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WEAK RIGIDITY OF COMPACT NEGATIVELY CURVED MANIFOLDS

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Let M and M' be simply connected, complete Riemannian manifolds of nonpositive sectional curvature and let Γ and Γ' be properly discontinuous groups of isometries acting freely on M and M' respectively such that M/Γ and M'/Γ' are compact. Let $\theta: \Gamma \to \Gamma'$ be an isomorphism. There exists a pseudo-isometry $\phi: M \to M'$ such that $\phi(\gamma x) = \theta(\gamma)\phi(x)$ for all γ in Γ and x in M. The question is whether this pseudo-isometry ϕ can be extended to a homeomorphism $\bar{\phi}$ between the boundaries $M(\infty)$ and $M'(\infty)$ of M and M' respectively. This homeomorphism is further required to be equivariant with respect to the isomorphism θ. This extendability is called the weak rigidity of compact nonpositively curved manifolds. In this paper, this weak rigidity question is answered affirmatively if M is a simply connected, complete Riemannian manifold of negative sectional curvature and M' is a noncompact symmetric space of rank one. If M and M' are noncompact symmetric spaces without direct factors of closed one or two dimensional geodesic subspaces, then this weak rigidity is proved by G. D. Mostow and is a part of his important strong rigidity theory of compact, locally symmetric Riemannian manifolds. This paper is motivated by this theory of Mostow.

1. Statements of theorems. In [7], Mostow has proved the following remarkable result.

THEOREM. (Mostow) *Let Y be a locally symmetric Riemannian manifold. The fundamental group $\pi_1(Y)$ determines Y uniquely up to an isometry and a choice of normalizing constants, provided that Y has no closed one or two dimensional geodesic subspaces which are direct factors locally.*

According to the uniformization theorem, every compact Riemann surface Y of genus greater than one has its universal covering manifold analytically equivalent to the unit disk with the Poincaré metric. Therefore $\pi_1(Y)$ may be identified with a discrete and cocompact Γ of PSL $(2, R)$. It is well known that two compact Riemann surfaces Y and Y' of the same genus have isomorphic fundamental groups Γ and Γ' but need not be analytically equivalent. This is the reason for the proviso in Mostow's theorem.

In this paper, one has the following two theorems which form a part of the investigation suggested by Mostow in his introduction of [7] that the theory of pseudo-isometries may be useful in other contexts. The theory of quasi-conformal mappings over the complex numbers C will be discussed in a forthcoming paper.

THEOREM 1. *Let M and M' be simply connected, complete Riemannian manifolds of nonpositive sectional curvature and let Γ and Γ' be properly discontinuous groups of isometries acting freely on M and M' respectively such that M/Γ and M'/Γ' are compact. Let $\theta: \Gamma \to \Gamma'$ be an isomorphism. Then there is a pseudo-isometry $\phi: M \to M'$ such that $\phi(\gamma x) = \theta(\gamma)\phi(x)$ for all γ in Γ and x in M, that is ϕ is a Γ-space morphism and a pseudo-isometry.*

Note that the proviso in Mostow's theorem is not required in this and the next theorems. In the next theorem, one investigates whether this Γ-space morphism can be extended to the boundaries $M(\infty)$ and $M'(\infty)$ equivariantly so that the extension is a homeomorphism. Here the boundary $M(\infty)$ of a simply connected complete Riemannian manifold M of nonpositive sectional curvature is defined by Eberlein and O'Neill [4].

THEOREM 2. *Let M be a simply connected, complete Riemannian manifold of negative sectional curvature and let M' be a noncompact symmetric space of rank one. Then the Γ-space morphism and pseudo-isometry $\phi: M \to M'$ in Theorem 1 can be extended to a Γ-space homeomorphism $\bar{\phi}: M(\infty) \to M'(\infty)$.*

The manifold M is required to have negative sectional curvature. Since it admits a cocompact covering group Γ, the sectional curvature is bounded above by a negative constant. According to [5], fixed points of all the elements of Γ are dense in the limit set of Γ which is actually the full boundary $M(\infty)$. The density of fixed points for an arbitrary manifold of nonpositive sectional curvature admitting a cocompact covering group Γ is unknown to the author. If the density is known, then Theorem 2 may be extended to manifolds of nonpositive sectional curvature and symmetric spaces of arbitrary rank. Furthermore, M' is required to be symmetric, because a basic inequality in [7] (§6) does not seem to be extendable to manifolds of nonpositive sectional curvature.

