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THE COMMON FIXED POINT THEORY OF SINGLEVALUED MAPPINGS AND MULTIVALUED MAPPINGS

SHIGERU ITOH AND WATARU TAKAHASHI

First, in a locally convex topological vector space, a theorem is proved which extends fixed point theorems by Lau and Fan-Glicksberg. In a strictly convex Banach space, a similar result is obtained, which is a generalization of the fixed point theorem by Bohnenblust-Karlin. In a Banach space which satisfies Opial's condition, a fixed point theorem is given that generalizes both results by Holmes-Lau-Lim and Lami Dozo. In a uniformly convex Banach space, a similar theorem is considered which extends Lim's fixed point theorem. Finally, the existence of common fixed points of a quasi-nonexpansive mapping and a multivalued nonexpansive mapping is established by an elementary constructive method in a Hilbert space. In many cases, preliminary results on nonexpansive or quasi-nonexpansive retractions are obtained which play crucial roles in proving the above theorems.

1. Introduction. De Marr [11] proved that if G is a commutative family of nonexpansive mappings on a compact convex subset K of Banach space, then G has a common fixed point in K. Then results for nonexpansive mappings on weakly compact convex subsets appeared. Browder [5] proved a fixed point theorem for a single nonexpansive mapping on a bounded closed convex subset of a Hilbert space, while Browder [6] and Göhde [19] on a bounded closed convex subset of a uniformly convex Banach space. Kirk [23] obtained a general form of the similar result for a single nonexpansive mapping on a weakly compact convex subset K of a Banach space in the case that K has normal structure. Since then, various fixed point theorems for nonexpansive mappings were given by Belluce and Kirk [2, 3], Takahashi [33, 34], Mitchell [31], Kirk [24], Holmes and Lau [21], Dotson [12], Lau [26], Bruck [9, 10] and Lim [27], etc. Among them, Bruck obtained interesting characterizations of fixed point sets of nonexpansive mappings. There were also Dotson's results [13] on fixed points of quasi-nonexpansive mappings. Lim [28] proved that if K is a weakly compact convex subset of a Banach space and K has normal structure, then K has complete normal structure. Hence combining this with a theorem of Holmes and Lau [21], it follows that if G is a left reversible semigroup of nonexpansive mappings on a weakly compact convex subset K of a

Banach space and K has normal structure, then there is a common fixed point of G in K.

On the other hand, fixed point theorems for multivalued upper semicontinuous mappings were proved by Bohnenblust and Karlin [4], Glicksberg [17], Fan [15], Browder [8], and Takahashi [35], etc. And fixed point theorems for multivalued nonexpansive mappings were given by Markin [32], Lami Dozo [25], Assad and Kirk [1] and Lim [29], etc. Contrary to singlevalued cases, results for multivalued nonexpansive mappings on weakly compact convex subsets (which have normal structure) of general Banach spaces are not yet obtained up to the present.

In [22] we examined the existence of common fixed points for a singlevalued mapping and a multivalued mapping. In this paper we give various common fixed point theorems for families of singlevalued nonexpansive or quasi-nonexpansive mappings and multivalued upper semicontinuous or nonexpansive mappings. These generalize both results in singlevalued and multivalued cases simultaneously. We also obtain some theorems on nonexpansive or quasi-nonexpansive retractions. At first, in §3 we prove theorems for semigroups of nonexpansive mappings and multivalued upper semicontinuous mappings in locally convex topological vector spaces, while in §4 those for families of nonexpansive or quasi-nonexpansive mappings and multivalued nonexpansive mappings in Banach spaces. Finally, in §5 we give more precise results in Hilbert spaces.

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2. Preliminaries. Let X be a topological space, 2^X the family of all subsets of X, T a mapping of X into 2^X such that Tx is nonempty for all $x \in X$. T is called *upper semicontinuous* if for each closed subset C of X, $T^{-1}(C) = \{x \in X: Tx \cap C \neq \emptyset\}$ is closed.

Let E be a locally convex topological vector space, Q a family of continuous seminorms that generates the topology of E. Let X be a nonempty subset of E, f a mapping of X into E. Denote by $F(f)$ (which may be empty) the set of fixed points of f in X. f is called *Q-nonexpansive with respect to M* (a nonempty subset of X) if for any $p \in Q$, $p(fx - fu) \leqq p(x - u)$ whenever $x \in X$ and $u \in M$. If $M = X$, then f is called *Q-nonexpansive*, and if $M = F(f)$, then f is called *Q-quasi-nonexpansive* respectively.

