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TRANSVERSE WHITEHEAD TRIANGULATIONS

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Suppose M and N are PL manifolds and $f: M \to N$ is a proper PL map. Triangulate M and N so that f is simplical and let X be the dual complex in N. Then for each open simplex σ in X, $f^{-1}(\sigma)$ is a PL submanifold of M, so the stratification of N by the open simplices of X pulls back to a stratification of M. In other words, any such PL map can be regarded as a map of combinatorially stratified sets in which each n-stratum of therange is a disjoint union of copies of R^n. Here we prove the analogous theorem for a smooth map $f: M \to N$ between smooth manifolds.

An essentially similar (but simplified, since 1.1 is obvious) version of our proof would also apply to PL maps between PL manifolds, so our main theorem applies in the PL category as well. The theorem will be used elsewhere [2] to show that Cohen's notion of transverse cellularity [1] may be applied in the smooth category as well.

Let N be a smooth n-manifold imbedded in some high-dimensional Euclidean space R^N. An imbedding $h: X \to N$ of a simplical complex X into N is called a smooth imbedding if $h^{-1}(\partial N)$ is a subcomplex and, for every k-simplex σ of X, there is a neighborhood U of $h(\sigma)$ in R^N and a diffeomorphism $g: U \to R^k \times R^{N-k}$ such that $gh: \sigma \to R^k \times \{0\}$ is the linear map of σ onto the standard k-simplex $\Delta^k \subset R^k$. (The use of R^N is solely to avoid a special discussion of ∂N.) If X is a combinatorial manifold and h is a homeomorphism, then h is called a smooth triangulation of N. Combinatorial triangulations of smooth manifolds always exist (see e.g., [4]).

If $f: M \to N$ is a smooth map of manifolds, a smoothly imbedded complex $h: X \to N$ is said to be transverse to f over a closed k-simplex σ in X if the composition $p_2 gf: M \to R^{N-k}$ has no critical points near $f^{-1}(h(\sigma))$. In particular, $f^{-1}(h(\mathring{\sigma}))$ is a smooth submanifold of M. The definition is independent of the choice of U, g, or the imbedding of N in R^N. Our goal is the proof of

THEOREM 0.1. *Let $f: M \to N$ be a proper smooth map of smooth manifolds, $X \subset N$ a smoothly imbedded simplicial complex, $K \subset X$ a subcomplex of X transverse to f.*

If $X \cap \partial N \subset K$ or ∂N is transverse to f then there is an ambient diffeotopy $h_t: N \to N$, fixed near K, from the identity h_0 to a map h_1 such that $h_1(X)$ is transverse to f. Moreover, the diffeotopy h_t

may be made arbitrarily small in any Riemannian metric on N.

In particular, if $f: M \to N$ is any smooth map between closed smooth manifolds, then N has a smooth triangulation transverse to f, and so f may be regarded as a strata preserving map of smooth stratified spaces in which each stratum of the range is a disjoint union of copies of R^n.

Our result is in fact somewhat stronger: if σ is a closed simplex, $f^{-1}h_1(\sigma)$ will be a topological manifold with smooth interior and boundary $f^{-1}h(\partial\sigma)$ (see Remark 1.2).

Some notation. Let $r: R^k \to R$ be the map $r(x_1, \cdots, x_k) = (\Sigma x_i^2)^{1/2}$, $\alpha B^k = r^{-1}[0, \alpha]$, for $\alpha > 0$. If M is a manifold, $\mathring{M}$ denotes interior of M. Let $\alpha I^k = \alpha\mathring{B}^1 \times \cdots \times \alpha\mathring{B}^1$ (k-times). For any X, the identity map $X \to X$ is denoted id_X.

1. Collaring smooth maps near $\partial\Delta^k$. Let Δ^k be a k-simplex in R^k with barycenter at the origin. Suppose $f: V \to R^k \times R^n$ is a proper smooth map of smooth manifolds and suppose the complex $\partial\Delta^k$ is transverse to f.

