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A SINGULAR NONLINEAR BOUNDARY VALUE PROBLEM

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We consider the singular nonlinear boundary value problem (BVP)

$$\ddot{y} - y + yF(y^2, t) = 0\ , \qquad t \in (0, \infty) \tag{1}$$

$$y(0) = 0,\ \lim_{t\to\infty} y(t) = 0,\ \dot{y}(0)\ \text{exists},\ y(t) > 0\ \text{on}\ (0, \infty). \tag{2}$$

The problem is singular in a second way, in that we will allow $F(\eta, t)$ to have a singularity at $t = 0$.

The problem is motivated by a nuclear model due to Takahashi [6], his equation (after some simplifications by Synge [5]) was $\ddot{x} + (2/t)\dot{x} = x - x|x|^{k-1}$ with $k = 2$; Nehari [2] wrote $x(t) = t^{-1}y(t)$ to transform this equation into

$$\ddot{y} - y + y\frac{|y|^{k-1}}{t^{k-1}} = 0\ . \tag{3}$$

Nehari showed that the BVP (3) (2) has a (not necessarily unique) solution for $1 < k \leqq 4$. If one drops the requirement that $\dot{y}(0)$ exist, then there is a solution for $1 < k < 5$. Sansone used techniques entirely different from those of Nehari in an exhaustive study [4] in which he showed that (3) (2) has a unique solution for $1 < k < 5$; he used an extension of a counterexample of Nehari to show that there is no solution of (3) (2) for $k \geqq 5$.

Ryder [3] extended the variational techniques of Nehari, as developed in [1], [2], to the more general problem (1) (2). However, his results when applied to the special case of (3) (2) only yield existence for $1 < k < 4$; if one drops the requirement that $\dot{u}(0)$ exist, then his techniques prove existence for $1 < k < 5$.

In this paper we improve the results of Ryder, with the result that when we specialize to the BVP (3) (2) we prove existence for the full range $1 < k < 5$, thus improving Nehari's results as well. *Throughout the remainder of this paper we will assume*:

(I) $F(\eta, t) \in C([0, \infty) \times (0, \infty))$, $F(\eta, t) > 0$ for $\eta > 0$.

(II) $\exists\delta > 0$ such that for each $t > 0$, $\eta^{-\delta}F(\eta, t)$ is strictly increasing in η on $[0, \infty)$. In particular, $\lim_{\eta\to 0}\eta^{-\delta}F(\eta, t)$ exists for $t > 0$.

(III) $\lim_{t\to\infty} F(c^2, t) = 0$ for any constant c.

(IV) For some fixed $\varepsilon > 0$ (hence for all smaller values of ε) $\int_0^1 t^{1-\varepsilon}F(c^2t, t)dt$ converges for any constant c.

The first three assumptions coincide with those of Ryder. In place of IV, he assumes $\int_0^1 t^{(1/2)-\varepsilon}F(c^2t, t)dt$ converges for all sufficiently small $\varepsilon > 0$ (he uses IV to prove weaker results). Our version of IV allows F to be more strongly singular at $t = 0$.

Our approach is the same as that of Ryder, which in turn is based on that of Nehari, although we present our facts in a different order. However, since Ryder in turn refers to two different papers of Nehari for certain details, we feel it is necessary for the reader's sake to present a complete development. Also, our improvements come into Ryder's proofs in such a way that it is easier to do the entire proof. In order to help the continuity of the presentation, we have relegated all proofs to an appendix. The crucial new idea is Lemma 4.

We define $G(y^2, t) = \int_0^{y^2} F(\eta, t)d\eta$, and consider the variational problem

$$\min_A J(y) \equiv \min_A \int_0^\infty [\dot{y}^2 + y^2 - G(y^2, \tau)]d\tau \ , \tag{4}$$

$$A = \left\{ y \,|\, y(0) = 0,\ y(t) \not\equiv 0,\ y \in D'[0, \infty),\ y(t) \geqq 0,\ \int_0^\infty (\dot{y}^2 + y^2)dt = \int_0^\infty y^2F(y^2, \tau)d\tau \right\} .$$

Note that admissible functions are differentiable at $t = 0$, and that $J(y)$ might be an improper integral at $t = 0$ since G might be singular there.