Let B be a Banach space, X a nonempty subset of B, f a mapping of X into B. f is said to be *k-contraction* (where $0 \leqq k < 1$) if for any x, $y \in X$, $\|fx - fy\| \leqq k\|x - y\|$. f is said to be *nonexpansive with respect to M* (a nonempty subset of X) if for each $x \in X$, $u \in M$, $\|fx - fu\| \leqq \|x - u\|$. If $M = X$, f is said to be *nonexpan-*

sive, and if $M = F(f)$, f is said to be *quasi-nonexpansive*. f is said to be *generalized nonexpansive* if there exist nonnegative real numbers a, b, c with $a + 2b + 2c \leqq 1$ such that for any $x, y \in X$,

$$\begin{aligned}||fx - fy|| \leqq a||x - y|| + b\{||x - fx|| + ||y - fy||\} \\ + c\{||x - fy|| + ||y - fx||\} .\end{aligned}$$

Note that if a generalized nonexpansive mapping has fixed points, then it is quasi-nonexpansive. Let T be a mapping of X into 2^X such that for each $x \in X$, Tx is nonempty bounded closed and let D be the Hausdorff metric on nonempty bounded closed subsets of B induced by the norm of B. If for any $x, y \in X$, $D(Tx, Ty) \leqq ||x - y||$, T is called *nonexpansive*.

Let G be a semitopological semigroup, that is, G is a semigroup with a Hausdorff topology such that the semigroup operation $G \times G \to G$ by $(s, t) \to st$ $(s, t \in G)$ is separately continuous. G is said to be *left reversible* if any two nonempty closed right ideals of G have nonvoid intersection. Let $C(G)$ be the Banach algebra of all continuous bounded realvalued functions on G with sup norm. For each $t \in G$, define the operators r_t, l_t on $C(G)$ by $(r_t h)(s) = h(st)$, $(l_t h)(s) = h(ts)$ for all $s \in G$ and $h \in C(G)$. Let A be a subspace of $C(G)$ containing the constant function 1. An element m of the dual space A^* of A is called a *mean* if $m(1) = ||m|| = 1$. For any $h \in C(G)$, denote $r_G h = \{r_t h: t \in G\}$. Then $AP(G) = \{h \in C(G): r_G h$ is precompact in $C(G)\}$ is a left and right translation invariant (i.e., $l_t(AP(G)) \subset AP(G)$, $r_t(AP(G)) \subset AP(G)$ for all $t \in G$) closed subalgebra of $C(G)$ containing 1. A mean m on $AP(G)$ is called a *left invariant mean* if $m(l_t h) = m(h)$ for all $h \in AP(G)$ and $t \in G$. If G is left reversible, then $AP(G)$ has a left invariant mean (cf. [21, 26]).

An *action* of a semitopological semigroup G on a topological space X is a mapping $G \times X \to X$ such that $(st)x = s(tx)$ for all $s, t \in G$ and $x \in X$, where tx denotes the image of (t, x). The action is called *separately continuous* if the mapping $G \times X \to X$ is separately continuous. If X is a subset of a Banach space B (a locally convex space E), then the action of G on X is called *(Q-)nonexpansive* if for each $s \in G$, the mapping of X into X defined by $x \to sx$ $(x \in X)$ is (Q-)nonexpansive.

Let X be a nonempty subset of a locally convex topological vector space E (or a Banach space B), f a mapping of X into X, G a family of mappings of X into X, T a mapping of X into 2^X such that Tx is nonempty for every $x \in X$. We denote by Gx the set $\{gx: g \in G\}$ for any $x \in X$, and by $F(G)$ (which may be nonempty) the set of common fixed points of G in X. Let Y be a empty subset of X, then we denote by $bd_X Y$ the relative boundary of Y with respect to X, that is, $bd_X Y = \mathrm{cl}(Y) \cap \mathrm{cl}(X \backslash Y)$, where $\mathrm{cl}(Z)$ is the closure of Z. Y is said to be *f-invariant* if $f(Y) \subset Y$. Y is said to be *G-invariant* if Y is g-invariant for all $g \in G$. f and T is said to *commute* if for each $x \in X$, $f(Tx) \subset T(fx)$. f and T is said to *commute weakly* if for any $x \in X$, $f(bd_X Tx) \subset T(fx)$. G and T is said to commute (weakly) if each $g \in G$ and T commute (weakly). Let G be a semitopological semigroup acting on X, then we also denote by $F(G)$ the set of fixed points of the semigroup of mappings $J = \{x \to sx (x \in X): s \in G\}$. G and T is said to commute (weakly) if J and T commute (weakly). Y is called G-invariant if Y is J-invariant. A mapping r of X into X is called a *retraction* if $r^2 = r$.