PROPOSITION 1.1. *For some $0 < \alpha < 1$ and $\varepsilon > 0$ there exist*
(1) *a smooth $m - n - 1$ manifold L*
(2) *a diffeomorphism $c: \mathring{B}^k \to \mathring{\Delta}^k$*
(3) *a diffeomorphism*

$$\bar{c}: L \times (\alpha, 1) \times \varepsilon B^n \longrightarrow f^{-1}[c(\mathring{B}^k - \alpha B^k) \times \varepsilon B^n]$$

such that

$$\begin{array}{ccc} L \times (\alpha, 1) \times \varepsilon B^n & \xrightarrow{f\bar{c}} & R^k \times R^n \\ \big\downarrow{\scriptstyle p_2} & & \big\downarrow{\scriptstyle (rc^{-1}) \times \mathrm{id}_{R^n}} \\ (\alpha, 1) \times \varepsilon B^n & \hookrightarrow & R \times R^n \end{array}$$

commutes.

Proof. *Case* 1: $n = 0$.

Proof of case 1: Let $w: R^k \to \tau_{R^k}$ be the smooth vector field $\mathrm{grad}(r)$. The trajectories of w are the rays from the origin, so, since Δ^k is convex, there is a unique trajectory through each point of $\partial\Delta^k$.

For each j-simplex σ of $\partial\Delta^k$ let $h_\sigma^0: R^j \to R^k$ be a linear imbedding such that $\sigma \subset \text{image } h_\sigma^0$. By Picard's theorem, there is a unique smooth map $h_\sigma: R^j \times R \to R^k$ such that $h_\sigma | R^j \times 0 = h_\sigma^0$ and dh_σ carries the vector field $\text{grad}(p_2: R^j \times R \to R)$ to w. If τ is a $(k-1)$-simplex then h_τ is an imbedding onto a neighborhood of τ in R^k.

Order the $(k-1)$-simplices (faces) $\tau_0, \cdots, \tau_k$ of $\partial\Delta^k$. Any j-simplex σ of $\partial\Delta^k$ is contained in $k-j$ faces $\tau_{i_1}, \cdots, \tau_{i_{k-j}}$, $i_l < i_{l+1}$. Define the map q_{i_l} from a neighborhood of τ_{i_l} to R by $q_{i_l} = p_2 h_{\tau_{i_l}}^{-1}$ and on a neighborhood U_σ of σ define $q_\sigma: U_\sigma \to R^{k-j}$ by $(q_\sigma)_l = q_{i_l}$. Observe:

(1) Since f is transverse to $\partial\Delta^k$ we may choose U_σ so small that $q_\sigma f: V \to R^{k-j}$ has no critical values near 0 in R^{k-j}.

(2) $dq_\sigma(w) = \text{grad}(p_1 + \cdots + p_{k-j})$ where $p_i: R^{k-j} \to R$ is projection on the ith factor.

(3) If $\sigma \subset \sigma'$, then $q_{\sigma'} | U_\sigma$ is just q_σ followed by a projection.

Claim. There is a smooth vector field v on V near $f^{-1}(\partial\Delta^k)$ such that for every j-simplex σ of $\partial\Delta^k$, and y sufficiently close to $f^{-1}(\sigma)$, $d(q_\sigma f)(v(y)) = d(q_\sigma)(w(f(y)))$.

Proof of claim. Use induction over simplices. Suppose v has been defined near $f^{-1}((j-1)\text{-skeleton})$, $0 \leqq j \leqq k-1$. Let σ be a j-simplex. Then, by (2) and (3) above, $d(q_\sigma f)(v) = dq_\sigma(w) = \text{grad}$ $(p_1 + \cdots + p_{k-j})$ near $f^{-1}(\partial\sigma)$. By (1) there is a vector field v_σ defined on $f^{-1}(U_\sigma)$ such that $d(q_\sigma f)(v_\sigma) = \text{grad}(p_1 + \cdots + p_{k-j})$. Let $\varphi: f^{-1}(U_\sigma) \to [0, 1]$ be a smooth map with support where v is defined and such that $\varphi = 1$ near $f^{-1}(\partial\sigma)$. Then $\varphi v + (1-\varphi)v_\sigma$ is an appropriate extension of v near σ, completing the inductive step and so verifying the claim.