LEMMA 1. *If y satisfies $y \in D'[0, \infty)$, $y(0) = 0$, $y(t) \not\equiv 0$, $y(t) \geqq 0$, and if*

$$\sigma^2 \equiv \int_0^\infty (\dot{y}^2 + y^2)dt \quad \textit{exists} \ , \tag{5}$$

then $\int_0^\infty y^2F(y^2, t)dt$ and $\int_0^\infty G(y^2, t)dt$ both exist, and $\exists \alpha > 0$ such that $\alpha y \in A$.

Proof. Appendix.

Thus the last condition in the definition of A may be viewed as a normalization.

LEMMA 2. $\mathrm{Inf}_A\, J(y) = \lambda \geqq 0$. *There exists a minimizing sequence $\{y_n\} \subset A$ that converges on $[0, \infty)$, uniformly on compact subintervals, to $y(t) \in C[0, \infty)$, and $J(y_n) \geqq \delta(\delta + 1)^{-1}\sigma_n^2$, where δ is defined in* II.

Proof. Appendix. We do not claim $y \in A$ or even $y \in D'[0, \infty)$, thus $J(y)$ may not exist. Throughout the remainder of this paper, $\{y_n\}$ will be a minimizing sequence.

LEMMA 3. *For each $y_n(t)$, and any constant $\alpha_n > 0$, the function*

$$(6)\qquad u_n(t) = \alpha_n e^{-t}\int_0^t (\sinh\tau) y_n F(y_n^2\tau),\, d\tau + \alpha_n \sinh t \int_t^\infty e^{-\tau} y_n F(y_n^2, \tau) d\tau$$

is in $C^1[0, \infty) \cap C^2(0, \infty)$ and solves

$$(7)\qquad \ddot{u}_n - u_n + \alpha_n y_n F(y_n^2, t) = 0\ ,\ t \in (0, \infty)\ ,\ u_n(0) = 0 = \lim_{t\to\infty} u_n(t).$$

Also, (8) $\lim_{t\to 0} \dot{u}_n(t)$ *exists,* $\lim_{t\to\infty} \dot{u}_n(t) = 0$. *We can choose α_n so that $u_n \in A$, and we will then have $\{\alpha_n\}$ bounded, and $J(u_n) \leqq J(y_n)$ for all n. Furthermore, $J(u_n) = J(y_n)$ if and only if $u_n \equiv y_n$. Finally, $\exists u_0 \in C[0, \infty)$ such that $\lim_{n\to\infty} u_n(t) = u_0(t)$ for $t \in [0, \infty)$, uniformly on compact subsets.*

Proof. Appendix. Note that the BVP for $u_n(t)$ may be singular at $t = 0$. Since $\{\alpha_n\}$ is bounded, we shall assume henceforth, by using a subsequence if necessary, that $\lim_{n\to\infty} \alpha_n = \alpha_0$.

Lemma 3 implies that $\{u_n\}$ is a minimizing sequence of admissible functions. Thus we can repeat the procedure described in Lemma 3, starting with $\{u_n\}$ as our new minimizing sequence. We can do this any finite number of times. We will still call the solution u_n, and the last-used minimizing sequence $\{y_n\}$.

LEMMA 4. *We can iterate the procedure of Lemma 3 a finite number of times, to obtain $y_n(t) = 0(t)$, uniformly in n, as $t \to 0$.*

Proof. Appendix. If is interesting that Sansone also uses an iterative procedure in a completely different context to get the full parameter range $1 < k < 5$. (Cf. [4], pp. 22-29.)

By Lemma 4, we may assume throughout the remainder of this paper that $y_n(t) = 0(t)$ as $t \to 0$, *uniformly in n.*

LEMMA 5. *$\{\dot{u}_n(t)\}$ is a Cauchy sequence, uniformly on $[0, \infty)$, hence $\lim_{n\to\infty} \dot{u}_n(t) = \dot{u}_0(t)$ exists on $[0, \infty)$.*

Proof. Appendix.