The proof of Mostow's theorem [7] consists of the following three steps:

(1) Let Y and Y' be locally symmetric Riemannian manifolds, let X and X' denote their universal covering manifolds whose isometry

groups are G and G' respectively and let $\pi_1(Y) \cong \Gamma \cong \pi_1(Y')$. Then there is a Γ.space pseudo-isometry $\phi: X \to X'$.

(2) The Γ-space pseudo-isometry ϕ has continuous boundary values ϕ_0.

(3) ϕ_0 is induced by an analytic isomorphism of the semisimple Lie groups G and G'.

2. Nonpositively curved manifolds. Let M be a simply connected, complete Riemannian manifold of nonpositive sectional curvature. In [4], Eberlein and O'Neill have defined the boundary $M(\infty)$ of M to be the set of asymptotic classes of geodesics in M. The isometries of M can be classified into three classes: (1) elliptic, (2) parabolic and (3) axial [4]. In this paper, we shall assume that M admits a properly discontinuous group Γ of isometries acting freely on M such that M/Γ is compact. In this case, every element of Γ is axial. Under the restriction that the sectional curvature is bounded above by a negative constant (this restriction is valid for Theorem 2), every axial element of $I(M)$, the isometry group of M, has two fixed points in $M(\infty)$ and translates the infinite geodesic joining these two points. If M is symmetric, then these axial elements are those semisimple elements of G whose polar parts are nontrivial ([7], §2).

In [6], Lawson and Yau have proved that for the compact manifold $N = M/\Gamma$ there exists an abelian subgroup A of rank in Γ if and only if there exists a flat k-torus immersed isometrically and totally geodesically in N. For symmetric spaces, the maximal rank is called the rank of the symmetric space, however, in the general case, various maximal ranks may be different. If the sectional curvature is bounded above by a negative constant, then there exists only flat 1-torus [4]. Thus, for Theorem 2, we only have to consider infinite geodesics instead of flat totally geodesic subspaces and Mostow's arguments in [7] can be simplified in our discussion. The geometry of symmetric spaces developed in [7] has its counterpart in simply connected, complete Riemannian manifolds of nonpositive sectional curvature. In particular, §§3, 5, and 7 of [7] can be carried over to our investigation. For any subset S of a metric space and for any nonnegative real number v, we denote by $T_v(S)$ the subset of points lying within a distance less than v of S. Because of the law of cosines, $T_v(S)$ is convex if S is convex. Let F be a flat totally geodesic subspace of M. Then any point p in M has a point $\pi(p)$ in F such that $d(p, \pi(p)) = d(p, F)$. We call the map $\pi: M \to F$ the orthogonal projection of M onto F. Given two sets A and B, the Hausdorff distance $hb(A, B)$ is defined by

$$hb(A, B) = \inf\{v \leqq \infty;\ A \subset T_v(B),\ B \subset T_v(A)\}\ .$$

It is obvious that, given two infinite geodesics σ_1 and σ_2, $hd(\sigma_1, \sigma_2) < \infty$ if and only if σ_1 and σ_2 are asymptotic.

3. The basic approximation. Let M be a simply connected, complete Riemannian manifold of nonpositive sectional curvature and let M' be a symmetric space of noncompact type. Let Γ and Γ' be properly discontinuous groups of isometries acting freely on M and M' respectively such that M/Γ and M'/Γ' are compact. Let $\theta: \Gamma \to \Gamma'$ be an isomorphism. The following lemma extends Lemma 13.1 of [7].