Now we give some results in Banach spaces.

PROPOSITION 1. *Let K be a nonempty closed convex subset of a Banach space B, f a mapping of K into B such that $M = \{y \in K: \|fy - y\| = \min\{\|fy - x\|: x \in K\}\}$ is nonempty and f is nonexpansive with respect to M. Then M is a closed set on which f is continuous. Furthermore, if B is strictly convex and f is isometric on M, then $w = ku + (1-k)v (u, v \in M, 0 < k < 1)$ implies that $fw = kfu + (1-k)fv$.*

Proof. It is obvious that f is continuous at each point of M. We show that M is closed. Let $\{y_n\}$ be a sequence of M which converges to $z \in K$. For any $c > 0$, take $m = m(c)$ such that $\|y_m - z\| < c$, then

$$\begin{aligned} \|fz - z\| &\leqq \|fz - fy_m\| + \|fy_m - y_m\| + \|y_m - z\| \\ &\leqq \|y_m - z\| + \|fy_m - y_m\| + \|y_m - z\| \\ &< 2c + \|fy_m - y_m\| . \end{aligned}$$

For each $x \in K$,

$$\begin{aligned} \|fy_m - x\| &\leqq \|fy_m - fz\| + \|fz - x\| \\ &\leqq \|y_m - z\| + \|fz - x\| \\ &< c + \|fz - x\| , \end{aligned}$$

hence

$$\begin{aligned} \|fy_m - y_m\| &= \min\{\|fy_m - x\|: x \in K\} \\ &\leqq c + \inf\{\|fz - x\|: x \in K\} . \end{aligned}$$

Therefore, it follows that

$$\|fz - z\| < 3c + \inf\{\|fz - x\|: x \in K\} .$$

Since c is arbitrary, we have $\|fz - z\| \leqq \inf\{\|fz - x\|: x \in K\}$ and $z \in M$. Thus M is closed.

Suppose B is strictly convex and f is isometric on M. Let

$u, v \in M$, $0 < k < 1$ and $w = ku + (1-k)v$. By assumption

$$||u - v|| = ||fu - fv|| \quad \text{and} \quad ||fw - fv|| \leqq ||w - v|| .$$

Thus

$$\begin{aligned} ||u - w|| &= ||u - v|| - ||w - v|| \\ &\leqq ||fu - fv|| - ||fw - fv|| \\ &\leqq ||fu - fw|| \leqq ||u - w|| . \end{aligned}$$

Hence $||fu - fw|| = ||u - w||$. Similarly we have $||fv - fw|| = ||v - w||$. Therefore, we obtain

$$\begin{aligned} ||fu - fw|| + ||fw - fv|| &= ||u - w|| + ||w - v|| \\ &= ||u - v|| = ||fu - fv|| . \end{aligned}$$

Since B is strictly convex, it follows that $fw = kfu + (1-k)fv$.

For nonexpansive mappings a similar result to the second part of Proposition 1 was given by Edelstein [14]. As a direct consequence of Proposition 1, we have the following result which was due to Dotson [13].

COROLLARY 1. *Let K be a closed convex subset of a strictly convex Banach space, f a quasi-nonexpansive mapping of K into K. Then $F(f)$ is a nonempty closed convex set on which f is continuous.*

Proof. Since $f(K) \subset K$, $F(f)$ equals the set M as in Proposition 1. Thus, for any $u, v \in F(f)$, $0 < k < 1$ with $w = ku + (1-k)v$, we have $fw = kfu + (1-k)fv = ku + (1-k)v = w$. This implies that $F(f)$ is convex.

The following was given by Bruck [10].

PROPOSITION 2. *Let X be a Hausdorff topological space, S a semigroup of mappings of X into X. If S is compact in the topology of pointwise convergence on X and for each $x \in X$, there exists a common fixed point of S in Sx, then there is in S a retraction of X onto $F(S)$.*

3. Fixed point theorems in topological vector spaces. Throughout this section, let K be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E, Q a family of continuous seminorms that generates the topology of E. The notion of nonexpansiveness always means Q-nonexpansiveness and we omit the term Q.

