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MULTIPLICATIVE LINEAR FUNCTIONALS OF STEIN ALGEBRAS

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Let $(X, \mathcal{O}_X)$ be a Stein analytic space, and let $\mathcal{O}(X)$ denote the space of global sections of $\mathcal{O}_X$ endowed with its usual Frechet topology. The question of the continuity of complex valued multiplicative linear functionals of $\mathcal{O}(X)$ will be studied. The main result can be stated as follows: Theorem: Let $(X, \mathcal{O}_X)$ be a Stein space, and let $\alpha: \mathcal{O}(X) \to \mathrm{C}$ be a multiplicative linear functional. Suppose one can find an analytic subset $Y \subset X$ such that all the connected components of both Y and $X-Y$ are finite dimensional. Then α must be continuous. More generally, suppose that one can find a sequence of analytic subsets of $X, X=Y_0 \supset Y_1 \supset \cdots \supset Y_n = \emptyset$, such that for any $i, 0 \leqq i < n$, all the connected components of $Y_i - Y_{i+1}$ are finite dimensional. Then α must be continuous.

This paper resulted from an attempt to understand the claim made without proof in [5] that if $(X, \mathcal{O}_X)$ is a Stein space, and if $\lambda: (X, \mathcal{O}_X) \to \mathrm{Spec}\ (\mathcal{O}(X))$ is the natural morphism, then the pair $((X, \mathcal{O}_X), \lambda)$ is an analytic C-cover of $\mathrm{Spec}\,(\mathcal{O}(X))$. (See [5] for definitions.) In particular, all multiplicative linear functionals of $\mathcal{O}(X)$ would have to be continuous for this to be true. Michael proved the continuity of such functionals in case X is a domain of holomorphy in C^n [7]. (He in fact conjectured the continuity of all multiplicative linear functionals on any Frechet algebra [7].) A result of Arens [1] guarantees the desired continuity in case X can be embedded as a closed subspace of some C^n. Forster [3] proved the desired continuity in case X is finite dimensional. My result is a generalization of Forster's. Markoe [6] gave a weaker extension of Forster's result. He showed continuity under the assumption that $Sg(X)$, the singular locus of X, is finite dimensional. This follows from my result with $Y_1 = Sg(X)$ and $n = 2$. Finally, let me note that an advantage of the techniques of this paper is that they expose the elementary nature of Forster's theorem. They provide a proof which, unlike those in [3] and [6], does not depend on the deep existence of a proper map from a finite dimensional Stein space to some Euclidean space.

1. Preliminaries. Let X be a Stein space. (In what follows I will write X rather than $(X, \mathcal{O}_X)$ for analytic spaces as long as this leads to no ambiguity.) If $\mathcal{F}$ is a coherent analytic sheaf on

X then $\mathscr{F}(X)$, the space of global sections of $\mathscr{F}$, has a naturally defined Frechet space topology. I will not repeat the definition of that topology here, but I will mention some basic facts about it. (For more details see [2].)

(1.1) If $\mathscr{F} \to \mathscr{G}$ is a homomorphism of coherent analytic sheaves, then the induced map $\mathscr{F}(X) \to \mathscr{G}(X)$ is continuous.

(1.2) If X is reduced, then the topology on $\mathscr{O}(X)$ is the topology of uniform convergence on compact subsets of X.

We get:

PROPOSITION 1.3. *Let X be a Stein space, let Y be any analytic subspace of X, and let $r_{X,Y}: \mathscr{O}(X) \to \mathscr{O}(Y)$ be the canonical restriction map. Then $r_{X,Y}$ is a surjective, continuous, open map.*

Proof. The surjectivity follows from Cartan's Theorem B; the continuity follows from (1.1). The openness then follows from the Frechet open mapping theorem.

COROLLARY 1.4. *Let X be a Stein space, and let Y be an analytic subspace of X. Suppose $\alpha: \mathscr{O}(X) \to C$ and $\beta: \mathscr{O}(Y) \to C$ satisfy $\alpha = \beta \circ r_{X,Y}$. Then α is continuous if and only if β is continuous.*

PROPOSITION 1.5. *Let X be a Stein space and let X_{red} be its reduction. Then if $\alpha: \mathscr{O}(X) \to C$ is any multiplicative linear functional, then there is a multiplicative linear functional $\beta: \mathscr{O}(X_{red}) \to C$ satisfying $\alpha = \beta \circ r_{X,X_{red}}$.*

Proof. We only need to show that for any $f \in \mathscr{O}(X)$ which is also a section of the nilpotent ideal sheaf of X we have $\alpha(f) = 0$. If not, then $g = f - \alpha(f)$ would be a unit in $\mathscr{O}(X)$ satisfying $\alpha(g) = 0$. But this would imply $\alpha = 0$, a contradiction.

As an immediate consequence of Corollary 1.4 and Proposition 1.5 we get

COROLLARY 1.6. *Let X be a Stain space. Every multiplicative linear functional on $\mathscr{O}(X)$ is continuous if and only if every multiplicative linear functional on $\mathscr{O}(X_{red})$ is continuous.*

This allows for a convenient simplification of the problem. The next result is useful for inductive arguments.