LEMMA 6. *$u_0 \in A$, $\lim_{n\to\infty} J(u_n) = J(u_0)$, and $\lambda \equiv \inf_A J(y) = \lim_{n\to\infty} J(u_n) = J(u_0) > 0$, hence u_0 is minimizing and nontrivial.*

Proof. Appendix.

LEMMA 7. $\lim_{n\to\infty} \alpha_n \equiv \alpha_0 = 1$, *and* $u_0(t)$ *solves the BVP* (1) (2).

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APPENDIX. *Proofs.* Throughout this section, K, L, M will denote various unimportant constants needed in the course of arguments.

Proof of Lemma 1. For $0 \leqq y \in D'[0, \infty)$ with $y(0) = 0$, we have, for t small

$$y(t) = \dot{y}(0)[t + o(t)] \leqq Kt\ , \qquad K = K(y)\ ; \tag{10}$$

while standard inequalities imply

$$y^2(t) = \left(\int_0^t \dot{y}(\tau)d\tau\right)^2 \leqq t\int_0^t \dot{y}^2(\tau)d\tau \leqq t\int_0^t (\dot{y}^2 + y^2)d\tau \equiv t\sigma^2(t) \leqq t\sigma^2\ , \tag{11}$$

$$y^2(t) = 2\int_0^t y(\tau)\dot{y}(\tau)d\tau \leqq \int_0^t (y^2 + \dot{y}^2)d\tau \equiv \sigma^2(t) \leqq \sigma^2\ ; \tag{12}$$

(11) is useful for t small, (12) for t large. Now if σ^2 exists for a given y, then $\int_a^T y^2F(y^2, \tau)d\tau = \left(\int_a^1 + \int_1^T\right)[y^2F(y^2, \tau)]d\tau$, so we can use (11) in the first integral, and (12) in the second, to obtain

$$\int_a^T y^2F(y^2, \tau)d\tau \leqq \int_a^1 \sigma^2\tau F(\sigma^2\tau, \tau)d\tau + \int_1^T y^2F(\sigma^2, \tau)d\tau\ .$$

The first integral on the right converges as $a \to 0$, by IV, and the second is bounded above by $\sigma^2 \max_{[1,\infty)} F(\sigma^2, t)$. For $\int_0^\infty G(y^2, \tau)d\tau$, we note that (13) $G(y^2, \tau) = \int_0^{y^2} F(\eta, \tau)d\eta \leqq y^2F(y^2, \tau)$. We now define $H(\alpha) = \left[\int_0^\infty y^2F(\alpha^2y^2, \tau)d\tau\right]\Big/\sigma^2$. (This integral is easily shown to converge.) By II,

$$0 < \alpha < 1 \Rightarrow H(\alpha) \leqq \alpha^{2\delta}H(1)\ ,$$
$$1 \leqq \alpha \Rightarrow \alpha^{2\delta}H(1) \leqq H(\alpha)\ ,$$

which implies that $H(\alpha)$ maps $(0, \infty)$ onto $(0, \infty)$. Thus $H(\alpha) = 1$ for exactly one $\alpha > 0$ (H is strictly increasing). Then αy will satisfy the normalization condition.

Proof of Lemma 2. If y is admissible, σ^2 exists by the normalization condition in the definition of A, thus $\int_0^\infty G(y^2, \tau)d\tau$ exists by Lemma 1, and $J(y)$ may be computed. By (12),

$$J(y) = \sigma^2 - \int_0^\infty G(y^2, \tau)d\tau \geqq \sigma^2 - \int_0^\infty y^2F(y^2, \tau) = 0 .$$

Clearly there exists a minimizing sequence of nontrivial functions $\{y_n\} \subset A$ such that $J(y_n) \to \lambda \equiv \inf_A J(y) \geqq 0$. Therefore, there exists a number $\rho^2 > 0$ such that $J(y_n) \leqq \rho^2$ for all n.