THEOREM 1. *Let G be a family of mappings of K into K such that $F(G)$ is nonempty and each $g \in G$ is nonexpansive with respect to $F(G)$. If any G-invariant closed convex subset of K has a common fixed point of G, then there exists a quasi-nonexpansive retraction r of K onto $F(G)$ for which every G-invariant closed convex subset of K is r-invariant.*

Proof. Define $S = \{s: K \to K: s$ is nonexpansive with respect to $F(G)$, $F(s) \supset F(G)$ and any G-invariant closed convex subset of K is s-invariant$\}$. Then $G \subset S$. We show that S is compact in the topology of pointwise convergence. Fix $v \in F(G)$. For each $x \in K$, put

$$Wx = \{y \in K: p(y - v) \leqq p(x - v) \quad \text{for all} \quad p \in Q\} ,$$

then Wx is nonempty compact convex. In fact, for any $s \in S$, $p(sx - v) \leqq p(x - v)$, hence $sx \in Wx$. S can be regarded as a subset of the product topological space $W = \prod_{x \in K} Wx$. Since W is compact and the topology on W is that of pointwise convergence, it suffices to show that S is closed in W. Suppose that $\{s_i\}$ is a net in S which converges to s in W. For any $x \in K$, $u \in F(G)$, $p \in Q$, we have

$$p(sx - u) = \lim_i p(s_i x - u) \leqq p(x - u)$$

and

$$su = \lim_i s_i u = u .$$

It is obvious that if C is a G-invariant closed convex subset of K, then $s(C) \subset C$. Hence $s \in S$ and S is closed in W. It can be seen that S is a semigroup and for any $s, t \in S$, $k(0 \leqq k \leqq 1)$, $ks + (1-k)t \in S$. This implies that for each $x \in K$, Sx is compact convex and G-invariant. By assumption Sx has a common fixed point of G, that is, Sx has a common fixed point of S. Therefore, by Proposition 2 there exists in S a retraction r of K onto $F(S) = F(G)$. Since $F(r) = F(G)$, r is quasi-nonexpansive.

COROLLARY 2. *Let f be a continuous quasi-nonexpansive mapping of K into K, then there exists a quasi-nonexpansive retraction r of K onto $F(f)$ such that each f-invariant closed convex subset of K is r-invariant.*

Proof. Since f is continuous, any f-invariant closed convex subset of K has a fixed point of f by Tychonoff's fixed point theorem.

For a nonexpansive action of a semitopological semigroup on K, we obtain the following.

THEOREM 2. *Let G be a semitopological semigroup acting on K such that $AP(G)$ has a left invariant mean. If the action of G on K is separately continuous and nonexpansive, then there exists a nonexpansive retraction r of K onto $F(G)$ such that every G-invariant closed convex subset of K is r-invariant.*

Proof. $F(G)$ is nonempty by Lau's theorem [26]. Put $S = \{f: K \to K: f$ is nonexpansive with $F(f) \supset F(G)$ and any G-invariant closed convex subset of K is f-invariant$\}$, then as in the proof of Theorem 1, S is a semigroup and compact in the topology of pointwise convergence. For each $x \in K$, Sx is a G-invariant compact convex subset of K, so by [26] again there exists a common fixed point of G in Sx which is also a fixed point of S. Thus, by Proposition 2 there is in S a retraction r of K onto $F(S) = F(G)$.

Now we give a common fixed point theorem for a semigroup of nonexpansive mappings and a multivalued upper semicontinuous mapping.

THEOREM 3. *Let G be a semitopological semigroup acting on K for which $AP(G)$ has a left invariant mean. Suppose the action of G on K is separately continuous and nonexpansive. Let T be an upper semicontinuous mapping of K into 2^K such that for each $x \in K$, Tx is nonempty compact convex. If G and T commute, then there exists an element $z \in K$ such that $gz = z \in Tz$ for all $g \in G$.*

Proof. By Theorem 2 there exists a nonexpansive retraction r of K onto $F(G)$ for which every G-invariant closed convex subset of K is r-invariant. Define a mapping S of K into 2^K by $Sx = T(rx)$ $(x \in K)$, then S is upper semicontinuous. By Fan-Glicksberg's fixed point theorem [15], [17] there is a $v \in K$ such that $v \in Sv$. Since for any $g \in G$, $g(Sv) = g(T(rv)) \subset T(rv) = Sv$, it follows that $r(Sv) \subset Sv$ and in particular $rv \in Sv$. Put $z = rv$, then we have $gz = z \in Tz$ for all $g \in G$.

If G is generated by a single mapping and T is a singlevalued mapping, then the following holds.

