . The circles ∂D_2 and ∂D may be coordinatized so that $p|_{D_2}$ is the map $(e^{i\theta}, t) \rightarrow (e^{2i\theta}, t)$, $0 \leq t \leq 1$, where $\partial D_2 = \{(e^{i\theta}, 1)\}$. Since f maps ∂D homeomorphically onto R_B , since $\gamma f_*[\partial D] \in \Sigma_3$ is a transposition, and since $h\{p^{-1}(x)\} = f(x)$ for $x \in \partial D$, we see that h must be constant and equal to B on ∂D_1 , and that h must map ∂D_2 homeomorphically onto a circle \hat{C} in S^2 , where the circle \hat{C} contains the points A and C and bounds a disc \hat{D} centered at the antipodal point of B . We extend h to D_1 by setting it equal to B and we extend h to D_2 by considering \hat{D} to be the cone on its boundary with cone point, the antipodal point of B and defining $h(p, t) = (h(p), t)$. That is, we just take the cone on h . It follows from the definitions, now, that h is well defined, continuous, and that $(p, h): Y \rightarrow X \times S^2$ is a locally flat embedding.

6. **Remarks.** The same methods used in proving Theorems 3, 4, and 5 can be applied to cyclic covering spaces in all dimensions. The following theorem is probably well known but I do not know of a reference.

THEOREM 6. *Let $p: X \rightarrow Y$ be a k -fold cyclic branched covering space of compact n -manifolds. Let the branch set B be a properly embedded locally flat orientable submanifold of Y . Assume $H_1(Y) = H_2(Y) = 0$.*

Then there is a locally flat embedding $e: X \rightarrow Y \times D^2$ such that the following diagram is commutative

$$\begin{array}{ccc}
 & Y \times D^2 & \\
 e \nearrow & & \downarrow \pi \\
 X & \xrightarrow{p} & Y
 \end{array}
 .$$

Proof. The proof is similar but much easier. R is replaced by the set of k -tuples of equally spaced points on S^1 . This space is just S^1 , so there is no obstruction theory to do as the coefficient groups vanish and f is obtained immediately.

This result can be used to show, for example, that certain closed orientable n -manifolds (those that do not embed in E^{n+2}) are not branched cyclic covers of S^n , branched over a locally flat, oriented $n - 2$ manifold.

Conversations with Bob Little and Allan Edmonds, and correspondence with Jose Montesinos were very helpful to me in writing this paper. In particular, Little pointed out to me the argument that not every W^4 can be embedded in $S^4 \times S^2$.

REFERENCES

1. R. H. Fox, *A quick trip through knot theory, topology of 3-manifolds and related topics*, Ed. M. K. Fort Jr., Prentice Hall.
2. H. M. Hilden, *Three fold branched coverings of S^3* , Amer. J. Math., **98** (1976), 989-997.
3. H. M. Hilden and J. M. Montesinos, *A method of constructing 3-manifolds and its application to the computation of the μ -invariant*. To appear.
4. J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Annals of Math. Studies 76, Princeton University Press.
5. J. M. Montesinos, *Three manifolds as 3-fold branched covers of S^3* , Quart. J. Math. Oxford Ser., vol. **27** (1976).
6. ———, *Una familia infinita de nudos representados no separables*, Revista Mathematica Hisp-Amer., **32** (1972), 158-166.
7. ———, *4-manifolds, 3-fold covering spaces and ribbons*, preprint.
8. L. P. Neuwirth, *Knot Groups*, Annals of Math. Studies 56, Princeton University Press.
9. C. P. Rourke and B. J. Sanderson, *Block bundles I*, Ann. of Math., **87** (1968), 1-28.
10. E. H. Spanier, *Algebraic Topology*, McGraw-Hill, N. Y., N.Y.

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