Now by II and the normalization condition for A, if $y \in A$,

$$\int_0^\infty G(y^2, \tau)d\tau = \int_0^\infty\left[\int_0^{y^2} \eta^\delta\eta^{-\delta}F(\eta, \tau)d\eta\right]d\tau \leqq \int_0^\infty y^{-2\delta}F(y^2, \tau)\left[\int_0^{y^2} \eta^\delta d\eta\right]d\tau$$
$$\leqq (\delta + 1)^{-1}\int_0^\infty y^2F(y^2, \tau)d\tau = (\delta + 1)^{-1}\sigma^2 ,$$

thus $J(y) = \sigma^2 - \int_0^\infty G(y^2, \tau)d\tau \geqq \delta(\delta + 1)^{-1}\sigma^2$. For the minimizing sequence $\{y_n\}$, we conclude that

$$\rho^2 \geqq J(y_n) \geqq \delta(\delta + 1)^{-1}\sigma_n^2 > \delta(\delta + 1)^{-1}\int_0^\infty \dot{y}_n^2 d\tau .$$

Since $y_n(0) = 0$, the Ascoli-Arzela theorem applies, and by using a subsequence if necessary we can assume $\lim_{n\to\infty} y_n(t) = y(t)$ exists, the convergence is uniform on compact subintervals, and $y \in C[0, \infty)$.

Now $J(y_n) \leqq \rho^2$ for all n implies $\sigma^2 \leqq ((\delta + 1)/\delta)\rho^2$, so by relabelling the constant we can write, by (11), (12),

$$(14) \qquad \text{(a)} \quad y_n^2(t) \leqq \rho^2 t \qquad \text{(b)} \quad y_n^2(t) \leqq \rho^2 \quad \text{uniformly in } n .$$

Proof of Lemma 3. Following Ryder, we rewrite (6) as $u_n(t) = \alpha_n e^{-t}Q(t) + \alpha_n(\sinh t)R(t)$. We can use (14a), IV, and the fact that $\sinh t = 0(t)$ for t small to conclude that

$$(15) \quad Q(t) \equiv \int_0^t(\sinh \tau)y_nF(y_n^2, \tau)d\tau \leqq Kt^{1/2+\varepsilon}\int_0^t \tau^{1-\varepsilon}F(\rho^2\tau, \tau)d\tau = o(t^{1/2+\varepsilon})$$

as $t \to 0$, *uniformly in* n. If we use (10) instead of (14a), and II, we get

$$(16) \quad Q(t) \leqq L_nt^{1+\varepsilon}\int_0^t \tau^{1-\varepsilon}F(K_n^2\tau^2, \tau)d\tau \leqq M_nt^{1+\varepsilon+\delta}\int_0^t \tau^{1-\varepsilon}F(\tau, \tau)d\tau .$$

Thus $Q(t) = O_n(t^{1+\varepsilon+\delta})$, which is considerably stronger (for a given n) then the uniform bound (15). To estimate $R(t)$ for t small, we use (14), I, III, and IV to obtain

$$
\begin{aligned}
(\sinh t)R(t) &\equiv \sinh t\left(\int_t^1 + \int_1^\infty\right)(e^{-\tau}y_nF(y_n^2, \tau)d\tau) \\
&\leqq (\sinh t)t^{\varepsilon-1/2}\int_t^1 \tau^{1/2-\varepsilon}\rho\tau^{1/2}F(\rho^2\tau, \tau)d\tau + \sinh t\int_1^\infty e^{-\tau}\rho^2F(\rho^2, \tau)d\tau \\
&\leqq Kt^{1/2+\varepsilon}\int_t^1 \tau^{1-\varepsilon}F(\rho^2\tau, \tau)d\tau + Lt\max\nolimits_{[1,\infty)}F(\rho^2, \tau)\int_1^\infty e^{-\tau}d\tau \\
&= O(t^{1/2+\varepsilon}) + O(t)\ ,
\end{aligned}
\tag{17}
$$

as $t \to 0$, uniformly in n. If we use the nonuniform bound (10) we get $\sinh tR(t) = O_n(t)$.