LEMMA 1.7. *Let X be a Stein space, and let $\alpha: \mathscr{O}(X) \to C$ be a nonzero multiplicative linear functional. Let $f \in \ker \alpha$. Then* *the coherent ideal sheaf generated by f defines a nonempty Stein subspace $V(f) \subset X$. Moreover, there is a multiplicative linear functional $\beta: \mathcal{O}(V(f)) \to C$ satisfying $\alpha = \beta \circ r_{X,V(f)}$.*

Proof. If $V(f)$ were empty then the germ of f at every point would be a unit, and this would imply that f is a unit in $\mathcal{O}(X)$. But then we would have $\alpha = 0$, a contradiction.

To prove the existence of β we need only show that every section of the coherent ideal sheaf generated by f is an element of $\ker \alpha$. But by Cartan's Theorem B every such section is a multiple of f in $\mathcal{O}(X)$, and the result follows.

COROLLARY 1.8. *Let X be a Stein space. Suppose $X = \amalg X_i$, the disjoint union of a family $\{X_i\}_{i \in I}$ of open Stein subspaces of X. Then if $\alpha: \mathcal{O}(X) \to C$ is a multiplicative linear functional there is a $j \in I$ and a multiplicative linear functional $\beta: \mathcal{O}(X_j) \to C$ satisfying $\alpha = \beta \circ r_{X,X_j}$.*

Proof. Since X is second countable we may assume that I is a set of integers. We may also assume that $\alpha \neq 0$ since for $\alpha = 0$ the result is trivial.

Define $f \in \mathcal{O}(X)$ by $f|_{X_i} = i$. We have $\alpha(f) \in I$ since otherwise we would have $V(f - \alpha(f)) = \varnothing$ contradicting Lemma 1.7. Setting $j = \alpha(f)$ it is clear that $X_j = V(f - \alpha(f))$ and the result follows from Lemma 1.7.

From Corollary 1.4 and Corollary 1.8 we get

COROLLARY 1.9. *Let X be a Stein space. Suppose $X = \amalg X_i$, the disjoint union of a family $\{X_i\}_{i \in \mathcal{I}}$ of open Stein subspaces of X. Then every multiplicative linear functional of $\mathcal{O}(X)$ is continuous if and only if every multiplicative linear functional of $\mathcal{O}(X_i)$ is continuous for all $i \in \mathcal{I}$.*

2. Continuity of multiplicative linear functionals. I begin this section by proving Forster's theorem.

THEOREM 2.1. *Let X be a finite dimensional Stein space. Then every multiplicative linear functional of $\mathcal{O}(X)$ is continuous.*

Proof. The proof proceeds by induction on $\dim X$. If $\dim X = 0$ we use Corollary 1.9 and Corollary 1.6 to reduce to the case X is connected and reduced. But then X is the reduced point and $\mathcal{O}(X) = C$. The result is trivial in this case (since the only multiplicative linear functionals on C are the identity and the zero map).

Now suppose $\dim X > 0$ and that the result has been established for all Stein spaces of dimension $< \dim X$. Again we may assume that X is connected and reduced. By Cartan's Theorem B we may find an $f \in \mathcal{O}(X)$ which is constant on no irreducible component of X. Let $\alpha: \mathcal{O}(X) \to C$ be a nonzero multiplicative linear functional. Then $f - \alpha(f)$ is not constant on any irreducible component of X so that $\dim V(f - \alpha(f)) < \dim X$. Applying Lemma 1.7 we get β: $\mathcal{O}(V(f - \alpha(f))) \to C$ satisfying $\alpha = \beta \circ r_{X, V(f-\alpha(f))}$. It follows from Proposition 1.3 and from the induction hypothesis that α is continuous. This completes the induction step.

From Theorem 2.1 and Corollary 1.9 we get

COROLLARY 2.2. *Let X be a Stein space and suppose every connected component of X is finite dimensional, then every multiplicative linear functional is continuous.*

I now prove my generalization of Forster's theorem.

THEOREM 2.3. *Let X be a Stein space. Suppose that one can find a sequence of analytic subsets of X, $X = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \varnothing$, such that for any i, $0 \leqq i < n$, all the connected components of $Y_i - Y_{i+1}$ are finite dimensional. Then every multiplicative linear functional $\alpha: \mathcal{O}(X) \to C$ is continuous.*

Proof. The proof proceeds by induction on n. If $n = 1$ then all the connected components of X are finite dimensional and the result follows from Corollary 2.2.

Now suppose $n > 1$ and that the result has been established for all Stein spaces admitting the desired type of sequence of analytic subsets, but of length $< n$.

By Corollary 1.6 we may suppose that X and all of the Y_i's have the structure of reduced Stein spaces. If $\alpha(f) = 0$ for all $f \in \mathcal{O}(X)$ which vanish on Y_1 then one can find a $\beta: \mathcal{O}(Y_1) \to C$ satisfying $\alpha = \beta \circ r_{X, Y_1}$. It follows from the induction hypothesis and from Proposition 1.3 that α is continuous in this case.

Otherwise, we can find an $f \in \mathcal{O}(X)$ vanishing on Y_1 for which $\alpha(f) \neq 0$. Then $V(f - \alpha(f))$ is disjoint from Y_1. Thus, every connected component of $V(f - \alpha(f))$ is contained in a connected component of $Y_0 - Y_1$, and thus is finite dimensional. By Lemma 1.7 we may find a multiplicative linear functional $\beta: \mathcal{O}(V(f - \alpha(f))) \to C$ satisfying $\alpha = \beta \circ r_{X, V(f-\alpha(f))}$. It now follows from Corollary 2.2 and Proposition 1.3 that α is continuous in this case as well. This completes the induction step.

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