LEMMA 1. *Let F be a Γ-compact flat in M, let A be the stabilizer of Γ, and let $A' = \theta(A)$. Then there is a unique Γ'-compact flat F' in M' stable under A'.*

A flat is a maximal, flat totally geodesic subspace in M. A flat is Γ-compact if it is invariant under a maximal abelian subgroup A of Γ. The proof of this lemma follows from the flat torus theorem of Lawson and Yau [6]. In this paper, we only need the lemma for infinite geodesics which are invariant under axial elements of Γ and are the axes of these axial elements respectively. Recall that the fixed points of these axial elements or the end points of these geodesics are dense in the boundary $M(\infty)$.

LEMMA 2. (*The basic approximation*) *Let F be a Γ-compact flat in M and F' be the unique flat in M' stabilized by $\theta(A)$. Then there is a constant v depending only on k and b but not on the particular choice of the Γ-compact flat F such that $hd(\phi(F), F') \leqq v$, where hd denotes the Hausdorff distance between F and F'.*

Proof. Let $\pi: M \to F$ and $\pi': M' \to F'$ denote the orthogonal projections of M onto F and M' onto F'. Note that π and π' are A and A' equivariant respectively. The topological argument of Mostow [7] still implies that $\pi'(\phi(F)) = F'$. Also $\pi(T_b(\phi^{-1}(F')) = F$. Let

$$d = \sup_{x \in F} d(\phi(x), F')$$
$$d' = \sup_{x \in F'} d(\phi(F), x) .$$

Then, Mostow's estimates are valid in his proof in [7] except particular attention needs to be paid to Lemma 6.4. of [7] which gives a basic inequality for symmetric spaces. Since we have assumed that M' is symmetric, this basic inequality holds. We do not know how to prove this inequality for arbitrary negatively curved manifolds.

4. The boundary map. The cone topology on the boundary

$M(\infty)$ has been introduced by Eberlein and O'Neill on [4]. Let x be a point in M. A sequence $\{z_n\}$ in $M(\infty)$ converges to z in $M(\infty)$ if and only if the angles $\{\sphericalangle_x(z_n, z)\}$ converges to 0 as n approaches to ∞. That is, the infinite geodesics $\{\sigma_n\}$ (σ_n is an infinite geodesic joining x to z_n) converges to the infinite geodesic σ joining x to z with the topology of uniform convergence on compact sets in M. Consequently, §14 of [7] makes sense.

LEMMA 3. *Under the hypotheses of* §3, *for each infinite geodesic* σ *in* M, *there is an unique infinite geodesic* σ' *in* M' *such that* $hd(\phi(\sigma), \sigma') \leqq v$, *where* v *is given in Lemma* 2. *The map* $\bar{\phi}: \sigma \to \sigma'$ *induces a homeomorphism from the boundary* $M(\infty)$ *onto the boundary* $M'(\infty)$. *The induced homeomorphism is still denoted by* $\bar{\phi}$.

Proof. Recall that the fixed points of (axial) elements of Γ and Γ' are dense in the boundaries $M(\infty)$ and $M'(\infty)$ respectively. Let us denote by S the set of fixed points of elements in Γ in $M(\infty)$. For each σ representing z in S, let $\bar{\phi}(\sigma)$ denote the unique infinite geodesic in M' such that $hd(\phi(\sigma), \bar{\phi}(\sigma)) \leqq v$. In order to prove that $\bar{\phi}$ is the restriction to S of a continuous map of $M(\infty)$ into $M'(\infty)$, it suffices to prove: that if z in $M(\infty)$ and $\{z_n\}$ is a sequence in S converging to z, then $\{\bar{\phi}(z_n)\}$ is convergent in $M'(\infty)$. Let $B_s(x)$ and $B'_s(\phi(x))$ denote the balls of radius s and centers x in M and $\phi(x)$ in M' respectively. Let σ_n denote the infinite geodesic joining x to z_n and assume that x is in σ. By hypothesis, $hd(\sigma_n \cap B_s(x), \sigma \cap B_s(x)) \to 0$ as $n \to \infty$. Since ϕ is continuous and $\phi(B_s(x)) \supset B'_{s/k}(\phi(x))$ for all $s \geqq b$, we have

$$hd(\phi(\sigma_n) \cap B'_s(\phi(x)), \phi(\sigma) \cap B'_s(\phi(x))) \longrightarrow 0$$