COROLLARY 3. *Let f be a continuous mapping of K into K, g a nonexpansive mapping of K into K. If f and g commute, then there exists a $z \in K$ such that $fz = gz = z$.*

4. Fixed point theorems in Banach spaces. In this section we consider various common fixed point theorems for singlevalued mappings and multivalued mappings in Banach spaces. At first, we have the following theorem for a family of quasi-nonexpansive mappings and an upper semicontinuous multivalued mapping on a compact convex subset of a strictly convex Banach space.

THEOREM 4. *Let K be a nonempty compact convex subset of a strictly convex Banach space, G a family of mappings of K into K for which $F(G)$ is nonempty and every $g \in G$ is nonexpansive with respect to $F(G)$, T an upper semicontinuous mapping of K into 2^K such that for each $x \in K$, Tx is nonempty closed convex. If G and T commute weakly, then there exists a point $z \in K$ such that $gz = z \in Tz$ for all $g \in G$.*

Proof. By Corollary 1 $F(G)$ is closed convex. Choose any point $u \in F(G)$. For each $x \in F(G)$, Tx is nonempty closed convex, hence there exists a unique element $v \in bd_K Tx$ nearest to u. Since for any $g \in G$, $g(bd_K Tx) \subset T(gx) = Tx$ and g is nonexpansive with respect to $F(G)$, it follows that $\|gv - u\| \leqq \|v - u\|$ and $gv = v$. Thus we have $Tx \cap F(G) \neq \emptyset$ for all $x \in F(G)$. Now define a multivalued mapping S of $F(G)$ into $2^{F(G)}$ by $Sx = Tx \cap F(G)$ $(x \in F(G))$. Then it is obvious that S is upper semicontinuous. By the fixed point theorem of Bohnenblust and Karlin [4], we obtain a point $z \in F(G)$ such that $z \in Sz$. Hence $gz = z \in Tz$ for all $g \in G$.

COROLLARY 4. *Let K be a nonempty compact convex subset of a strictly convex Banach space, f a continuous generalized nonexpansive mapping of K into K, T an upper semicontinuous mapping of K into 2^K such that for any $x \in K$, Tx is nonempty closed convex. If f and T commute weakly, then there exists an element $z \in K$ for which $fz = z \in Tz$.*

Proof. By Schauder's fixed point theorem, $F(f)$ is nonempty, thus f is quasi-nonexpansive.

On a weakly compact convex subset of a strictly convex Banach space, an analogous result to Theorem 1 holds without any assumption.

THEOREM 5. *Let K be a nonempty weakly compact convex subset of a strictly convex Banach space, G a family of mappings of K into K for which $F(G)$ is nonempty and any $g \in G$ is nonexpansive with respect to $F(G)$. Then there exists a quasi-nonexpansive retraction r of K onto $F(G)$ such that each G-invariant closed convex subset of K is r-invariant.*

Proof. We make use of methods employed by Bruck [9, 10]. Put $S = \{s: K \to K: s$ is nonexpansive with respect to $F(G)$, $F(s) \supset F(G)$ and every G-invariant closed convex subset of K is s-invariant$\}$. Then $G \subset S$. It is obvious that S is a semigroup of mappings of K into K. We show that S is compact in the topology of pointwise weak convergence on K. Fix an element $v \in F(G)$. For each $x \in K$, denote $Wx = \{y \in K: ||y - v|| \leqq ||x - v||\}$. Then since for any $s \in S$, $||sx - v|| \leqq ||x - v||$, $Sx \subset Wx$ and Wx is closed convex. Since K is weakly compact, Wx is weakly compact. S is a subset of the product topological space $W = \prod_{x \in K} Wx$ (each Wx is endowed with the weak topology). W is compact and the topology of W is that of pointwise weak convergence, hence it is sufficient to prove that S is closed in W. Let $\{s_i\}$ be a net in S which converges to s in W, then for any $x \in K$, $u \in F(G)$,

$$\begin{aligned} ||sx - u|| &= ||w - \lim_i (s_i x - u)|| \\ &\leqq \liminf_i ||s_i x - u|| \leqq ||x - u|| , \end{aligned}$$

and

$$su = w - \lim s_i u = u .$$

For any G-invariant closed convex subset C of K, C is also weakly closed, hence $s(C) \subset C$. These imply that $s \in S$ and S is closed in W. Now, for any $x \in K$, consider Sx. Then since for each $s, t \in S$ and $0 \leqq k \leqq 1$, $ks + (1 - k)t \in S$ and S is a semigroup, Sx is a G-invariant closed convex subset of K. Since B is strictly convex, there is a unique point $w \in Sx$ such that $||w - v|| = \min\{||w - y||: y \in Sx\}$. For any $s \in S$, $sw \in Sx$ and $||sw - v|| \leqq ||w - v||$, hence $sw = w$. By Proposition 2 there exists a retraction $r \in S$ of K onto $F(S) = F(G)$. Since $F(r) = F(G)$, r is quasi-nonexpansive.