Applying II and (14b), we get for t large and $0 < T < t$,

$$
e^{-t}Q(t) \leqq Ke^{-t}\int_0^T \tau\rho\tau^{1/2}F(\rho^2\tau, \tau)d\tau + Le^{-t}\int_T^t e^{\tau}\rho F(\rho^2, \tau)d\tau\ . \tag{18}
$$

If we choose T so that $F(\rho^2, \tau) < \varepsilon$ for $\tau > T$, the above inequality implies $\limsup_{t\to\infty} e^{-t}Q(t) \leqq L\rho\varepsilon$, thus $\lim_{t\to\infty} e^{-t}Q(t) = 0$. For $R(t)$ we get

$$
(\sinh t)R(t) \leqq e^t\int_t^\infty e^{-\tau}\rho^2F(\rho^2, \tau)d\tau \leqq \rho^2\max_{[t,\infty)}F(\rho^2, \tau) = o(1) \quad \text{as} \quad t \to \infty\ . \tag{19}
$$

It is easy to see that $\dot{u}_n(t) = -e^{-t}Q(t) + (\cosh t)R(t)$, and that $u_n(t)$ solves the differential equation in (7) on $(0, \infty)$. The boundary conditions are easily verified, using the estimates above. Using the (nonuniform) estimates $\sinh tR(t) = O_n(t)$ as $t \to 0$, we see that $R(t) = O_n(1)$, and we can conclude that $\dot{u}_n(0)$ exists, since the integral defining $R(0)$ must either diverge to $+\infty$ or exist as a real number. Also, $\lim_{t\to\infty}\dot{u}_n(t) = 0$ follows easily from the above estimates. If we alternately multiply (7) by $u_n(t)$, $y_n(t)$, and integrate by parts, we obtain respectively

$$
s_n^2(T) \equiv \int_0^T(\dot{u}_n^2 + u_n^2)d\tau = \alpha_n\int_0^T u_ny_nF(y_n^2, \tau)d\tau + u_n(T)\dot{u}_n(T)\ , \tag{20}
$$

$$
\int_0^\infty(\dot{y}_n\dot{u}_n + y_nu_n)d\tau = \alpha_n\int_0^\infty y_n^2F(y_n^2, \tau)d\tau = \alpha_n\int_0^\infty(\dot{y}_n^2 + y_n^2)d\tau = \alpha_n\sigma_n^2\ . \tag{21}
$$

Now by (11) and (14a),

$$
\begin{aligned}
&\int_0^1 u_ny_nF(y_n^2, \tau)d\tau \leqq s_n(T)\rho\int_0^1 \tau F(y_n^2, \tau)d\tau\ , \quad \text{and similarly} \\
&\int_1^T u_ny_nF(y_n^2, \tau)d\tau \leqq \max_{[1,\infty)}F(\rho^2, \tau)\int_1^T u_ny_n \leqq K\left(\int_1^T u_n^2\right)^{1/2}\left(\int_1^T y_n^2\right)^{1/2} \\
&\qquad \leqq K\rho s_n(T)\ .
\end{aligned}
$$

Then (20) implies $s_n^2(T) \leqq L_ns_n(T) + u_n(T)\dot{u}_n(T)$, so $(s_n^2(T) - L_n/2)^2 \leqq (L_n^2/4) + u_n(T)\dot{u}_n(T)$. Therefore $\int_0^\infty(\dot{u}_n^2 + u_n^2)d\tau \equiv s_n^2$ exists. We then conclude from Lemma 1 that αu_n is admissible for some $\alpha > 0$. This just corresponds to an appropriate choice of α_n in the integral (6) that defines u_n, and we so choose α_n. In exactly the same way as for $\{y_n\}$, we can then show $\lim_{n\to\infty} u_n(t) = u_0(t)$ exists.

