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FIXED POINT THEOREMS IN LOCALLY CONVEX SPACES

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Let C be a convex subset of a nuclear locally convex space that is also an F-space. Suppose $T: C \to C$ is nonexpansive and $\{v_n\}$ is given by the Mann iteration process. It is shown that if $\{v_n\}$ is bounded, T has a fixed point. Also, a sequence $\{y_n\}$ can be constructed such that $y_n \to y$ weakly where $Ty = y$. If C is a linear subspace and T is linear, then $\lim y_n = y$.

1. Introduction. With a few exceptions, the nonnormable locally convex spaces encountered in analysis are nuclear spaces. Precupanu [8]-[11] studied those locally convex spaces whose locally convex spaces whose generating family of seminorms satisfy the parallelogram law, and he called them *H-locally convex* spaces. Precupanu [9] observed that they include all nuclear spaces. This is immediate from Corollary 1, page 102 of [13]. Such a space that is also complete will be called a *generalized Hilbert space*. Theorem 2 generalizes a theorem of Reich [12] which generalizes a result of Dotson and Mann [2]. Reich's ingenious proof is modified to apply in this setting. Theorem 4 generalizes a result of Dotson [1]. His approach to the proof is used, but substantial changes are needed in the details.

Let X be a T_2 locally convex space generated by a family $\{\rho_\alpha : \alpha \in \Delta\}$ of continuous seminorms. The function $\rho: X \to R^\Delta$ is defined by

$$(\rho(x))(\alpha) = \rho_\alpha(x)\,, \qquad x \in X\,, \qquad \alpha \in \Delta\,.$$

ρ satisfies the axioms of norm. The topology t_ρ generated by ρ is the original topology where a t_ρ neighborhood of x is of the form

$$\Omega(x, U) = \{y: \rho(x-y) \in U\}$$

where U is a neighborhood of zero in R^Δ. Thus ρ norms X over R^Δ. A mapping T from X into X is *nonexpansive* if $\rho(Tx - Ty) \leqq \rho(x-y)$ for all $x, y \in X$; that is, $\rho_\alpha(Tx-Ty) \leqq \rho_\alpha(x-y)$ for all $x, y \in X$ and $\alpha \in \Delta$.

We look at the Mann iteration process. Let C be a convex subset of X and suppose T maps C into C. Suppose $A = [a_{nk}]$ is an infinite matrix satisfying:

$$a_{nk} \geqq 0 \quad \text{for all } n \text{ and } k\,,$$
$$a_{nk} = 0 \quad \text{for} \quad k > n\,,$$

$$\sum_{k=1}^{n} a_{nk} = 1 \quad \text{for all } n, \text{ and}$$

$$\lim_{n\to\infty} a_{nk} = 0 \quad \text{for all } k .$$

If $x_1 \in C$, $v_1 = a_{11}x_1$, $x_2 = Tv_1$, $v_2 = a_{21}x_1 + a_{22}x_2$, $x_3 = Tv_2$ and, in general,

$$v_n = \sum_{k=1}^{n} a_{nk}x_k \quad \text{and} \quad x_{n+1} = Tv_n .$$

Thus for $n \geqq 2$,

$$v_n = a_{n1}x_1 + \sum_{k=2}^{n} a_{nk}Tv_{k-1} .$$

2. Results. In the remainder of this paper, C will denote a convex subset of a T_2 locally convex space (X, t) and $T: C \to C$ is nonexpansive.

THEOREM 1. *If $Tp = p$ for some p in C, the sequences $\{x_n\}$ and $\{v_n\}$ are bounded.*

Proof. t is given by the family $\{\rho_\alpha\}$ of seminorms and it suffices to show that for each $\alpha \in \Delta$, $\{\rho_\alpha(x_n)\}$ is bounded. The proof given in [2] carries over if you replace $||x_n||$ by $\rho_\alpha(x_n)$.

THEOREM 2. *Suppose that every closed bounded convex subset of C has the fixed point property for nonexpansive mappings. If for some x_1 in C the sequence $\{v_n\}$ is bounded, then T has a fixed point.*

Proof. Let $y \in C$ and set

$$R_\alpha = \limsup_n \rho_\alpha(y - v_n) .$$

R_α is finite since $\{v_n\}$ is bounded. Let

$$K_\alpha = \{z \in C: \limsup_n \rho_\alpha(z - v_n) \leqq R_\alpha\} .$$

K_α is a closed in t_{ρ_α} and, therefore, closed in t for every $\alpha \in \Delta$. Let

$$\begin{aligned} K &= \cap\{K_\alpha: \alpha \in \Delta\} \\ &= \{z \in C: \limsup_n \rho_\alpha(z - v_n) \leqq R_\alpha \text{ for all } \alpha \in \Delta\} . \end{aligned}$$

$K \neq \phi$ since $y \in K$. K is closed bounded and convex. If $z \in K$ implies $Tz \in K$, it follows that T has a fixed point. Let $z \in K$ and $\alpha \in \Delta$. For each $\varepsilon > 0$, there exists $N = N(\varepsilon, \alpha)$ such that $\rho_\alpha(z - v_n) < R_\alpha + \varepsilon$ for all $n \geqq N$. For $n > N + 1$,

$$\begin{aligned}\rho_\alpha(Tz - v_n) &= \rho_\alpha\Big(\sum_{k=1}^{n} a_{nk}Tz - a_{n1}x_1 - \sum_{k=2}^{n} a_{nk}Tv_{k-1}\Big)\\ &\leqq a_{n1}\rho_\alpha(Tz - x_1) + \sum_{k=2}^{n} a_{nk}\rho_\alpha(Tz - Tv_{k-1})\\ &\leqq a_{n1}\rho_\alpha(Tz - x_1) + \sum_{k=2}^{n} a_{n\alpha}\rho_k(z - v_{k-1})\\ &\leqq a_{n1}\rho_\alpha(Tz - x_1) + \sum_{k=2}^{N} a_{nk}\rho_\alpha(z - v_{k-1})\\ &\quad + \sum_{k=N+1}^{n} a_{nk}(R_\alpha + \varepsilon) = h(n) + R_\alpha + \varepsilon\end{aligned}$$

where $\lim_n h(n) = 0$. Hence $Tz \in K_\alpha$ for each α and, therefore, $Tz \in K$.