Now choose $\gamma > 0$ so small that for any j-simplex σ of $\partial\Delta^k$, $q_\sigma^{-1}(\gamma I^{k-j})$ is contained in U_σ, $q_\sigma: U_\sigma \to R^{k-j}$ is nonsingular over γI^{k-j} and v is defined on $(q_\sigma f)^{-1}(\gamma I^{k-j})$. Without loss of generality, let $U_\sigma = q_\sigma^{-1}(\gamma I^{k-j})$, so $d(q_\sigma f)(v(y)) = d(q_\sigma)(w(f(y)))$ throughout $f^{-1}(U_\sigma)$.

Let $\mu: R \to [-1, 1]$ be a smooth map such that $\mu(x) = -x/|x|$ for $|x| \geqq \gamma/2$, $\mu(x) = -x$ for x near 0, and $\mu(-\gamma/2, \gamma/2) \to (-1, 1)$ is a diffeomorphism. Define $\rho_{k-j}: R^{k-j} \to R$ by $\rho_{k-j}(x_1, \cdots, x_{k-j}) = \prod_{i=1}^{k-j} \mu(x_i)$. For each j-simplex σ of $\partial\Delta^k$, define $\rho_\sigma = \rho_{k-j} q_\sigma: U_\sigma \to R$. Notice that if $\sigma \subset \sigma'$, then ρ_σ coincides with $\rho_{\sigma'}$ except perhaps where $|q_{\sigma_i}(x)| < \gamma/2$, some $0 < i < k-j$. In particular ρ_σ coincides with $\rho_{\sigma'}$ except well within U_σ. We may therefore consistently define a smooth $\rho: R^k \to [0, 1]$ as follows. Let $U = \bigcup_\sigma U_\sigma$, σ in $\partial\Delta^k$, and let $\sigma(x)$ denote the lowest dimensional simplex of $\partial\Delta^k$ for which $U_{\sigma(x)}$ contains x.

(i) If x is in $\Delta^k - U$, let $\rho(x) = 1$

(ii) If x is in $R^k - (\Delta^k \cup U)$, let $\rho(x) = -1$

(iii) If x is in U, $\rho(x) = \rho_{\sigma(x)}x$.

Clearly $\rho^{-1}(0) = \partial\Delta^k$ and $\rho|\Delta^k > 0$. Define new vector fields w' and v' over R^k and $f^{-1}(U)$ by $w'(x) = \rho(x)w(x)$, $v'(y) = \rho(f(y))v(y)$ for x in R^k and y in $f^{-1}(U)$. Just as for w and v we have, for any j-simplex σ of $\partial\Delta^k$ and y in $f^{-1}(U)$, $d(q_\sigma f)(v'(y)) = dq_\sigma(w'(f(y)))$. In other words, suppose we define the vector field u_σ in R^{k-j} by $u_\sigma = \rho_{k-j}\ \mathrm{grad}(p_1+\cdots+p_{k-j})$. Then, in fact, we have $d(q_\sigma f)(v'(y)) = dq_\sigma(w'(f(y))) = u_\sigma(q_\sigma f(y))$.

Choose $\varepsilon>0$ so small that any point s in $\rho^{-1}(\varepsilon)$ satisfies $q_\tau(s) > -\gamma/2$ for some face τ of $\partial\Delta^k$. Then for $\sigma = \sigma(s)$,

$$\begin{aligned} d(\rho_{k-j})(u_\sigma(q_\sigma(s))) &= \rho_{k-j}(q_\sigma(s)) \cdot \sum_{i=1}^{k-j} \frac{\partial\rho_{k-j}}{dx_i} \\ &= \rho_{k-j} \cdot \left(\sum_{i=1}^{k-j} \frac{\mu'(q_{\sigma_i})}{\mu(q_{\sigma_i})}\right) \cdot \prod_{i=1}^{k-j} \mu(q_{\sigma_i}) = \left(\sum_{i=1}^{k-j} \frac{\mu'(q_{\sigma_i})}{\mu(q_{\sigma_i})}\right)\rho_{k-j}^2 < 0 \end{aligned}$$