COROLLARY 5. *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space, f a continuous generalized nonexpansive mapping of K into K. Then there exists a quasi-nonexpansive retraction r of K onto $F(f)$ such that any f-invariant closed convex subset of K is r-invariant.*

Proof. By the fixed point theorem of Goebel, Kirk, and Shimi [18] $F(f)$ is nonempty, hence f is quasi-nonexpansive.

REMARK 1. Let r be a quasi-nonexpansive retraction of a subset K of a Banach space into itself, then for any $x, y \in K$, it follows that $||rx - ry|| \leqq 1/2\{||x - ry|| + ||y - rx||\}$. This is a special form of generalized nonexpansive mapping.

If K is a subset of a Banach space and K has normal structure, then the following holds. For the definition of normal structure and related results, see for example Lim [28].

THEOREM 6. *Let K be a nonempty weakly compact convex subset of a Banach space, G a left reversible semitopological semigroup acting on K. If the action of G on K is separately continuous, nonexpansive and K has normal structure, then there exists a nonexpansive retraction r of K onto $F(G)$ such that every G-invariant closed convex subset of K is r-invariant.*

Proof. By theorems of Holmes and Lau [21], Lim [28], $F(G)$ is nonempty. Define $S = \{f: K \to K: f$ is nonexpansive with $F(f) \supset F(G)$ and every G-invariant closed convex subset of K is f-invariant$\}$. Then S is a semigroup and compact in the topology of pointwise weak convergence as in the proof of Theorem 5. Moreover, for any $x \in K$, Sx is G-invariant and weakly compact, hence there is a common fixed point of G in Sx. This point is also a fixed point of S. Therefore Proposition 2 implies that there exists in S a retraction r of K onto $F(S) = F(G)$.

The following theorem is a common fixed point theorem for a left reversible semigroup of nonexpansive mappings and a multivalued nonexpansive mapping in a Banach space which satisfies Opial's condition. Concerning results related to Opial's condition, we refer the reader to [20, 25].

THEOREM 7. *Let K be a nonempty weakly compact convex subset of a Banach space which satisfies Opial's condition, G a left reversible semitopological semigroup acting on K, T a nonexpansive mapping of K into 2^K such that for each $x \in K$, Tx is nonempty compact convex. Suppose the action of G on K is separately continuous and nonexpansive. If G and T commute, then there exists an element $z \in K$ such that $gz = z \in Tz$ for all $g \in G$.*

Proof. K has normal structure by a theorem of Gossez and Lami Dozo [20]. Hence, by Theorem 6 there is a nonexpansive retraction r of K onto $F(G)$ for which every G-invariant closed convex subset of K is r-invariant. Define a mapping S of K into 2^K by $Sx = T(rx)$ $(x \in K)$, then S is nonexpansive. Thus, there exists a fixed point v of S in K by Lami Dozo's fixed point theorem [25]. Since G and T commute, Sv is G-invariant, hence $rv \in Sv$. Let $z = rv$. Then it follows that $gz = z \in Tz$ for all $g \in G$.

In a uniformly convex Banach space we can prove the following common fixed point theorem for a family of quasi-nonexpansive mappings and a multivalued nonexpansive mapping.

THEOREM 8. *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space, G a family of mappings of K into K for which $F(G)$ is nonempty and each $g \in G$ is nonexpansive with respect to $F(G)$, T a nonexpansive mapping of K into 2^K, where for any $x \in K$, Tx is nonempty compact convex. If G and T commute weakly, then there exists a point $z \in K$ such that $gz = z \in Tz$ for all $g \in G$.*