as $n \to \infty$. Let v_1 be any number such that $v_1 > v$. Let $\mathscr{F}_s$ denote the set of all infinite geodesics σ' in M' such that

$$hd(\sigma' \cap B'_s(\phi(x)), \phi(\sigma) \cap B'_s(\phi(x))) \leqq v_1 .$$

Then $\bar{\phi}(\sigma_n)$ are all in $\mathscr{F}_s$ for all sufficiently large subscripts of the sequences $\{\sigma_n\}$ of axes of elements of Γ. The family $\{\mathscr{F}_s; s > 0\}$ is a nested family with a nonempty intersection (see p. 104 of [7]). Let σ' be an element of the intersection $\cap \mathscr{F}_s (s > 0)$. Then

$$hd(\sigma' \cap B'_s(\phi(x)), \phi(\sigma) \cap B'_s(\phi(x))) \leqq v_1$$

for all $s > 0$. Thus $hd(\sigma', \phi(\sigma)) \leqq v_1$ for all $v_1 > v$. Hence $hd(\sigma', \phi(\sigma)) \leqq v$. If σ'' is another element of the intersection $\cap \mathscr{F}_s (s > 0)$. Then $hd(\sigma'', \phi(\sigma)) \leqq v$, and $hd(\sigma', \sigma'') \leqq 2v$. It follows that σ' and σ'' are asymptotic and the sequence $\{\bar{\phi}(z_n)\}$ converges to a single point in

$M'(\infty)$. By density, we have that, for each z in $M(\infty)$, which is represented by σ, $hd(\bar{\phi}(\sigma), \phi(\sigma)) \leqq v$.

Furthermore for each γ in Γ and each infinite geodesic σ in M, we have $hd(\phi(\gamma\sigma), \theta(\gamma)\bar{\phi}(\sigma)) = hd(\theta(\gamma)\phi(\sigma), \theta(\gamma)\bar{\phi}(\sigma)) = hd(\phi(\sigma), \bar{\phi}(\sigma))$. Hence $hd(\bar{\phi}(\gamma\sigma), \theta(\gamma)\bar{\phi}(\sigma)) \leqq hd(\bar{\phi}(\gamma\sigma), \phi(\gamma\sigma)) + hd(\phi(\gamma\sigma), \theta(\gamma)\bar{\phi}(\sigma)) \leqq 2v$. Consequently, $\bar{\phi}(\gamma\sigma)$ and $\theta(\gamma)\bar{\phi}(\sigma)$ are asymptotic and represent the same point $\bar{\phi}(\gamma z) = \theta(\gamma)\bar{\phi}(z)$ in $M'(\infty)$.

The map $\bar{\phi}$ is injective. For given distinct points z_1 and z_2 represented by σ_1 and σ_2 such that $hd(\sigma_1, \sigma_2)$ is ∞. Then $hd(\phi(\sigma_1), \phi(\sigma_2))$ is ∞ since ϕ is a pseudo-isometry. Therefore $hd(\bar{\phi}(\sigma_1), \bar{\phi}(\sigma_2)) = \infty$. Thus $\bar{\phi}(z_1) \neq \bar{\phi}(z_2)$.

To prove that $\bar{\phi}$ is a homeomorphism of $M(\infty)$ onto $M'(\infty)$, we consider the map $\bar{\phi}'\colon M'(\infty) \to M(\infty)$ induced by the pseudo-isometric Γ'-map $\phi'\colon M' \to M$. Let z in S and let σ represent z. If γ is the axial element translating σ, then $\bar{\phi}(\sigma)$ is the infinite geodesic in M' translated by $\theta(\gamma)$. Moreover $\bar{\phi}'\bar{\phi}(\sigma)$ is the infinite geodesic in M translated by γ. Therefore $\bar{\phi}'\bar{\phi}(z) = z$ for all z in S. Hence $\bar{\phi}'\bar{\phi} =$ identity. Similarly $\bar{\phi}\bar{\phi}' =$ identity. $\bar{\phi}$ is a homeomorphism from $M(\infty)$ onto $M'(\infty)$.

Thus the proof of Theorem 2 is completed.

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