since $\mu(q_{\sigma_i}) > 0$ in $\mathring{\Delta}^k$, $\mu'(q_{\sigma_i}) \leqq 0$ and for at least one term (where $q_{\sigma_i} = q_\tau$), $\mu'(q_{\sigma_i}) < 0$. Then, for $f(y) = s$, $d(\rho f)v'(y) = (d\rho)w'(s) = d(\rho_{k-j}q_\sigma)w(s) = d(\rho_{k-j})U_\sigma(q_\sigma(s)) < 0$, so ρ and ρf are transverse to ε. Define S and L to be the smooth suqmanifolds $\rho^{-1}(\varepsilon)$ and $f^{-1}(S)$ of Δ^k and V respectively. Since $(d\rho)w' < 0$ at all s in $S = \rho^{-1}(\varepsilon)$, each trajectory of w' intersects S precisely once. Similarly, each trajectory of v' intersects L precisely once.

Picard's theorem then provides smooth imbeddings $c_0\colon S\times[0, \infty)\to \mathring{\Delta}^k$ and $\bar{c}_0\colon L\times[0, \infty)\to f^{-1}(\mathring{\Delta}^k)$ whose trajectories $c_0(s\times[0, \infty))$ and $\bar{c}_0(l\times[0, \infty))$ have tangent vectors w' and v' respectively.

Claim 2. Diagram A below commutes.

$$\begin{tikzcd} L\times[0,\infty) \arrow[rr, "c_0^{-1}f\bar{c}_0"] \arrow[dr, "p_2"'] & & S\times[0,\infty) \arrow[dl, "p_2"] \\ & {[0,\infty)} & \end{tikzcd}$$

DIAGRAM A

Proof of Claim 2. Unfortunately, $f_*(v') \neq w'$, so the proof is not immediate. For (l, t_0) in $L\times[0, \infty)$, σ any j-simplex in $\partial\Delta_k$, let $*_\sigma$ denote the following condition:

l has a neighborhood $\tilde{L}$ in L such that for some codimension 1 manifold $\tilde{M}$ in R^{k-j} transverse to u_{k-j}, $c_0(\tilde{L}\times t_0)\subset(q_\sigma f)^{-1}(\tilde{M})$.

Then notice:

(a) If $\sigma = \sigma(f(\bar{c}_0(l, t_0)))$, so $d(q_\sigma f)v' = u_\sigma$, then the unit flow of $\tilde{L}\times t_0$ in $L\times[0, \infty)$ is mapped by $q_\sigma f\bar{c}_0$ to the flow of $\tilde{M}$ along u_{k-j}, so $*_\sigma$ will continue to hold for $t \geqq t_0$ as long as $\sigma = \sigma(f(\bar{c}_0(l, t)))$.

(b) If σ is a simplex in $\sigma' \subset \partial\Delta^k$, then $*_\sigma$ holds wherever $\rho_\sigma = \rho_{\sigma'}$, i.e., except well within U_σ. Indeed, q_σ is just $q_{\sigma'}$ composed with a projection p. Let $\tilde{M} = p^{-1}(\tilde{M}')$, where $\tilde{M}'$ is the manifold of condition $*_{\sigma'}$. Since we assume $\rho_{\sigma'} = \rho_\sigma$, $dp(u_\sigma) = u_{\sigma'}$ so u_σ is transverse to $\tilde{M}$.

(c) If $\sigma = \sigma(f\bar{c}(l, 0))$, then $*_\sigma$ holds for $(l, 0)$. Indeed, take $\tilde{M}$ to be $\rho_{k-j}^{-1}(\varepsilon)$; we showed above that u_σ is transverse to $\tilde{M}$ and defined L so that it coincides with $(q_\sigma f)^{-1}(\tilde{M})$ near $\bar{c}(l, 0)$.