Proof. Let $x \in F(G)$. Then for any $g \in G$, $g(bd_K Tx) \subset T(gx) = Tx$ and g is nonexpansive with respect to $F(G)$, hence it follows that $Tx \cap F(G) \neq \emptyset$, since a unique point $w \in bd_K Tx$ nearest to x is a common fixed point of G. Define a multivalued mapping S of $F(G)$ into $2^{F(G)}$ by $Sx = Tx \cap F(G)$ $(x \in F(G))$. Then S is nonexpansive. In fact, for any $x, y \in F(G)$,

$$\begin{aligned} D(Sx, Sy) &= \max\{\sup_{u \in Sx} d(u, Sy), \sup_{v \in Sy} d(v, Sx)\} \\ &= \max\{\sup_{u \in Sx} d(u, Ty), \sup_{v \in Sy} d(v, Tx)\} \\ &\leqq \max\{\sup_{u \in Tx} d(u, Ty), \sup_{v \in Ty} d(v, Tx)\} \\ &= D(Tx, Ty) \leqq \|x - y\| , \end{aligned}$$

where $d(b, A) = \inf\{\|b - a\| : a \in A\}$. Now by Lim's fixed point theorem [29], there exists an element $z \in F(G)$ for which $z \in Sz$. Thus we obtain $gz = z \in Tz$ for all $g \in G$.

As direct consequences of Theorem 8, we have the following corollaries.

COROLLARY 6. *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space, G a left reversible semitopological semigroup acting on K, T a nonexpansive mapping of K into 2^K such that for each $x \in K$, Tx is nonempty compact convex. Suppose the action of G on K is separately continuous and nonexpansive. If G and T commute weakly, then there is a $z \in K$ such that $gz = z \in Tz$ for all $g \in G$.*

Proof. Browder [6] showed that K has normal structure, hence $F(G)$ is nonempty by theorems of Holmes and Lau [21] and Lim [28].

COROLLARY 7. *Let K be a nonempty bounded closed convex sub-*

set of a uniformly convex Banach space, f a continuous generalized nonexpansive mapping of K into K, T a nonexpansive mapping of K into 2^K, where for any $x \in K$, Tx is nonempty compact convex. If f and T commute weakly, then there exists an element $z \in K$ such that $fz = z \in Tz$.

Proof. f is quasi-nonexpansive, since $F(f)$ is nonempty by the theorem of Goebel, Kirk, and Shimi [18].

5. Fixed point theorems in Hilbert spaces. Throughout this section let H be a Hilbert space. For the sake of simplicity we assume that H is real. For each nonempty closed convex subset M of H, we denote by P_M the projection of H onto M, and recall that P_M is nonexpansive (cf. Phelps [32]).

THEOREM 9. *Let K be a nonempty closed convex subset of H, f a continuous quasi-nonexpansive mapping of K into K and $M = F(f)$. Then for each compact convex subset C of K such that $f(bd_K C) \subset C$, we have $P_M(C) \subset C$.*

Proof. Suppose C is a compact convex subset of K such that $f(bd_K C) \subset C$. Take a sequence $\{k_n\}$ of real numbers for which $0 < k_n < 1$ and $k_n \to 0$ as $n \to \infty$. Fix an element $w \in C$ and for each n, define a mapping f_n of C into K by

$$f_n x = k_n w + (1 - k_n) fx \quad (x \in C) ,$$

then since f_n is continuous and C is compact, by a theorem of Fan [16] there exists a point $y_n \in C$ such that

$$\|f_n y_n - y_n\| = \min \{\|f_n y_n - x\| : x \in C\} .$$

Since $f(bd_K C) \subset C$ implies $f_n(bd_K C) \subset C$, we obtain $f_n y_n = y_n$. We may assume that $\{y_n\}$ converges to some $v \in C$. This v is a fixed point of f. In fact, choose any $u \in F(f)$, then by quasi-nonexpansiveness of f we have $\|fy_n - u\| \leqq \|y_n - u\|$. Hence $\{fy_n\}$ is bounded since C is bounded (compact). This implies that $\|y_n - fy_n\| = k_n \|w - fy_n\| \to 0$ as $n \to \infty$. Thus the right hand side of the inequality

$$\|fv - v\| \leqq \|fv - fy_n\| + \|fy_n - y_n\| + \|y_n - v\|$$

tends to 0 as $n \to \infty$ and we obtain $fv = v$. Now we show that $P_M w = v$ by methods employed by Browder [7]. We have

$$y_n - P_M w = \frac{1 - k_n}{k_n}(fy_n - y_n) + w - P_M w ,$$

hence

$$\|y_n - P_M w\|^2 = \frac{1-k_n}{k_n}(fy_n - y_n, y_n - P_M w) + (w - P_M w, y_n - P_M w) .$$

Since f is quasi-nonexpansive, we obtain

$$((I-f)y_n - (I-f)P_M w, y_n - P_M w) \geqq 0 ,$$

where I is the identity mapping on H. Thus

$$(fy_n - y_n, y_n - P_M w) \leqq 0 .$$

Also, since $(w - P_M w, P_M w - v) \geqq 0$, we have

$$(w - P_M w, y_n - P_M w) \leqq (w - P_M w, y_n - v) .$$

Therefore it follows that

$$\|y_n - P_M w\|^2 \leqq (w - P_M w, y_n - v) .$$

Since $\{y_n\}$ converges to v, $P_M w = v \in C$.