Using (21) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}(22)\qquad \left[\alpha_n \int_0^\infty y_n^2 F(y_n^2, \tau)d\tau\right]^2 &< \int_0^\infty (\dot{u}_n^2 + u_n^2)d\tau \int_0^\infty (\dot{y}_n^2 + y_n^2)d\tau \\ &= \int_0^\infty u_n^2 F(u_n^2, \tau)d\tau \int_0^\infty y_n^2 F(y_n^2, \tau)d\tau\ ,\end{aligned}$$

which implies $\alpha_n^2 \int_0^\infty y_n^2 F(y_n^2, \tau)d\tau \leqq \int_0^\infty u_n^2 F(u_n^2, \tau)d\tau$. Now by (20) and the normalization condition,

$$\begin{aligned}\left[\int_0^\infty u_n^2 F(u_n^2, \tau)d\tau\right]^2 &= \left[\int_0^\infty (\dot{u}_n^2 + u_n^2)d\tau\right]^2 = \alpha_n^2\left[\int_0^\infty u_n y_n F(y_n^2, \tau)d\tau\right]^2 \\ &\leqq \alpha_n^2 \int_0^\infty u_n^2 F(y_n^2, \tau)d\tau \int_0^\infty y_n^2 F(y_n^2, \tau)d\tau\ ;\end{aligned}$$

combining the two above results, we get

$$(23)\qquad \int_0^\infty u_n^2 F(u_n^2, \tau)d\tau \leqq \int_0^\infty u_n^2 F(y_n^2, \tau)d\tau\ .$$

Since $F(\eta, t)$ is strictly increasing in η, the function $G(\eta, t) = \int_0^\eta F(\gamma, t)d\gamma$ is strictly convex and

$$(24)\qquad \int_0^T G(u_n^2, \tau)d\tau \geqq \int_0^T G(y_n^2, \tau)d\tau + \int_0^T (u_n^2 - y_n^2)F(y_n^2, \tau)d\tau\ ,$$

where equality holds if and only if $u_n^2(\tau) \equiv y_n^2(\tau)$ on $[0, T]$. Therefore, using (23), we obtain

$$\int_0^\infty [u_n^2 F(u_n^2, \tau) - G(u_n^2, \tau)]d\tau \leqq \int_0^\infty [y_n^2 F(y_n^2, \tau) - G(y_n^2, \tau)]d\tau\ .$$

But $J(w) = \int_0^\infty [w^2 F(w^2, \tau) - G(w^2, \tau)]d\tau$ for $w \in A$, hence, the above inequality implies $J(u_n) \leqq J(y_n)$. If $J(u_n) = J(y_n)$, then (24) reduces to an equality, and $u_n(t) \equiv y_n(t)$ on $[0, \infty)$.

Finally, we shall show that $\{\alpha_n\}$ is bounded. We have by (22)

$$(25)\qquad \alpha_n^2 \leqq \left[\int_0^\infty y_n^2 F(y_n^2, \tau)d\tau\right]^{-1} \int_0^\infty u_n^2 F(u_n^2, \tau)d\tau\ .$$

Using (14) (since $\{u_n\}$ is minimizing), I and III, we obtain

$$\int_0^\infty u_n^2 F(u_n^2, \tau)d\tau \leqq \rho^2 \int_0^1 \tau F(\rho^2\tau, \tau)d\tau + \rho^2 \max_{[1,\infty)} F(\rho^2, \tau)\ ,$$

so the numerator in (25) is bounded above. To bound the denominator below, we use the normalization condition (11), and (12), to obtain

$$\sigma_n^2 = \int_0^\infty (\dot{y}_n^2 + y_n^2)d\tau = \int_0^\infty y_n^2 F(y_n^2, \tau)d\tau \leqq \int_0^1 \sigma_n^2 \tau F(\sigma_n^2 \tau, \tau)d\tau + \sigma_n^2 \max_{[1,\infty)} F(\sigma_n^2, \tau) ,$$

thus

$$1 \leqq \int_0^1 \tau F(\sigma_n^2 \tau, \tau)d\tau + \max_{[1,\infty)} F(\sigma_n^2, \tau) \leqq K\sigma_n^{2\delta} . \tag{26}$$

Therefore $\sigma_n^2 \geqq M > 0$ for all n, and this implies that the denominator in (25) is bounded below, since $\int_0^\infty y_n^2 F(y_n^2, \tau)d\tau = \sigma_n^2$.