(d) Since the trajectories of u_σ never increase their distance from $0 \in R^{k-j}$ it follows that if $\sigma = \sigma(f\bar{c}_0(l, t_0))$, then $\sigma(f\bar{c}_0(l, t)) \subset \sigma$ for all $t \geqq t_0$.

Combining $a - d$, it follows that condition $*_\sigma$ holds for any (l, t) when $\sigma = \sigma(f\bar{c}_0(l, t))$.

Now, by definition, A commutes over $0 \in [0, \infty)$. The set of values $t \in [0, \infty)$ over which A commutes is clearly closed; we show that is also open. Let t_0 be a point such that $c_0^{-1}f\bar{c}_0(L \times t_0) = S \times t_0$. Choose any l in L and let $\sigma = \sigma(f\bar{c}_0(l, t_0))$ be a j-simplex.

Then there is a neighborhood $\tilde{L}$ of l and a codimension one manifold $\tilde{M}$ of R_{k-j} transverse to u_σ such that $\bar{c}_0(\tilde{L} \times t_0) \subset (q_0 f)^{-1}(\tilde{M})$. Then $f^{-1}(\tilde{M})$ contains a neighborhood $\tilde{S}$ of $f\bar{c}_0(l, t_0)$ in $S \times t_0$, but since $dq_\sigma(w') = u_\sigma$, the unit upward flow of $\tilde{S}$ in $S \times [0, \infty)$ is mapped by $q_\sigma c_0$ to the unit upward flow of $\tilde{M}$ under u_σ. By condition (a) above, the unit upward flow of $\tilde{L}$ in $L \times [0, \infty)$ then is mapped by $c_0^{-1}f\bar{c}_0$ to the unit upward flow of $c_0^{-1}f\bar{c}_0(\tilde{L})$ in $S \times [0, \infty)$. Hence A commutes near (l, t_0). Since f is proper, L is compact. Thus a repetition of our argument near a finite number of points l, shows A commutes over a neighborhood of t_0. Hence A commutes everywhere, verifying Claim 2.

It remains only to show that c extends to an imbedding of $\mathring{B}^k$ in $\mathring{\Delta}^k$. Each trajectory of w' intersects both $S \times 0$ and the boundary of a small ball about 0 in Δ^k exactly once. It is then a classical result that the ball can be smoothly deformed so that the interior of a collar of its boundary coincides with $S \times (0, \infty)$, giving an extension of c over the rest of Δ^k.

Case 2. $n > 0$.

Proof of Case 2. Since f is transverse to $\partial\Delta^k$ the map $p_2 f: V \to R^n$ is transverse to 0 near $\partial\Delta^k$. Then there is a neighborhood U of $\partial\Delta^k$ in R^k such that $f^{-1}(U)$ is a smooth submanifold of V, and $f \mid f^{-1}(U) \to U$ is transverse to $\partial\Delta^k$. Apply Case 1 to get $\bar{c}: L \times (0, 1) \to f^{-1}(U)$, $c: S \times (0, 1) \to U$ such that $c^{-1}f\bar{c}$ commutes with projection to $(0, 1)$. Extend c to an imbedding $c: S \times (0, 1) \times R^n \to U \times R^n$ by crossing with id_{R^n}.

Since $c^{-1}f$ is transverse to $S\times(0,1)$, it follows from classical tubular neighborhood theory that $\bar{c}$ extends to a map $c: L\times(0,1)\times \varepsilon B^n \to V$ such that $c^{-1}f\bar{c}$ commutes with projection to $(0,1)\times\varepsilon B^n$.

REMARK 1.2. Since each trajectory of w' (resp. v') lies in a trajectory of w (resp. v) and each point of $\partial\Delta^k$ (resp. $f^{-1}(\partial\Delta^k)$) lies in a unique trajectory of w (resp. v), each point of $\partial\Delta^k$ (resp. $f^{-1}(\partial\Delta^k)$) is the limit point of a unique trajectory of w' (resp. v'). Therefore the smooth imbeddings $c: S\times[0,1)\to\Delta^k$, $\bar{c}: L\times[0,1)\to f^{-1}(\Delta^k)$ given by $c(s,t) = c_0(s, t/1-t)$, $\bar{c}(l,t) = \bar{c}_0(l, t/1-t)$ extend to topological collars $c: S\times[0,1]\to\Delta^k$ and $\bar{c}: L\times[0,1]\to f^{-1}(\Delta^k)$ of $\partial\Delta^k$ and $f^{-1}(\partial\Delta^k)$ respectively.