Similarly we have the following

THEOREM 10. *Let K be a nonempty closed convex subset of H, f a nonexpansive mapping of K into K for which $M = F(f)$ is nonempty. Then for any bounded closed convex subset C of K such that $f(bd_K C) \subset C$, $P_M(C) \subset C$ holds.*

Proof. Take a sequence $\{k_n\}$ of real numbers such that $0 < k_n < 1$ and $k_n \to 0$ as $n \to \infty$. Fix $w \in C$ and for any n, define a mapping f_n of C into K by

$$f_n x = k_n w + (1 - k_n) fx \quad (x \in C) ,$$

then, since f_n is $(1-k_n)$-contraction and P_C is nonexpansive, $P_C f_n$ is a $(1-k_n)$-contraction mapping of C into C. Hence there exists a unique fixed point $y_n \in C$ of $P_C f_n$, that is,

$$\|f_n y_n - y_n\| = \min\{\|f_n y_n - x\| : x \in C\} .$$

Since $f(bd_K C) \subset C$ implies $f_n(bd_K C) \subset C$, we have $f_n y_n = y_n$. We may assume that $\{y_n\}$ converges weakly to some $v \in C$. The rest of the proof proceeds as in the proof of Theorem 9 by using methods in Browder's paper [7]. In conclusion, we obtain that $fv = v$ by demiclosedness of $I - f$ (cf. [12, Remark 3]) and $P_M w = v$.

PROPOSITION 3. *Let K be a nonempty bounded closed convex subset of H, f a nonexpansive mapping of K into H. Then there exists* *a point* $z \in K$ *such that* $\|fz - z\| = \min\{\|fz - x\|: x \in K\}$.

Proof. Since $P_M f$ is a nonexpansive mapping of K into K, by Browder's fixed point theorem [5] there exists a $z \in K$ such that $P_K fz = z$. For this z, we have the desired equality.

REMARK 2. We do not know whether Proposition 3 is true when K is only required to be weakly compact convex and to have normal structure in a Banach space.

Now we can prove a common fixed point theorem for a quasi-nonexpansive mapping and a multivalued nonexpansive mapping in a Hilbert space.

THEOREM 11. *Let K be a nonempty bounded closed convex subset of H, f a continuous mapping of K into H, T a nonexpansive mapping of K into 2^K, where for any $x \in K$, Tx is nonempty compact convex. Suppose $M = \{x \in K: \|fx - x\| = \min\{\|fx - y\|: y \in K\}\}$ is nonempty and f is nonexpansive with respect to M. If for each $x \in K$, $P_K f(bd_K Tx) \subset T(P_K fx)$, then there exists an element $z \in K$ such that $\|fz - z\| = \min\{\|fz - x\|: x \in K\}$ and $z \in Tz$.*

Proof. It is obvious that $M = F(P_K f)$. For any $x \in K$, $u \in M$, we have

$$\begin{aligned}\|P_K fx - u\| &= \|P_K fx - P_K fu\| \\ &\leqq \|fx - fu\| \leqq \|x - u\| .\end{aligned}$$

Hence $P_K f$ is a quasi-nonexpansive mapping of K into K. Define a mapping S of K into 2^K by $Sx = T(P_M x)$ $(x \in K)$. Then S is non-expansive and has a fixed point v in K. Since

$$\begin{aligned}P_K f(bd_K Sv) &= P_K f(bd_K T(P_M v)) \\ &\subset T(P_K f(P_M v)) = T(P_M v) = Sv ,\end{aligned}$$

by Theorem 9 it follows that $P_K(Sv) \subset Sv$. In particular $P_M v \in Sv$. Denote $z = P_M v$. Then we have $P_K fz = z \in Tz$.

REMARK 3. Theorem 11 is also a corollary to Theorem 8, but the proof given above is a constructive one. Compare this with the proofs of Theorem 3 and Theorem 7.

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DEPARTMENT OF INFORMATION SCIENCES
TOKYO INSTITUTE OF TECHNOLOGY
OH-OKAYAMA, MEGURO-KU, TOKYO 152, JAPAN