Proof of Lemma 4. Without loss of generality, we shall assume that $\lim \alpha_n = \alpha_0$ exists for any specific given minimizing sequence $\{y_n\}$ and corresponding solutions of (7), $\{u_n\}$. Since the members of the first minimizing sequence satisfy $y_n(t) \leqq \rho t^{1/2}$ on $[0, 1]$ and $\{\alpha_n\}$ is bounded, (15) and (17) imply $u_n(t) = O(t^{1/2+\varepsilon})$ as $t \to 0$, uniformly in n. Now suppose we have iterated $p \geqq 0$ times and obtained $u_n(t) = O(t^{1/2+\mu})$ with $1/2 + \mu < 1$. We then parallel the derivation of (15), (17) to write

$$Q(t) \leqq K\int_0^t \tau\tau^{1/2+\mu} F(L^2\tau^{1+2\mu}, \tau)d\tau \leqq K_2 \int_0^t \tau^{\mu(2\delta+1)+3/2} F(\tau, \tau)d\tau = O(t^{1/2+\nu}) ,$$

where

$$\nu = \mu(1 + 2\delta) .$$

Clearly we can make $\nu > 1$ after a finite number of such iterations, beginning with $\mu = \varepsilon$.

Proof of Lemma 5. We may assume, using Lemma 4, that $y_n(t) = O(t)$ as $t \to 0$, uniformly in n. Now $\dot{u}_n(t) = -e^{-t}Q_n(t) + (\cosh t)R_n(t)$; where we have introduced subscripts to indicate dependence on y_n. It is now easy to show that $\{Q_n\}$, $\{R_n\}$ converge uniformly on $[0, \infty)$, hence so does $\{\dot{u}_n(t)\}$; for example,

$$|R_n(t) - R_m(t)| \leqq \left(\int_t^1 + \int_1^\infty\right) e^{-\tau} |y_n F(y_n^2, \tau) - y_m F(y_m^2, \tau)| d\tau \leqq \int_0^{t_0} e^{-\tau} M\tau F(M^2\tau^2, \tau) + \int_{t_0}^T e^{-\tau} O(1)d\tau + \int_T^\infty e^{-\tau} 2\rho F(\rho^2, \tau)d\tau , \tag{27}$$

where we have used Lemma 4 ($y_n(\tau) \leqq M\tau$) in the first integral, and the $O(1)$ in the middle integral is as $n, m \to \infty$, uniformly on a fixed interval $[t_0, T]$ (since $\{y_n\}$ is uniformly Cauchy there). We choose t_0 so small that the first integral is less than $\varepsilon/3$, T so large that the last is less than $\varepsilon/3$ (*both are uniform in* m, n), then let $m, n \to \infty$ in the middle.

Proof of Lemma 6. By standard arguments, Lemma 5 implies

$$\lim_{n\to\infty} u_n(t) = u_0(t) , \qquad \lim_{n\to\infty} \dot{u}_n(t) = \dot{u}_0(t) ,$$

uniformly on compact subsets. However, uniform estimates like those in (27) enable us to conclude $\lim J(u_n) = J(u_0)$ (see Ryder, pp. 489–490 for details). By the remarks following (26), we have $\int_0^\infty \dot{y}_n^2 + y_n^2 = \int_0^\infty y_n^2 F(y_n^2, \tau)d\tau \geqq A^2 > 0$, and by Lemma 2, $J(y_n) \geqq A^2 > 0$.

Proof of Lemma 7. We pick $u_0(t)$ for the function $y(t)$ appearing in the BVP (7). Then $u(t)$ as defined by (6) must satisfy $J(u) \leqq J(u_0)$ which implies $J(u) = J(u_0)$, thus $u(t) \equiv u_0(t)$. Therefore $u_0(t)$ solves (7) with $y(t) = u_0(t)$, and some $\alpha_0 > 0$. In particular $\lim_{t\to 0} u_0(t) = \lim_{t\to\infty} u_0(t) = 0$. By (20) (letting $T \to \infty$),

$$\sigma_0^2 \equiv \int_0^\infty (\dot{u}_0^2 + u_0^2)d\tau = \alpha_0 \int_0^\infty u_0^2 F(u_0^2, \tau)d\tau ,$$

then by the normalizing condition for A, $\alpha_0 = 1$. Thus $u_0(t)$ solves (1) (2).

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