2. Proof of the theorem.

First consider the following special case.

LEMMA 2.1. *Let $f: M\to R^k\times R^n$ be a proper smooth map transverse to $\partial\Delta^k = \partial\Delta^k\times 0\subset R^k\times 0$ and let δ: $R^k\times R^n\to(0,\infty)$ be continuous. Then there is an ambient diffeotopy h_t: $R^k\times R^n\to R^k\times R^n$, fixed outside a compact set in $\mathring{\Delta}^k\times R^n$, from the identity h_0 to a map h_1 such that $h_1(\Delta^k)$ is transverse to f. Furthermore $d(h_t(x),x)<\delta(x)$, $0\leqq t\leqq 1$.*

Proof. Sinɔe h_t will be fixed outside a compactum, we may assume δ is constant.

Let ε, L, c, $\bar{c}$, α be as in 1.1. With no loss of generality, let $\varepsilon=\delta$. Since critial values of $p_2f: M\to R^n$ are meager, by Sard's theorem, there is a regular value y_0 in $\varepsilon\mathring{B}^n$. Let ψ_t, $0\leqq t\leqq 1$ be a diffeotopy of id R^n with support in $\varepsilon\mathring{B}^n$ carrying 0 to y_0. Let $\mu: R\to[0,1]$ be a smooth map such that $\mu(x)=1$ for x near $(-\infty,\alpha]$, $\mu(x)=0$ for x near $[1,\infty)$. Define

$$h_t: c(B^k)\times R^n \longrightarrow c(B^k)\times R^n$$

by $h_t(c(x),y) = (c(x), \psi_{t\mu(|x|)}(y))$. Extend h_t by the identity to the rest of $R^k\times R^n$.

We claim $h_1(\Delta^k)$ is transverse to f. Certainly $h_1c(\alpha B^k)$ is transverse to f, for $h_1c(\alpha B^k) = \alpha B_k\times\psi_1(y) = \alpha B_k\times y_0$. Since $\mu(|x|)=0$ for $|x|$ near 1, h is fixed near $\partial\Delta^k$, so, by hypothesis, $h_1(\Delta^k)$ is transverse to f near $\partial\Delta^k$.

Define $\bar{h}$: $\bar{c}(L\times(\alpha,1)\times\varepsilon B^n)\to\bar{c}(L\times(\alpha,1)\times\varepsilon B^n)$ by $\bar{h}(\bar{c}(l,r,y)) = \bar{c}(l,r,\psi_{\mu(r)}(y))$. Then $h_1^{-1}f\bar{h} = f$, which is transverse to $c(\mathring{B}^k-\alpha B^k)$, by 1.1. Since $\bar{h}$ is a diffeomorphism, $h_1^{-1}f$ is also transverse to $c(\mathring{B}^k-\alpha B^k)$, so f is transverse to $h_1c(\mathring{B}^k-\alpha B^k)$, completing the proof.

Proof of 0.1.

Case 1: $X \cap \partial N \subset K$.

The proof is a straightforward induction over simplices of $X - K$; suppose h_t has been constructed so that f is now transverse to the $(k-1)$-skeleton. Apply 2.1 to a neighborhood of each k-simplex, the neighborhoods chosen so that f is already transverse to the k-simplices wherever neighborhoods overlap. This completes the inductive step, hence the proof in this case.

Case 2. *∂N is transverse to f.*

Apply Case 1 first to $f \mid f^{-1}(\partial N)$ isotoping ∂N until the subcomplex $X \cap \partial N$ is transverse to $f \mid f^{-1}(\partial N)$. Extend to an isotopy of N. Since f is transverse to ∂N, after the isotopy f will be transverse to $X \cap \partial N$. This reduces the problem to the previous case.

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