THEOREM 3. *Suppose X is a nuclear locally convex space that is also an F-space. If for some x_1 in C the sequence $\{v_n\}$ is bounded, then T has a fixed point in C. In this case, $S_\lambda^n x_1 \to y$ weakly where $Ty = y$ and $S_\lambda = \lambda I + (1 - \lambda)T$, $0 < \lambda < 1$.*

Proof. Let K be as in Theorem 2. K is a closed bounded convex subset of C and, therefore, K is weakly sequentially compact. Also, $T(K) \subseteqq K$. By Theorem 2 of [4], T has a fixed point in K. Applying Theorem 9 of [4] to K gives the last part of the theorem.

REMARK. Theorem 3 is valid in any generalized Hilbert space in which closed, bounded, and convex subsets are weakly sequentially compact. One would like to have strong convergence of some sequence to a fixed point of T. The next theorem shows that if T is linear and C is a linear subspace, you have the desired result. One can not obtain strong convergence without some additional conditions; however, one should be able to replace the linearity of T by some less restrictive condition.

THEOREM 4. *Suppose X is nuclear locally convex space that is also an F-space and C is a linear subspace of X. If for some x_1 in C the sequence $\{v_n\}$ is bounded and T is linear, there exists x_0 C such that $\lim_n S_\lambda^n x_0 = y_0$ where $Ty_0 = y_0$.*

Proof. Let $x_0 \in K$ where K is as in Theorem 3. $\{S_\lambda^n(x_0)\} \subseteqq K$ and it has a subsequence that converges weakly to y_0 in K. We show that the sequence $\{S_\lambda^n\}_n$ of linear operators is a system of almost invariant integrals for the semigroup $\{T^m: m = 0, 1, 2, \cdots\}$ and then apply the mean ergodic theorem of Eberlein [3] to obtain $\lim_n S_\lambda^n x_0 = y_0$ with $Ty_0 = y_0$.

(1) $S_\lambda^n: C \to C$ is linear.

(2) For each n and each x, $S_\lambda^n(x)$ is in the convex hull of $x, Tx, \cdots, T^n x$, since T is linear gives

$$S_\lambda^n = [\lambda I + (1-\lambda)T]^n = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j}(1-x)^j T^j .$$

(3) We show that $\{S_\lambda^n\}$ is an equicontinuous family. By a theorem of Banach [5, p. 169], it suffices to prove that $\{S_\lambda^n(x): n = 0, 1, 2, \cdots\}$ is bounded for every x in K. Thus it suffices to show that $\{\rho_\alpha(S_\lambda^n(x)): n = 1, 2, \cdots\}$ is bounded for every $\alpha \in \Delta$. This is true since

$$\begin{aligned} \rho_\alpha(S_\lambda^n x) &\leqq \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j \rho_\alpha(T_x^j) \\ &\leqq \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j \rho_\alpha(x) \\ &\leqq \rho_\alpha(x) . \end{aligned}$$

From the proof of Theorem 3, T has a fixed point in K. From Theorem 6 of [4], $S_\lambda^{n+1}x - S_\lambda^n x \to 0$. Now

$$\begin{aligned} S_\lambda^{n+1}x - S_\lambda^n x &= S_\lambda(S_\lambda^n x) - S_\lambda^n x \\ &= \lambda S_\lambda^n x + (1-\lambda)T(S_\lambda^n x) - S_\lambda^n x \\ &= (1-\lambda)(TS_\lambda^n x - S_\lambda^n x) . \end{aligned}$$

Thus $TS_\lambda^n x - S_\lambda^n x \to 0$. Since T is linear and continuous, $T^2 S_\lambda^n x - S_\lambda^n x = T(TS_\lambda^n x - S_\lambda^n x) - (TS_\lambda^n x - S_\lambda^n x) \to T(0) + 0 = 0$. Using induction, we have

(4a) $T^m S_\lambda^n x - S_\lambda^n x \to 0$ as $n \to \infty$ for every x in C and all $m = 0, 1, 2, \cdots$.

Since S_λ^n is a polynomial in T, $T^m S_\lambda^n = S_\lambda^n T^m$ and, using (4a), we have

(4b) $S_\lambda^n T^m x - S_\lambda^n x \to 0$ as $n \to \infty$ for every x in C and all $m = 0, 1, 2, \cdots$. Now, we apply the mean ergodic theorem to obtain the desired result.

REMARK. $\{S_\lambda^n x_1\}$ is a special sequence $\{v_n\}$ given by the Mann iteration process. Just let $a_{n1} = \lambda^{n-1}$, $a_{nj} = \lambda^{n-j}(1-\lambda)$ for $j = 2, 3, \cdots, n$, and $a_{nj} = 0$ for $j > n$, $n = 1, 2, 3, \cdots$.

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